A Remark on Generalized Jacobian Conjecture for $\mathbb{A}^2/G$

M. Miyanishi

RCMS, Kwansei Gakuin University

January 12, 2017

Contents of the talk and Introduction

Contents of the talk and Introduction

Contents of the talk and Introduction

1. Introduction
2. Generalized Jacobian Problem (GJC) for $\mathbb{A}^2/G$
3. The standard $\mathbb{A}^1_*$-fibration $p^\circ : (\mathbb{A}^2/G)^\circ \to C$

Contents of the talk and Introduction

1. Introduction
2. Generalized Jacobian Problem (GJC) for $\mathbb{A}^2/G$
3. The standard $\mathbb{A}^1_*$-fibration $p^\circ : (\mathbb{A}^2/G)^\circ \to C$
4. A quasi-étale endomorphism of $\mathbb{A}^2/G$

1. Introduction
2. Generalized Jacobian Problem (GJC) for $\mathbb{A}^2/G$
3. The standard $\mathbb{A}^1_*$-fibration $p^\circ : (\mathbb{A}^2/G)^\circ \to C$
4. A quasi-étale endomorphism of $\mathbb{A}^2/G$
5. When does φ preserve the standard $\mathbb{A}^1_*$-fibration?

§1. Introduction

G : a small finite subgroup of $\mathrm{GL}(2, \mathbb{C})$, $X = \mathbb{A}^2/G$, P_0 : the unique singular point of X. $\varphi : X \to X$: a quasi-étale endomorphism i.e., étale on the smooth part $X^\circ = X \setminus \{P_0\}$. $\pi : \mathbb{A}^2 \to X$: the quotient morphism. Then π is a quasi-universal covering, i.e., $\pi : \mathbb{A}^2 \setminus \{O\} \to X^\circ$ is the universal covering.

The generalized Jacobian conjecture (GJC) for X

φ is an automorphism.

- φ lifts to a (G, χ)-equivariant étale endomorphism $\widetilde{\varphi} : \mathbb{A}^2 \to \mathbb{A}^2$, i.e., $\widetilde{\varphi}(O) = O$ and $\widetilde{\varphi}(gx) = \chi(g)\widetilde{\varphi}(x)$ for $x \in \mathbb{A}^2, g \in G$, where $\chi : G \to G$ is a group automorphism.

§1. Introduction

G : a small finite subgroup of $\mathrm{GL}(2,\mathbb{C})$, $X = \mathbb{A}^2/G$, P_0 : the unique singular point of X. $\varphi : X \to X$: a quasi-étale endomorphism i.e., étale on the smooth part $X^\circ = X \setminus \{P_0\}$. $\pi : \mathbb{A}^2 \to X$: the quotient morphism. Then π is a quasi-universal covering, i.e., $\pi : \mathbb{A}^2 \setminus \{O\} \to X^\circ$ is the universal covering.

The generalized Jacobian conjecture (GJC) for X

φ is an automorphism.

- φ lifts to a (G,χ)-equivariant étale endomorphism $\widetilde{\varphi} : \mathbb{A}^2 \to \mathbb{A}^2$, i.e., $\widetilde{\varphi}(O) = O$ and $\widetilde{\varphi}(gx) = \chi(g)\widetilde{\varphi}(x)$ for $x \in \mathbb{A}^2, g \in G$, where $\chi : G \to G$ is a group automorphism.
- If $\chi = \mathbf{id}.$, then $\widetilde{\varphi}$ is G-equivariant. Further, φ extends to an étale endomorphism $\widehat{\varphi} : \widehat{X} \to \widehat{X}$ such that $\widehat{\varphi}$ restricted onto the exceptional locus is an automorphism, where $\widehat{X}$ is the minimal resolution of singularity of X.

Introduction continued

- X° has the standard $\mathbb{A}^1_*$-fibration $p : X^\circ \to \mathbb{P}^1$. If φ preserves the fibration p, φ and $\widetilde{\varphi}$ are automorphisms.

- X° has the standard $\mathbb{A}^1_*$-fibration $p : X^\circ \to \mathbb{P}^1$. If φ preserves the fibration p, φ and $\widetilde{\varphi}$ are automorphisms.
- We look for sufficient conditions with which φ preserves the standard $\mathbb{A}^1_*$-fibration.

§2. Generalized Jacobian Conjecture (GJC)

GJC. Let X be a complex normal affine algebraic surface and let $\varphi : X \to X$ be an étale endomorphism. Then φ is a finite morphism.

- GJC is almost positive for smooth surfaces which are close to $\mathbb{A}^2$, but not known for $\mathbb{A}^2$ itself. There are sporadic counterexamples to GJC. Recently, Dubouloz-Palka found moduli of counterexamples in the case X is an affine pseudo-plane.

GJC. Let X be a complex normal affine algebraic surface and let $\varphi : X \to X$ be an étale endomorphism. Then φ is a finite morphism.

- GJC is almost positive for smooth surfaces which are close to $\mathbb{A}^2$, but not known for $\mathbb{A}^2$ itself. There are sporadic counterexamples to GJC. Recently, Dubouloz-Palka found moduli of counterexamples in the case X is an affine pseudo-plane.
- An enodomorphism φ is *quasi-étale* if $\varphi^\circ : X^\circ \to X^\circ$ is étale, where X° is the smooth part of X. Conversely, given φ°, it extends uniquely to an endomorphism φ of X.

§2. Generalized Jacobian Conjecture (GJC)

GJC. Let X be a complex normal affine algebraic surface and let $\varphi : X \to X$ be an étale endomorphism. Then φ is a finite morphism.

- GJC is almost positive for smooth surfaces which are close to $\mathbb{A}^2$, but not known for $\mathbb{A}^2$ itself. There are sporadic counterexamples to GJC. Recently, Dubouloz-Palka found moduli of counterexamples in the case X is an affine pseudo-plane.
- An enodomorphism φ is *quasi-étale* if $\varphi^\circ : X^\circ \to X^\circ$ is étale, where X° is the smooth part of X. Conversely, given φ°, it extends uniquely to an endomorphism φ of X.
- Gurjar-Miyanishi [J. Math. Kyoto Univ. **48** (2008), 757–764.] showed that φ is an isomorphism if there exists a non-quotient singularity.

GJC. Let X be a complex normal affine algebraic surface and let $\varphi : X \to X$ be an étale endomorphism. Then φ is a finite morphism.

- GJC is almost positive for smooth surfaces which are close to $\mathbb{A}^2$, but not known for $\mathbb{A}^2$ itself. There are sporadic counterexamples to GJC. Recently, Dubouloz-Palka found moduli of counterexamples in the case X is an affine pseudo-plane.
- An enodomorphism φ is *quasi-étale* if $\varphi^\circ : X^\circ \to X^\circ$ is étale, where X° is the smooth part of X. Conversely, given φ°, it extends uniquely to an endomorphism φ of X.
- Gurjar-Miyanishi [J. Math. Kyoto Univ. **48** (2008), 757–764.] showed that φ is an isomorphism if there exists a non-quotient singularity.
- We consider here the case $X \cong \mathbb{A}^2/G$ with G a small subgroup of $\mathrm{GL}(2, \mathbb{C})$. This case is closely related to the G-equivariant case of the Jacobian conjecture for $\mathbb{A}^2$.

§2. Generalized Jacobian Conjecture (GJC)

GJC. Let X be a complex normal affine algebraic surface and let $\varphi : X \to X$ be an étale endomorphism. Then φ is a finite morphism.

- GJC is almost positive for smooth surfaces which are close to $\mathbb{A}^2$, but not known for $\mathbb{A}^2$ itself. There are sporadic counterexamples to GJC. Recently, Dubouloz-Palka found moduli of counterexamples in the case X is an affine pseudo-plane.
- An enodomorphism φ is *quasi-étale* if $\varphi^\circ : X^\circ \to X^\circ$ is étale, where X° is the smooth part of X. Conversely, given φ°, it extends uniquely to an endomorphism φ of X.
- Gurjar-Miyanishi [J. Math. Kyoto Univ. **48** (2008), 757–764.] showed that φ is an isomorphism if there exists a non-quotient singularity.
- We consider here the case $X \cong \mathbb{A}^2/G$ with G a small subgroup of $\mathrm{GL}(2, \mathbb{C})$. This case is closely related to the G-equivariant case of the Jacobian conjecture for $\mathbb{A}^2$.

Let $X = \mathbb{A}^2/G$ and let $X^\circ = X \setminus \{P_0\}$, where P_0 is the unique singular point. Suppose that G is non-cyclic. Then X° is embedded as an open set into a smooth $\mathbb{P}^1$-fiber space $p : V \to C \cong \mathbb{P}^1$ as $X^\circ = V - (S_0 + T_1 + T_2 + T_3) - (S_1 + R_1 + R_2 + R_3)$, where :

1. S_0 and S_1 are sections of the $\mathbb{P}^1$-fibration p.

§3. The standard $\mathbb{A}^1_*$-fibration $p^\circ : (\mathbb{A}^2/G)^\circ \to C$

Let $X = \mathbb{A}^2/G$ and let $X^\circ = X \setminus \{P_0\}$, where P_0 is the unique singular point. Suppose that G is non-cyclic. Then X° is embedded as an open set into a smooth $\mathbb{P}^1$-fiber space $p : V \to C \cong \mathbb{P}^1$ as $X^\circ = V - (S_0 + T_1 + T_2 + T_3) - (S_1 + R_1 + R_2 + R_3)$, where :

1. S_0 and S_1 are sections of the $\mathbb{P}^1$-fibration p.
2. $S_0 + T_1 + T_2 + T_3$ is the exceptional locus of the minimal resolution of the singular point P_0, and hence it is an admissible rational fork with negative definite intersection matrix.

§3. The standard $\mathbb{A}^1_*$-fibration $p^\circ : (\mathbb{A}^2/G)^\circ \to C$

Let $X = \mathbb{A}^2/G$ and let $X^\circ = X \setminus \{P_0\}$, where P_0 is the unique singular point. Suppose that G is non-cyclic. Then X° is embedded as an open set into a smooth $\mathbb{P}^1$-fiber space $p : V \to C \cong \mathbb{P}^1$ as $X^\circ = V - (S_0 + T_1 + T_2 + T_3) - (S_1 + R_1 + R_2 + R_3)$, where :

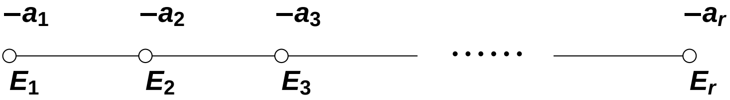
1. S_0 and S_1 are sections of the $\mathbb{P}^1$-fibration p.
2. $S_0 + T_1 + T_2 + T_3$ is the exceptional locus of the minimal resolution of the singular point P_0, and hence it is an admissible rational fork with negative definite intersection matrix.
3. There are three (-1)-curves F_1, F_2, F_3 such that $T_i + F_i + R_i$ is a linear chain and a degenerate $\mathbb{P}^1$-fiber ℓ_i of p with the unique (-1)-curve F_i for $i = 1, 2, 3$. Let m_i be the multiplicity of F_i in ℓ_i. Then the set $\{m_1, m_2, m_3\}$ is one of the Platonic triplets $\{2, 2, n\} (n \geq 2), \{2, 3, 3\}, \{2, 3, 4\}$ and $\{2, 3, 5\}$. The linear part R_i in ℓ_i is uniquely determined by $T_i + F_i$.

§3. The standard $\mathbb{A}^1_*$-fibration $p^\circ : (\mathbb{A}^2/G)^\circ \to C$

Let $X = \mathbb{A}^2/G$ and let $X^\circ = X \setminus \{P_0\}$, where P_0 is the unique singular point. Suppose that G is non-cyclic. Then X° is embedded as an open set into a smooth $\mathbb{P}^1$-fiber space $p : V \to C \cong \mathbb{P}^1$ as $X^\circ = V - (S_0 + T_1 + T_2 + T_3) - (S_1 + R_1 + R_2 + R_3)$, where :

1. S_0 and S_1 are sections of the $\mathbb{P}^1$-fibration p.
2. $S_0 + T_1 + T_2 + T_3$ is the exceptional locus of the minimal resolution of the singular point P_0, and hence it is an admissible rational fork with negative definite intersection matrix.
3. There are three (-1)-curves F_1, F_2, F_3 such that $T_i + F_i + R_i$ is a linear chain and a degenerate $\mathbb{P}^1$-fiber ℓ_i of p with the unique (-1)-curve F_i for $i = 1, 2, 3$. Let m_i be the multiplicity of F_i in ℓ_i. Then the set $\{m_1, m_2, m_3\}$ is one of the Platonic triplets $\{2, 2, n\} (n \geq 2), \{2, 3, 3\}, \{2, 3, 4\}$ and $\{2, 3, 5\}$. The linear part R_i in ℓ_i is uniquely determined by $T_i + F_i$.
4. There are no singular fibers of p other than the above three fibers ℓ_i.

The restriction $p^\circ = p|_{X^\circ} : X^\circ \to C$ is a Platonic $\mathbb{A}^1_*$-fiber space.
Suppose that G is a cyclic group of order n, $G = \{\zeta^i \mid 0 \le i < n\}$ with a primitive nth root ζ of 1. Take a system of coordinates $\{x, y\}$ on $\mathbb{A}^2$ so that G acts by ${}^\zeta(x, y) = (\zeta x, \zeta^d y)$, where $0 < d < n$ and $\gcd(n, d) = 1$. Then the minimal resolution of singularity at P_0 has the dual graph of the exceptional locus



where $E_i \cong \mathbb{P}^1$, $a_i \ge 2$ and $n/d = [a_1, a_2, \ldots, a_r]$ is the continued fraction expansion

$$\frac{n}{d} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_{r-1} - \cfrac{1}{a_r}}}}$$

Let $\widetilde{\varphi}$ be a G-equivariant étale endomorphism of $\mathbb{A}^2$ such that $\widetilde{\varphi}^{-1}(O) = O$, where O is the point of origin.

Lemma 3.1

Let $\widehat{\mathbb{A}}^2$ be the blowing-up of $\mathbb{A}^2$ at O and let E be the exceptional curve. Then $\widetilde{\varphi}$ induces a G-equivariant étale endomorphism $\widehat{\varphi} : \widehat{\mathbb{A}}^2 \to \widehat{\mathbb{A}}^2$ such that $\widehat{\varphi}$ is an automorphism on E. Furthermore, G acts on E via the natural action σ of $\mathrm{PGL}\,(2, \mathbb{C})$ on $\mathbb{P}^1$.

(1) If G is non-cyclic, the action (G, σ) has three nontrivial isotropy subgroups upto conjugacy which are cyclic groups of order m_1, m_2, m_3, where $\{m_1, m_2, m_3\}$ is a Platonic triplet. Hence, there exist three cyclic singularities on $E/G \hookrightarrow \widehat{\mathbb{A}}^2/G$. The minimal resolution of these singularities gives the lower part $S_0 + T_1 + T_2 + T_3$ of the normal completion V of X°, where S_0 is the proper transform of the curve E/G.

Let $\widetilde{\varphi}$ be a G-equivariant étale endomorphism of $\mathbb{A}^2$ such that $\widetilde{\varphi}^{-1}(O) = O$, where O is the point of origin.

Lemma 3.1

Let $\widehat{\mathbb{A}}^2$ be the blowing-up of $\mathbb{A}^2$ at O and let E be the exceptional curve. Then $\widetilde{\varphi}$ induces a G-equivariant étale endomorphism $\widehat{\varphi} : \widehat{\mathbb{A}}^2 \to \widehat{\mathbb{A}}^2$ such that $\widehat{\varphi}$ is an automorphism on E. Furthermore, G acts on E via the natural action σ of $\mathrm{PGL}\,(2, \mathbb{C})$ on $\mathbb{P}^1$.

(1) If G is non-cyclic, the action (G, σ) has three nontrivial isotropy subgroups upto conjugacy which are cyclic groups of order m_1, m_2, m_3, where $\{m_1, m_2, m_3\}$ is a Platonic triplet. Hence, there exist three cyclic singularities on $E/G \hookrightarrow \widehat{\mathbb{A}}^2/G$. The minimal resolution of these singularities gives the lower part $S_0 + T_1 + T_2 + T_3$ of the normal completion V of X°, where S_0 is the proper transform of the curve E/G.

(2) If G is cyclic and $d > 1$, there are two G-fixed points Q_0, Q_∞ on E such that the G-action is given by $(x, t) \mapsto (\zeta x, \zeta^{d-1} t)$ at Q_0 and $(u, y) \mapsto (\zeta^{n-d+1} u, \zeta^d y)$ at Q_∞,

where $\{x, y\}$ are coordinates $\mathbb{A}^2$, $t = y/x$ and $u = x/y$. If $\delta_1 = \gcd(n, d-1) > 1$, we consider the pair $(n/\delta_1, (d-1)/\delta_1)$ for $(n, d-1)$. Similarly, the pair $(n/\delta_2, e/\delta_2)$ for $(n-d+1, d)$, where $\xi = \zeta^d$, $\xi^e = \zeta^{n-d+1}$ $(0 < e < n)$ if $\delta_2 = \gcd(n, e) > 1$. Let S_0 be the proper transform of E/G and T_0, T_∞ the exceptional loci of the minimal resolution of the image points of Q_0, Q_∞. Then there exists a normal smooth completion V and a $\mathbb{P}^1$-fibration $p : V \to C$ such that
$X^\circ = V - (S_0 + T_0 + T_\infty) - (S_1 + R_0 + R_\infty)$, where:

Lemma 3.1 continued.

where $\{x, y\}$ are coordinates $\mathbb{A}^2$, $t = y/x$ and $u = x/y$. If $\delta_1 = \gcd(n, d-1) > 1$, we consider the pair $(n/\delta_1, (d-1)/\delta_1)$ for $(n, d-1)$. Similarly, the pair $(n/\delta_2, e/\delta_2)$ for $(n-d+1, d)$, where $\xi = \zeta^d$, $\xi^e = \zeta^{n-d+1}$ $(0 < e < n)$ if $\delta_2 = \gcd(n, e) > 1$. Let S_0 be the proper transform of E/G and T_0, T_∞ the exceptional loci of the minimal resolution of the image points of Q_0, Q_∞. Then there exists a normal smooth completion V and a $\mathbb{P}^1$-fibration $p : V \to C$ such that
$X^\circ = V - (S_0 + T_0 + T_\infty) - (S_1 + R_0 + R_\infty)$, where:

1. S_0 and S_1 are sections of p. Furthermore, $(S_0^2) = -1$.
2. There exist two degenerate fibers $T_0 + F_0 + R_0$ and $T_\infty + F_\infty + R_\infty$ of p, where F_0, F_∞ are (-1)-curves with respective multiplicities $m_0 = n/\delta_1$ and $m_\infty = n/\delta_2$.

Lemma 3.1 continued.

where $\{x, y\}$ are coordinates $\mathbb{A}^2$, $t = y/x$ and $u = x/y$. If $\delta_1 = \gcd(n, d-1) > 1$, we consider the pair $(n/\delta_1, (d-1)/\delta_1)$ for $(n, d-1)$. Similarly, the pair $(n/\delta_2, e/\delta_2)$ for $(n-d+1, d)$, where $\xi = \zeta^d$, $\xi^e = \zeta^{n-d+1}$ $(0 < e < n)$ if $\delta_2 = \gcd(n, e) > 1$. Let S_0 be the proper transform of E/G and T_0, T_∞ the exceptional loci of the minimal resolution of the image points of Q_0, Q_∞. Then there exists a normal smooth completion V and a $\mathbb{P}^1$-fibration $p : V \to C$ such that $X^\circ = V - (S_0 + T_0 + T_\infty) - (S_1 + R_0 + R_\infty)$, where:

1. S_0 and S_1 are sections of p. Furthermore, $(S_0^2) = -1$.
2. There exist two degenerate fibers $T_0 + F_0 + R_0$ and $T_\infty + F_\infty + R_\infty$ of p, where F_0, F_∞ are (-1)-curves with respective multiplicities $m_0 = n/\delta_1$ and $m_\infty = n/\delta_2$.
3. The parts R_0 and R_∞ have no (-1)-curves.

(3) If G is cyclic and $d = 1$, then the G-action on E is trivial and $X^\circ = V - (S_0 + S_1)$, where $V = \mathbb{F}_n$, $(S_0^2) = -n$ and $(S_1^2) = n$.

We call V the standard completion of $X^\circ := (\mathbb{A}^2/G)^\circ$, $p : V \to C$ the standard $\mathbb{P}^1$-fibration and $p^\circ : X^\circ \to C$ the standard $\mathbb{A}^1_*$-fibration, where $C \cong \mathbb{P}^1$. Let $D = V - X^\circ$ the boundary divisor of X°.

Lemma 3.2

The logarithmic canonical divisor $D + K_V$ is given as follows, where ℓ is a general fiber of the standard $\mathbb{P}^1$-fibration p.

$$D + K_V \sim \begin{cases} \ell - (F_1 + F_2 + F_3) & \text{Case } G \text{ is non-cyclic} \\ -F_0 - F_\infty & \text{Case } G \text{ is cyclic and } d > 1 \\ -2\ell & \text{Case } G \text{ is cyclic and } d = 1 \end{cases}$$

$X = \mathbb{A}^2/G$, $\varphi : X_u \to X_\ell$ a quasi-étale endomorphism and $\varphi^\circ : X_u^\circ \to X_\ell^\circ$ the induced étale endomorphism.

Lemma 4.1

The following assertions hold.

1. There exists an étale endomorphism $\widetilde{\varphi} : \mathbb{A}^2 \to \mathbb{A}^2$ such that $\widetilde{\varphi}^{-1}(O) = O$ and $\pi \cdot \widetilde{\varphi} = \varphi \cdot \pi$ for the quotient morphism $\pi : \mathbb{A}^2 \to X$. Furthermore, $\widetilde{\varphi}(gx) = \chi(g)\widetilde{\varphi}(x)$ for a group endomorphism χ of G.

$X = \mathbb{A}^2/G$, $\varphi : X_u \to X_\ell$ a quasi-étale endomorphism and $\varphi^\circ : X_u^\circ \to X_\ell^\circ$ the induced étale endomorphism.

Lemma 4.1

The following assertions hold.

1. There exists an étale endomorphism $\widetilde{\varphi} : \mathbb{A}^2 \to \mathbb{A}^2$ such that $\widetilde{\varphi}^{-1}(O) = O$ and $\pi \cdot \widetilde{\varphi} = \varphi \cdot \pi$ for the quotient morphism $\pi : \mathbb{A}^2 \to X$. Furthermore, $\widetilde{\varphi}(gx) = \chi(g)\widetilde{\varphi}(x)$ for a group endomorphism χ of G.
2. χ is an automorphism.

$X = \mathbb{A}^2/G$, $\varphi : X_u \to X_\ell$ a quasi-étale endomorphism and $\varphi^\circ : X_u^\circ \to X_\ell^\circ$ the induced étale endomorphism.

Lemma 4.1

The following assertions hold.

1. There exists an étale endomorphism $\widetilde{\varphi} : \mathbb{A}^2 \to \mathbb{A}^2$ such that $\widetilde{\varphi}^{-1}(O) = O$ and $\pi \cdot \widetilde{\varphi} = \varphi \cdot \pi$ for the quotient morphism $\pi : \mathbb{A}^2 \to X$. Furthermore, $\widetilde{\varphi}(gx) = \chi(g)\widetilde{\varphi}(x)$ for a group endomorphism χ of G.
2. χ is an automorphism.
3. Replacing φ by φ^N for some $N > 0$, we may assume that $\chi = \mathrm{id}$..

Lemma 4.2

(X, Q_0) a germ of a complex normal surface with the quotient singular point Q_0. $\varphi : X \to X$ an endomorphism such that $\varphi^{-1}(Q_0) = \{Q_0\}$ and $\varphi|_{X^\circ}$ is étale, where $X^\circ = X \setminus \{Q_0\}$. $\widehat{X}$ the minimal resolution of the singularity at Q_0. Then, after replacing φ by a suitable power of φ, φ extends to an étale endomorphism $\widehat{\varphi}$ of $\widehat{X}$.

Corollary 4.3

X and φ the same as in Lemma 4.1. $\widehat{X}$ the minimal resolution of singularity of X. Then, after replacing φ by its suitable power, φ extends to an étale endomorphism $\widehat{\varphi}$ of $\widehat{X}$ such that $\widehat{\varphi}$ preserves the exceptional locus of the singularity. Hence $\widehat{\varphi}$ induces an automorphism on the exceptional locus. This implies that the given quasi-étale endomorphism φ is a local isomorphism (hence étale) at P_0 as well.

Theorem 4.4

Let $\varphi : X \to X$ be a quasi-étale endomorphism of $X = \mathbb{A}^2/G$. Suppose that φ preserves the standard $\mathbb{A}^1_*$-fibration p°, i.e., there exists an endomorphism $\beta : C \to C$ such that $p^\circ \cdot (\varphi|_{X^\circ}) = \beta \cdot p^\circ$. Then φ is an automorphism.

Case G is non-cyclic. $m_1\Gamma_1, m_2\Gamma_2, m_3\Gamma_3$ multiple fibers of $p : X^\circ \to C \cong \mathbb{P}^1$. Replace φ by φ^N and assume that $\beta(P_i) = P_i$ for $P_i = p^\circ(\Gamma_i)$ and φ lifts to $\widetilde{\varphi} : \mathbb{A}^2 \to \mathbb{A}^2$ which is étale.

1. φ is étale $\Longrightarrow \beta$ is étale $\Longrightarrow \beta$ is an automorphism.

Theorem 4.4

Let $\varphi : X \to X$ be a quasi-étale endomorphism of $X = \mathbb{A}^2/G$. Suppose that φ preserves the standard $\mathbb{A}^1_*$-fibration p°, i.e., there exists an endomorphism $\beta : C \to C$ such that $p^\circ \cdot (\varphi|_{X^\circ}) = \beta \cdot p^\circ$. Then φ is an automorphism.

Case G is non-cyclic. $m_1\Gamma_1, m_2\Gamma_2, m_3\Gamma_3$ multiple fibers of $p : X^\circ \to C \cong \mathbb{P}^1$. Replace φ by φ^N and assume that $\beta(P_i) = P_i$ for $P_i = p^\circ(\Gamma_i)$ and φ lifts to $\widetilde{\varphi} : \mathbb{A}^2 \to \mathbb{A}^2$ which is étale.

1. φ is étale $\Longrightarrow \beta$ is étale $\Longrightarrow \beta$ is an automorphism.
2. Let $\overline{(p^\circ \cdot \pi^\circ)^{-1}(P_i)} = \ell_i^{(1)} \cup \cdots \cup \ell_i^{(r_i)}$, where $\ell_i^{(j)}$ is a line through the point O. Since $\varphi(\Gamma_i) = \Gamma_i$, $\widetilde{\varphi}$ preserves the set $\{\ell_i^{(j)} \mid 1 \le j \le r_i\}$.

Theorem 4.4

Let $\varphi : X \to X$ be a quasi-étale endomorphism of $X = \mathbb{A}^2/G$. Suppose that φ preserves the standard $\mathbb{A}^1_*$-fibration p°, i.e., there exists an endomorphism $\beta : C \to C$ such that $p^\circ \cdot (\varphi|_{X^\circ}) = \beta \cdot p^\circ$. Then φ is an automorphism.

Case G is non-cyclic. $m_1\Gamma_1, m_2\Gamma_2, m_3\Gamma_3$ multiple fibers of $p : X^\circ \to C \cong \mathbb{P}^1$. Replace φ by φ^N and assume that $\beta(P_i) = P_i$ for $P_i = p^\circ(\Gamma_i)$ and φ lifts to $\widetilde{\varphi} : \mathbb{A}^2 \to \mathbb{A}^2$ which is étale.

1. φ is étale $\Longrightarrow \beta$ is étale $\Longrightarrow \beta$ is an automorphism.
2. Let $\overline{(p^\circ \cdot \pi^\circ)^{-1}(P_i)} = \ell_i^{(1)} \cup \cdots \cup \ell_i^{(r_i)}$, where $\ell_i^{(j)}$ is a line through the point O. Since $\varphi(\Gamma_i) = \Gamma_i$, $\widetilde{\varphi}$ preserves the set $\{\ell_i^{(j)} \mid 1 \le j \le r_i\}$.
3. Choose ℓ_1, ℓ_2, ℓ_3 from $\cup_{i=1}^{3}\{\ell_i^{(j)} \mid 1 \le j \le r_i\}$ so that they are defined by $x = 0, y = 0$ and $x = y$. Then $\widetilde{\varphi}^*(x)$ and $\widetilde{\varphi}^*(y)$ are homogeneous polynomials. Since $\widetilde{\varphi}$ is étale, $\widetilde{\varphi}^*(x)$ and $\widetilde{\varphi}^*(y)$ are linear. Hence $\widetilde{\varphi}$ (and φ) is an automorphism.

§5. When does φ preserve the standard $\mathbb{A}^1_*$-fibration?

Y : normalization of X_ℓ in $\mathbb{C}(X_u)$, $\nu : Y \to X_\ell$: normalization morphism. By Zariski Main Theorem, $\nu \cdot \iota = \varphi$ for an open immersion $\iota : X_u \to Y$. Identify X_u with $\iota(X_u)$.

Lemma 5.1 (cf. Gurjar-Miyanishi [J. Math. Kyoto Univ., Th. 1.1])

$Y - X_u = \coprod_{i=1}^r A_i$ with $A_i \cong \mathbb{A}^1$. For each i, there exists at most one cyclic quotient singularity P_i lying on A_i. The exceptional locus B_i of the cyclic singularity at P_i meets the proper transform of A_i at the end component of B_i.

Remark 5.2

If no points of $Y - X_u$ map to the singular point P_0 of X, then $\varphi : X \to X$ is proper over P_0 and hence φ is an automorphism at P_0 by Corollary 4.3 provided φ is G-equivariant. Since $\varphi^{-1}(P_0) = P_0$ by the hypothesis, φ is then an automorphism. Since φ^N is G-equivariant for some N, φ^N is an automorphism, whence so is φ.

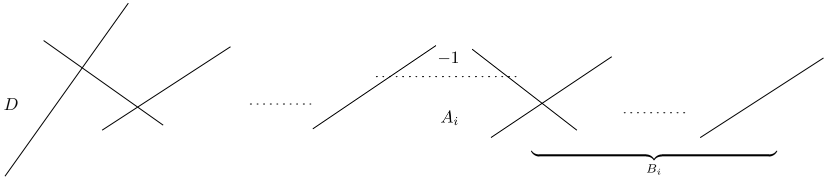
Case $\nu^{-1}(P_0) \cap (Y - X_u) \neq \emptyset$

We assume that φ lifts to a G-equivalent endomorphism $\widehat{\varphi}$ of $\mathbb{A}^2$.
$\widehat{Y}$: the minimal resolution of singularity of Y and $\alpha : \widetilde{Y} \to \widehat{Y}$: a minimal sequence of blowing-ups such that
(1) $\widehat{X}_u \hookrightarrow \widetilde{Y}$ an open immersion (cf. Corollary 4.3) and
(2) $\widetilde{\nu} : \widetilde{Y} \xrightarrow{\alpha} \widehat{Y} \longrightarrow \widehat{X}_\ell$ is a morphism
W : a smooth normal completion of $\widetilde{Y}$.
V_ℓ : the standard completion of X_ℓ°.
$\Phi : W \to V_\ell$ a projective morphism such that the diagram below is commutative

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & W \\ {\scriptstyle \widetilde{\nu}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\ \widehat{X}_\ell & \longrightarrow & V_\ell \end{array}$$

and W is obtained from V_u by a birational morphism $\rho : W \to V_u$ which is a composite of blowing up points on the boundary divisor $D = V_u - X_u^\circ$.

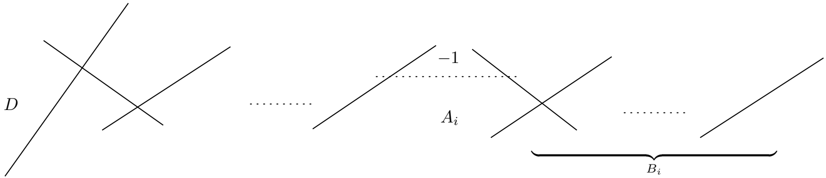
If Y has a cyclic singularity, $\widehat{Y}$ looks like



Let $\sigma : V'' \to V'$ be an intermediate single blowing-up in $\rho : W \to V_u$ with exceptional curve E.

1. If σ is subdivisional or a blow-up of a non-boundary point with E included in the new boundary divisor $D'' = \sigma'(D') + E$, then $D'' + K_{V''} = \sigma^*(D' + K_{V'})$.

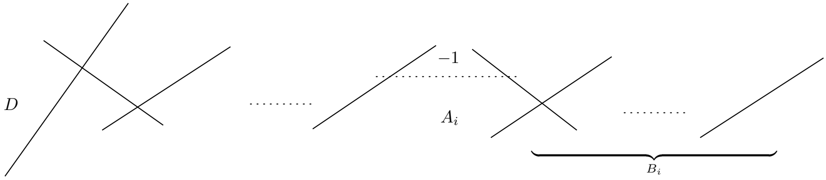
If Y has a cyclic singularity, $\widehat{Y}$ looks like



Let $\sigma : V'' \to V'$ be an intermediate single blowing-up in $\rho : W \to V_u$ with exceptional curve E.

1. If σ is subdivisional or a blow-up of a non-boundary point with E included in the new boundary divisor $D'' = \sigma'(D') + E$, then $D'' + K_{V''} = \sigma^*(D' + K_{V'})$.
2. If σ is subdivisional blowing-up with $E \not\subset D''$, then $D'' + K_{V''} = \sigma^*(D' + K_{V'}) - E$.

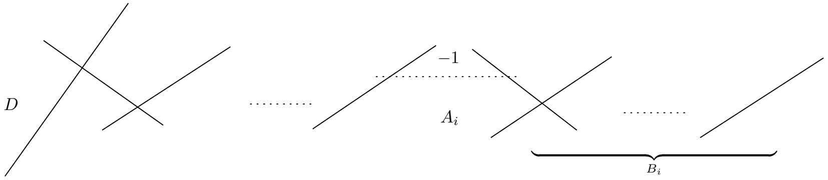
If Y has a cyclic singularity, $\widehat{Y}$ looks like



Let $\sigma : V'' \to V'$ be an intermediate single blowing-up in $\rho : W \to V_u$ with exceptional curve E.

1. If σ is subdivisional or a blow-up of a non-boundary point with E included in the new boundary divisor $D'' = \sigma'(D') + E$, then $D'' + K_{V''} = \sigma^*(D' + K_{V'})$.
2. If σ is subdivisional blowing-up with $E \not\subset D''$, then $D'' + K_{V''} = \sigma^*(D' + K_{V'}) - E$.
3. If σ is sprouting with $E \not\subset D''$, then $D'' + K_{V''} = \sigma^*(D' + K_{V'})$.

If Y has a cyclic singularity, $\widehat{Y}$ looks like



Let $\sigma : V'' \to V'$ be an intermediate single blowing-up in $\rho : W \to V_u$ with exceptional curve E.

1. If σ is subdivisional or a blow-up of a non-boundary point with E included in the new boundary divisor $D'' = \sigma'(D') + E$, then $D'' + K_{V''} = \sigma^*(D' + K_{V'})$.
2. If σ is subdivisional blowing-up with $E \not\subset D''$, then $D'' + K_{V''} = \sigma^*(D' + K_{V'}) - E$.
3. If σ is sprouting with $E \not\subset D''$, then $D'' + K_{V''} = \sigma^*(D' + K_{V'})$.
4. If σ is sprouting with $E \subset D''$, then $D'' + K_{V''} = \sigma^*(D' + K_{V'}) + E$.

Since $\widehat{Y} - \widehat{X}_u = \coprod_{i=1}^{r}(A_i + B_i)$, the cases (2) and (3) occur only for $A_i = E$. Y° minus the centers of blowing-ups of $\alpha : \widetilde{Y} \to \widehat{Y}$ is identified with an open set of W. Let Δ be the complement in W, which is an SNC divisor.

Lemma 5.3

We have

$$\Delta + K_W = \rho^*(D + K_{V_u}) + \sum_j E_j - \sum_{i=1}^{s} A_i \,, \qquad (1)$$

where E_j exhausts the total transforms on W of the exceptional curves of sprouting blowing-ups in the case (4) and A_i $(1 \le i \le s)$ (identified with the proper transform on W) does the irreducible components of $Y - X_u$ carrying the singular points of Y. Here $\sum_j E_j - \sum_{i=1}^{s} A_i$ is an effective divisor.

The logarithmic ramification divisor formula for $\Phi : (W, \Delta) \to (V_\ell, D)$ implies

$$\Delta + K_W = \Phi^*(D + K_{V_\ell}) + R, \tag{2}$$

where R is an effective divisor. Note that $\Phi^{-1}(D) = \Delta$.

Lemma 5.4

The divisor R is supported by the union of curves C on W such that C is contracted by Φ or $\Phi|_C : C \to \Phi(C)$ is ramifying. The curve C has coefficient zero if C is a component of Δ which is not contracted by Φ or possibly if C is a component of Δ which is contracted to an intersection point of two irreducible components of D.

Combining (1) and (2), we have

$$\Phi^*(D + K_{V_\ell}) = \rho^*(D + K_{V_u}) + \sum_j E_j - \sum_i A_i - R\,. \tag{3}$$

$H := \rho'(S_1)$. $p : W \to C$: the lift of the satandard $\mathbb{P}^1$-fibration $p : V_u \to C$. Except for H, all other components of $\sum_j E_j - \sum_i A_i - R$ are fiber components of p.

Theorem 5.5

If $\Phi(H) \neq$ a point, then φ preserves the standard $\mathbb{A}^1_*$-fibration.

By Lemma 5.4, $H \not\subset \mathrm{Supp}\,(R)$. Intersecting a general fiber ℓ of p with both sides of (3), $C \cdot (D + K_{V_\ell}) = 0$, where $C = \Phi(\ell)$.
Case G is non-cyclic. By Lemma 3.2,

$$D + K_{V_\ell} \sim \ell - (F_1 + F_2 + F_3) \geq \left(1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\right)\ell.$$

Hence $0 \geq \left(1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\right) \cdot (C \cdot \ell)$ and $C \cdot \ell \geq 0$. If $C \cdot \ell = 0$, C is a fiber of p on V_ℓ. If $C \cdot \ell > 0$, C meets the boundary components of $\ell_i - m_i F_i$ for some $i = 1, 2, 3$, where $F_i \subset \ell_i$. Since ℓ has only two places at the boundary and since $\Phi(\Delta) = D$, C has only two places at the boundary D, i.e., $C \cap S_0$ and $C \cap S_1$. This is a contradiction.

Remark 5.6

By Lemma 5.4, $\Phi(H) \neq$ a point implies $H \not\subset \mathrm{Supp}\,(R)$. The converse is not known except for the case G is cyclic and $d = 1$.

Let $B = \Phi^*(\ell)$ with ℓ a fiber of $p : V_\ell \to C$. Then $|B|$ is composed of a linear pencil.

Theorem 5.7

B is an irreducible curve and $|B|$ is an irreducible pencil. The following four conditions are equivalent.

1. $B \cdot (\Delta + K_W) = 0$.

Remark 5.6

By Lemma 5.4, $\Phi(H) \neq$ a point implies $H \not\subset \mathrm{Supp}\,(R)$. The converse is not known except for the case G is cyclic and $d = 1$.

Let $B = \Phi^*(\ell)$ with ℓ a fiber of $p : V_\ell \to C$. Then $|B|$ is composed of a linear pencil.

Theorem 5.7

B is an irreducible curve and $|B|$ is an irreducible pencil. The following four conditions are equivalent.

1. $B \cdot (\Delta + K_W) = 0$.
2. $B \cong \mathbb{P}^1$ and $B \cdot R = 0$.

Remark 5.6

By Lemma 5.4, $\Phi(H) \neq$ a point implies $H \not\subset \mathrm{Supp}\,(R)$. The converse is not known except for the case G is cyclic and $d = 1$.

Let $B = \Phi^*(\ell)$ with ℓ a fiber of $p : V_\ell \to C$. Then $|B|$ is composed of a linear pencil.

Theorem 5.7

B is an irreducible curve and $|B|$ is an irreducible pencil. The following four conditions are equivalent.

1. $B \cdot (\Delta + K_W) = 0$.
2. $B \cong \mathbb{P}^1$ and $B \cdot R = 0$.
3. The divisor Δ contains an irreducible component Δ_1 such that $\Delta_1 \cong S_1$ by Φ.

Remark 5.6

By Lemma 5.4, $\Phi(H) \neq$ a point implies $H \not\subset \mathrm{Supp}\,(R)$. The converse is not known except for the case G is cyclic and $d = 1$.

Let $B = \Phi^*(\ell)$ with ℓ a fiber of $p : V_\ell \to C$. Then $|B|$ is composed of a linear pencil.

Theorem 5.7

B is an irreducible curve and $|B|$ is an irreducible pencil. The following four conditions are equivalent.

1. $B \cdot (\Delta + K_W) = 0$.
2. $B \cong \mathbb{P}^1$ and $B \cdot R = 0$.
3. The divisor Δ contains an irreducible component Δ_1 such that $\Delta_1 \cong S_1$ by Φ.
4. φ is an automorphism.