On minimal 3-folds of general type with the geometric genus 1, 2 or 3

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Introduction

- Canonical stability index

$$r_s(V) = \min\{m_0 \mid \phi_m \text{ is birational for all } m \geq m_0\}$$

- Curve family $\mathcal{C}$ on X

$$\mathcal{C} = \{\tilde{g}(F) \mid F = \text{fibre of } f\}$$

$$\begin{array}{ccc} F & \longrightarrow & \bullet \\ Y & \xrightarrow[\text{fibration}]{f} & Z \\ {\scriptstyle \tilde{g}}\downarrow & & \\ C \subset X & & \end{array}$$

fibre = curve

canonical degree
$= (K_X \cdot C)$

- Theorem 0. (Bombieri) S minimal surface of general type. Then
  (1) $r_s(S) \leq 5$
  (2) $r_s(S) = 5 \iff S$ has a curve family $\mathcal{C}$ of genus 2 of canonical degree 1.

Introduction

X minimal 3-fold of general type

$p_g = h^0(\omega_X)$ ~ geometric genus

- $p_g \geqslant 4$, $r_s(X) \leqslant 5$ (Optimal)

curve family

$r_s(X) = 5$ (i.e. φ_4 non-birat.) $\Longleftrightarrow$
$$\begin{cases} p_g \geqslant 5, \ \exists\, \mathcal{C} \text{ of genus } 2 \text{ with the canonical degree } 1 \ \text{(Chen-D.-Q. Zhang)} \\ p_g = 4, \ \text{Characterized by (Chen - Q. Zhang)} \end{cases}$$

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Introduction and Examples

- $p_g = 3 \implies r_s(X) \leq 6$ (optimal)

  $r_s(X) = 6 \iff$?

- $p_g = 2 \implies r_s(X) \leq 8$ (optimal)

  $r_s(X) = 8 \iff X$ has a curve family $\mathcal{C}$ of genus 2 with canonical degree "$2/3$".

  $r_s(X) = 7 \iff$?

- $p_g = 1$. Chen-Chen $\Rightarrow r_s(X) \leq 18$

$$\left.\begin{array}{l} r_s(X) = 18 \\ r_s(X) = 17 \\ r_s(X) = 16 \end{array}\right\} \iff \text{? open}$$

- $p_g = 0$. Chen–Chen $\Rightarrow$ $r_3(X) \leqslant 61$
  $(r_3(X) \leqslant 57)$.

- Examples (Iano-Fletcher)

  1. $X_{12} \subset \mathbb{P}(1,1,1,2,6)$. $K^3 = 1$, $p_g = 3$
     $r_3(X_{12}) = 6$

  2. $X_{16} \subset \mathbb{P}(1,1,2,3,8)$, $K^3 = 1/3$, $p_g = 2$
     $r_3(X_{16}) = 8$

  3. $X_{14} \subset \mathbb{P}(1,1,2,2,7)$, $K^3 = 1/2$, $p_g = 2$
     $r_3(X_{14}) = 7$

Examples and Main Theorems

4. $X_{4,12} \subset \mathbb{P}(1,1,2,2,3,6)$, $K^3 = 2/3$, $p_g = 2$

   $r_s(X_{4,12}) = 6$

5. $X_{28} \subset \mathbb{P}(1,3,4,5,14)$, $K^3 = 1/3$, $p_g = 1$

   $r_s(X) = 14$.

(Joint with Yong Hu, Matteo Penegini)
(Genova)

<u>Theorem 1</u>. Assume $p_g(X) = 1$. Then

(1) $r_s(X) \leq 17$

(2) $r_s(X) \leq 16$ unless X belongs to one of the following types: (Table A)

- $B_X = \{4\times(1,2), (3,7), 3\times(2,5), (1,3)\}$, $K^3 = \frac{2}{105}$, $P_2 = 1$, $\chi(\mathcal{O}_X) = 1$;
- $B_X = \{4\times(1,2), (5,12), 2\times(2,5), (1,3)\}$, $K^3 = \frac{1}{60}$, $P_2 = 1$, $\chi(\mathcal{O}_X) = 1$;
- $B_X = \{7\times(1,2), (3,7), 2\times(1,3), (2,7)\}$, $K^3 = \frac{1}{42}$, $P_2 = 1$, $\chi(\mathcal{O}_X) = 1$;
- $B_X = \{7\times(1,2), (3,7), (1,3), (3,10)\}$, $K^3 = \frac{2}{105}$, $P_2 = 1$, $\chi(\mathcal{O}_X) = 1$;
- $B_X = \{7\times(1,2), 2\times(2,5), 2\times(1,3), (1,4)\}$, $K^3 = \frac{1}{60}$, $P_2 = 1$, $\chi(\mathcal{O}_X) = 1$.

Theorem 2. Assume $p_g(X) = 3$. Then

$r_s(X) = 6$ (Φ_5 non-birat.) $\iff$ ① $\dim \Phi_1(X) = 2$, $K_X^3 = 1$

$\implies$ ② $\dim \Phi_1(X) = 1$. $|K_X|$ is composed of a rational of $(1,2)$ surfaces and X belongs to one of the types in Table B.

$\overset{?}{\Longleftarrow}$

Theorem 2'. Assume $p_g(X) = 3$ and $X \notin$ Table B.

Then $r_s(X) = 6 \iff K_X^3 = 1$.

14	$41/30$	-1	6	$2\times[1,2], 2\times[1,3], 2\times[1,4], 1\times[1,5],$
15	$4/3$	-1	6	$2\times[1,2], 2\times[1,3], 2\times[1,4], 1\times[1,6],$
16	$17/12$	-1	6	$2\times[1,2], 2\times[1,3], 3\times[1,4],$
17	$4/3$	-1	6	$2\times[2,5], 1\times[1,3], 2\times[1,5],$
18	$83/60$	-1	6	$2\times[2,5], 1\times[1,3], 1\times[1,4], 1\times[1,5],$
19	$27/20$	-1	6	$2\times[2,5], 1\times[1,3], 1\times[1,4], 1\times[1,6],$
20	$41/30$	-1	6	$1\times[1,2], 1\times[2,5], 2\times[1,3], 2\times[1,5],$
21	$4/3$	-1	6	$1\times[1,2], 1\times[2,5], 2\times[1,3], 1\times[1,5], 1\times[1,6],$
22	$43/30$	-1	6	$2\times[2,5], 1\times[1,3], 2\times[1,4],$
23	$17/12$	-1	6	$1\times[1,2], 1\times[2,5], 2\times[1,3], 1\times[1,4], 1\times[1,5],$
24	$83/60$	-1	6	$1\times[1,2], 1\times[2,5], 2\times[1,3], 1\times[1,4], 1\times[1,6],$
25	$571/420$	-1	6	$1\times[1,2], 1\times[2,5], 2\times[1,3], 1\times[1,4], 1\times[1,7],$
26	$161/120$	-1	6	$1\times[1,2], 1\times[2,5], 2\times[1,3], 1\times[1,4], 1\times[1,8],$
27	$7/5$	-1	6	$2\times[1,2], 3\times[1,3], 2\times[1,5],$
28	$4/3$	-1	6	$2\times[1,2], 3\times[1,3], 2\times[1,6],$
29	$41/30$	-1	6	$2\times[1,2], 3\times[1,3], 1\times[1,5], 1\times[1,6],$
30	$47/35$	-1	6	$2\times[1,2], 3\times[1,3], 1\times[1,5], 1\times[1,7],$
31	$22/15$	-1	6	$1\times[1,2], 1\times[2,5], 2\times[1,3], 2\times[1,4],$
32	$29/20$	-1	6	$2\times[1,2], 3\times[1,3], 1\times[1,4], 1\times[1,5],$
33	$17/12$	-1	6	$2\times[1,2], 3\times[1,3], 1\times[1,4], 1\times[1,6],$
34	$39/28$	-1	6	$2\times[1,2], 3\times[1,3], 1\times[1,4], 1\times[1,7],$
35	$11/8$	-1	6	$2\times[1,2], 3\times[1,3], 1\times[1,4], 1\times[1,8],$
36	$49/36$	-1	6	$2\times[1,2], 3\times[1,3], 1\times[1,4], 1\times[1,9],$
37	$27/20$	-1	6	$2\times[1,2], 3\times[1,3], 1\times[1,4], 1\times[1,10],$
38	$59/44$	-1	6	$2\times[1,2], 3\times[1,3], 1\times[1,4], 1\times[1,11],$
39	$4/3$	-1	6	$2\times[1,2], 3\times[1,3], 1\times[1,4], 1\times[1,12],$

75	361/270	-1	6	$1 \times [1,2], 1 \times [2,5], 3 \times [1,3], 1 \times [1,27],$
76	187/140	-1	6	$1 \times [1,2], 1 \times [2,5], 3 \times [1,3], 1 \times [1,28],$
77	387/290	-1	6	$1 \times [1,2], 1 \times [2,5], 3 \times [1,3], 1 \times [1,29],$
78	4/3	-1	6	$1 \times [1,2], 1 \times [2,5], 3 \times [1,3], 1 \times [1,30],$
79	31/20	-1	6	$1 \times [1,2], 1 \times [2,5], 3 \times [1,3], 1 \times [1,4],$
80	43/30	-1	6	$1 \times [1,2], 2 \times [2,5], 2 \times [1,6],$
81	22/15	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,5], 1 \times [1,6],$
82	127/90	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,5], 1 \times [1,9],$
83	83/60	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,5], 1 \times [1,12],$
84	41/30	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,5], 1 \times [1,15],$
85	61/45	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,5], 1 \times [1,18],$
86	283/210	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,5], 1 \times [1,21],$
87	161/120	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,5], 1 \times [1,24],$
88	361/270	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,5], 1 \times [1,27],$
89	4/3	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,5], 1 \times [1,30],$
90	148/105	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,6], 1 \times [1,7],$
91	167/120	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,6], 1 \times [1,8],$
92	62/45	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,6], 1 \times [1,9],$
93	41/30	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,6], 1 \times [1,10],$
94	224/165	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,6], 1 \times [1,11],$
95	27/20	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,6], 1 \times [1,12],$
96	262/195	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,6], 1 \times [1,13],$
97	281/210	-1	6	$1 \times [1,2], 2 \times [2,5], 1 \times [1,6], 1 \times [1,14],$

131	41/30	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,5], 1 \times [1,12],$
132	1061/780	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,5], 1 \times [1,13],$
133	569/420	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,5], 1 \times [1,14],$
134	27/20	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,5], 1 \times [1,15],$
135	323/240	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,5], 1 \times [1,16],$
136	1369/1020	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,5], 1 \times [1,17],$
137	241/180	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,5], 1 \times [1,18],$
138	1523/1140	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,5], 1 \times [1,19],$
139	4/3	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,5], 1 \times [1,20],$
140	39/28	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,6], 1 \times [1,7],$
141	11/8	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,6], 1 \times [1,8],$
142	49/36	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,6], 1 \times [1,9],$
143	27/20	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,6], 1 \times [1,10],$
144	59/44	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,6], 1 \times [1,11],$
145	4/3	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,6], 1 \times [1,12],$
146	227/168	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,7], 1 \times [1,8],$
147	337/252	−1	6	$1 \times [1,3], 3 \times [1,4], 1 \times [1,7], 1 \times [1,9],$
148	4/3	−1	7	$4 \times [1,2], 7 \times [1,3],$
149	27/20	−1	7	$3 \times [1,2], 2 \times [2,5], 3 \times [1,3], 1 \times [1,4],$
150	4/3	−1	7	$4 \times [1,2], 1 \times [2,5], 4 \times [1,3], 1 \times [1,5],$
151	83/60	−1	7	$4 \times [1,2], 1 \times [2,5], 4 \times [1,3], 1 \times [1,4],$
152	41/30	−1	7	$5 \times [1,2], 5 \times [1,3], 1 \times [1,5],$
153	4/3	−1	7	$5 \times [1,2], 5 \times [1,3], 1 \times [1,6],$
154	17/12	−1	7	$5 \times [1,2], 5 \times [1,3], 1 \times [1,4],$
155	4/3	−1	7	$5 \times [2,5], 1 \times [1,3],$

Theorem 3

If $P_g(X)=2$ and $r_3(X)=7$, then one of the following holds:

(1) X is canonically fibred by $(2,3)$ surfaces of canonical degree 1;

(2) X is canonically fibred by $(1,2)$ surfaces and one of the following holds:

(a) $\xi=(K_X\cdot C)=\frac{2}{3}$, $C\in \mathrm{Mov}|K_F|$

(b) $\xi=\frac{4}{5}$

(c) $P_2(X)=5$, $\xi=1$, $\deg_C(F)=\frac{1}{2}$

(d) $P_4(X)=14$, $\xi=1$, $\deg_C(F)=\frac{1}{2}$

(e) $20\leqslant P_5(X)\leqslant 21$, $\xi=1$, $\deg_C(F)=\frac{1}{2}$

(f) $28\leqslant P_6(X)\leqslant 31$, $\xi=1$, $\deg_C(F)=\frac{1}{2}$

(g) X belongs to Table C (200 families).

(g) X belongs to Table C (all families).

Conversely, if X satisfies (1) or (2) and one of (a) ~ (f), then $r_3(X) = 7$.

<u>Theorem 3'</u> Assume $\rho_3(X) = 2$ and $X \notin$ Table C.

Then

$r_3(X) = 7$ (X_6 non-birat.) $\iff$ either (1) or (2) (a) ~ (f) holds.

Sketch of Proof

General assertion of birationality:

<u>Tool 1</u>: $(m_0, \zeta, |G|, \beta, \xi)$. For $m \in \mathbb{Z}_{>0}$,

$\alpha(m) > 2 \Rightarrow \varphi_{m,X}$ is birational.

Tool 2. Extension theorem (Kawamata type)

$|F|$ free. $m > 1$.

$$|m(K_X + F)|_{|F} = |mK_F|$$

$\Rightarrow$ Esmating β.

$\boxed{\text{Step 1}}$ Reduction to the study of $f: X' \to \mathbb{P}^1$

$F = (1, 2)$

- $p_g(X)=3$. Tool 1 $\Rightarrow$ $|K_{X'}|$ rational pencil

  $K_{X'} \geq 2F$

Tool 2 $\Rightarrow |5K_{X'}||_F \succcurlyeq |3(K_{X'}+F)||_F \succcurlyeq |3K_F|$

$\Rightarrow$ $F=(2,3)$ or $(1,2)$

$F=(2,3) \xRightarrow{\text{Tool 1}} \varphi_{5,X}$ is birational.

$F=(1,2) \Longrightarrow m_0=1.\ |G|=\mathrm{Mov}|K_F|,\ \beta=\frac{2}{3}.\ \underbrace{\xi=2}_{\xi=1}.$

$\Rightarrow \boxed{\alpha(5)=2}$.

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Sketch

- $p_g(X)=2$

Tool 2 $\Rightarrow |6K_{X'}||_F \succsim |3(K_{X'}+F)||_F = |3K_F|$

$\Rightarrow F=(2,3)$ or $(1,2)$.

$F=(2,3) \xrightarrow{\text{Tool 1}} \deg_c(F)=1$. $\Phi_{6,X}$ Non-birational

$F=(1,2) \Longrightarrow m_0=1,\ \xi=1.\ \beta=\frac{1}{2},\ |G|=\mathrm{Mov}|K_F|$

$\xi=\frac{2}{3}$

$\Rightarrow \alpha(6) \geq \frac{4}{3}$.

- $p_g(X)=1 \xrightarrow{\text{Chen-Chen}} \begin{cases} P_4(X)\geq 2 \\ P_4(X)=1,\ P_5(X)\geq 3 \end{cases}$

Tool 1 $\Rightarrow$ $\Phi_{16 \cdot X}$ non-birational implies

$$F=(1,2), \quad f: X' \longrightarrow \mathbb{P}^1.$$

<u>In general</u>

$f: X' \longrightarrow \mathbb{P}^1$. Fibre is a $(1,2)$ surface.

$q(X)=0$. $h^2(\mathcal{O}_X) = h^1(f_*\omega_{X'}) \leq 1$

$$\chi(\mathcal{O}_X) \leq 2 - p_g(X) = \begin{cases} -1 & p_g(X) = 3 \\ 0 & p_g(X) = 2 \\ 1 & p_g(X) = 1 \end{cases}$$

i.e. $\chi(\mathcal{O}_X)$ is bounded.

Step 2

$\forall m_1 > m_0,\ P_{m_1} \geq P_{m_0},\ \forall j \geq 0$

$$H^0(X', M_{m_1} - jF) \xrightarrow{\theta_{m_1,-j}} U_{m_1,-j} \subset H^0(F, M_{m_1}|_F)$$

$$H^0(F, M_{m_1}|_F - jC) \xrightarrow{\psi_{m_1,-j}} V_{m_1,-j} \subset H^0(C, M_{m_1}|_F)$$

$$u_{m_1,-j} = \dim U_{m_1,-j}$$

$$v_{m_1,-j} = \dim V_{m_1,-j}$$ $\Rightarrow$ decresing sequences

$$u_{m_1,0} \geq u_{m_1,-1} \geq u_{m_1,-2} \geq \cdots$$

$$v_{m_1,0} \geq v_{m_1,-1} \geq v_{m_1,-2} \geq \cdots$$

$$v_{m_1,0} \leq \begin{cases} m_1 - 1 & m_1 > 2 \\ 2 & m_1 = 2 \end{cases}$$

Observations:

$u_{m_1,0}$ is large $\Rightarrow$ either ξ is larger
or β is larger

$\Rightarrow$ Whenever P_{m_1} is larger, Φ_m is birational
$(m = 5, 6, 16)$

$\Rightarrow$ $P_2 \leq N_2$, $P_3 \leq N_3$, $\cdots$,

- $p_g = 2$. $P_2 \leq 4$, $P_3 \leq 8$, $P_4 \leq 13$, $P_5 \leq 19$, $P_6 \leq 31$
  $\chi(\omega_X) = -1, 0$, $\frac{1}{3} \leq K_X^3 \leq 4$

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Step 3. Classification to B_X

Using effective calculation of Chen-Chen,

To classify $\mathbb{B}(X) = \{B_X, P_2, \chi(\mathcal{O}_X)\}$

Weighted basket

$\Updownarrow$

birational family

$\Rightarrow$ Table A, B, C.

Thank you!