Holomorphic Dynamics
Calculus on currents.

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(Survey on ArXiv 1611.03598. Dinh–Sibony).

- Examples

1. $z \longrightarrow z^2 + c$ $\qquad z \in \mathbb{C}$

2. $(x,y) \longrightarrow (x^2 + ay + b, x)$. $\qquad (x,y) \in \mathbb{R}^2$ (resp $\mathbb{C}^2$

   higher dimensional analogues.

3. Compact Kähler manifold (X, ω).

$$X \overset{f}{\dashrightarrow} X \qquad \dim X = k.$$

f meromorphic (correspondance).

- Statistical long term behavior of orbits (points). objects of codim p.

  Distribution of periodic points and stable manifolds.

  Growth. Nevanlinna's theory.

  Chern – Griffith. $\sim$ 60's.

  Dynamical degrees in our context.

$$d_p(f) = \lim_n \left(\int (f^n)^* \omega^p \wedge \omega^{k-p} \right)^{1/n}$$

When $(f^n)^* = (f^*)^n$ in $H^{p,p}$.

Spectral radius.

? Limit exists ?

Growth and calculus.

entropy. $\Gamma_n = (x, f(x), \dots, f^{n-1}(x))$ in X^n

volume.

$$\int (f^{n_1})^* \omega \wedge \dots \wedge (f^{n_k})^* \omega$$

Singularities intersection.

Need for calculus and intersection theory.

Currents

$V \to (f^n)^*[V]$ V cycles.

a compact space containing cycles of codimension p.

$\mathcal{D}'^{(p,p)}(X)$.

Positive closed currents of bidegree (p,p).

Mass 1, normalisation.

$$\|T\| = \int T \wedge \omega^{k-p} = 1.$$

$$\frac{(f^n)^*[V]}{d_{p,n}} \xrightarrow{?}$$

<u>Th</u>. Dinh. S.

$$f : X \to X \quad \text{mero. dominant.}$$

Limit

$$d_p := \lim_n \left(\int (f^n)^* \omega^p \wedge \omega^{k-p} \right)^{1/n} \quad \text{exists.}$$

Bimeromorphic invariant

$$d_p(\pi^{-1} \circ f \circ \pi) = d_p(f).$$

$$h_a(f) := \max_{0 \leq p \leq k} \log d_p(f) \quad \text{algebraic}$$

entropy.

$$h_a(f) = \lim_n \frac{1}{n} \operatorname{Log} \operatorname{vol}(\Gamma_n).$$

$$h_t(f) \leq h_a(f).$$

For holomorphic maps. Gromov's idea for entropy.

Calculus.

<u>Th</u> (Dinh-S).

(X, ω) compact Kähler. $T \geq 0$ bidegree (p,p) closed.

$$T = T^+ - T^- \qquad \|T^\pm\| \leq c_X \|T\|.$$

$T^\pm$ are limits of <u>smooth</u> <u>positive</u> <u>closed</u> forms.

"movable" cycles.

Equidistribution a simple case
surfaces

Th (Fornaess. S) (Dinh. S)

(X, ω) compact Kähler suface
$f : X \circlearrowleft$ automorphism.

Assume $d_1 > 1$. Then. $d_1 = d$

$\left\{ \frac{(f^n)^* \omega}{d^n} \right\} \to c_+$ the class c_+ is very rigid.

very rigid : It contains a unic positive dd^c- closed current.

$(\mathbb{P}^2, \omega)$

$$T_+ := \lim \frac{(f^n)^* \omega}{d^n}$$

If $\phi^* T_+ = 0$ $\phi : \mathbb{C} \to X$

$$\frac{1}{T(r)} \int_0^r \frac{dt}{t} [\phi_*(D_t)] = \frac{1}{T(r)} \phi_* \left(\log^+ \frac{r}{|\zeta|} \right) \longrightarrow T_+$$

There is $A > 0$ if

T_n, ≥ 0 closed currents

class $\{T_n\} = c_n$

$$|\langle T_n - T_+ , \varphi \rangle| \leq A \left| \operatorname{Log} \|c_n - c_+\| \right| \|c_n - c_+\|^{1/2} \|\varphi\|_{C^2}$$

contains the rigidity. (unic closed current).

Explore rigidity of nef classes. $c^k = 0$

Arbitrary dimension

(X, ω) $\quad f : X \circlearrowleft$ automorphism. $d_q > d_{q-1}$.

T_+ Green curr. (q,q).

Assume $S \geq 0$ closed. (q,q).

There is $\alpha > 0$:

$$|\langle S - T_+, \varphi \rangle| \leq A \, \|c - c_+\|^\alpha \, \|\varphi\|_{C^2}.$$

φ test form $(k-q, k-q)$.

In particular $\{T_+\} = c_+$ is rigid.

Main tool. superpotentials

$$|\mathcal{U}_T(R)| \leq c \left(1 + \mathrm{Log}^+ \|R\|_{C^1}\right)$$

"exponential" estimates for quasi psh functions, $dd^c\varphi \geq -\omega$.

$$\int e^{-\alpha\varphi} < c.$$

$$\mathcal{U}_T(R) = \langle T, U_R \rangle \qquad dd^c U_R = R.$$

when $\{R\} = 0$

Periodic points.

Γ_n graph of f^n $\qquad f : X \circlearrowleft$ meromorph. dominant.

periodic points isolated points of $\Gamma_n \cap \Delta$

$$\Delta = \{(x,x) \,/\, x \in X\}.$$

$\underline{ex}$ $(z,w) \to (z+1, z^2+w)$ | only one period. pt at indeterminacy.

Lefschetz if $\dim \Gamma_n \cap \Delta = 0$.

$$\# Q_n = \{\Gamma_n\} \cup \{\Delta\}.$$

Question $\qquad d_n$ is mass of Γ_n. in general $\nearrow \infty$

$$\frac{1}{d_n}([\Gamma_n] \wedge [\Delta]) \xrightarrow{?}$$

Geometry of the limit.

$$\left(\operatorname{Lim} d_n^{-1} [\Gamma_n]\right) \wedge [\Delta] \qquad ?$$

equality.

is $\quad d_n^{-1} [\Gamma_n] \xrightarrow{?}$ convergence.

Does the intersection makes <u>sense</u>

even for $[\Gamma_n] \wedge [\Delta]$ $\qquad$ excess. dimension.

How much Γ_n concentrates around Δ_n.

Density of currents

- In algebraic geometry notion of multiplicity
- For currents near a point Lelong number

$$\frac{\|T\|_{B(a,r)}}{c_{2p}\, r^{2p}} \to \nu(T,a)$$

replace a point by a submanifold (diagonal)

This is the notion of density of currents along a subvariety + calculus on densities.

For a point in $\mathbb{C}^n$. 0.

$T \geq 0$ closed near 0. $A_\lambda : z \to \lambda z$. A_λ dilation by λ $|\lambda| \to \infty$

$$\{(A_\lambda)_* T\} \longrightarrow \nu(T,0)\,\{L\}$$

L cohomology class of a subspace.

Limits ~~are~~ of $(A_\lambda)_* T$ do not exist as $|\lambda| \to \infty$ but the cohomology class of all cluster points exists.

Th. (Dinh, S).

(X, ω) Kahler V submanifold of dim l. E normal Bundle to V.

$T \geq 0$ closed of bidegree (p,p)

τ admissible map from $V \subset X \longrightarrow V \subset \bar{E}$

$\bar{E}$ compactification.

$((A_\lambda)_* \tau_* T)_\lambda$ $\lambda \to \infty$ A_λ $z \to \lambda z$ on fibers of E

All cluster points ~~on~~ $\bar{E}$ are positive closed currents in the same cohomology in $H^{2p}(\bar{E}, \mathbb{C})$.

the cohomology class does not depend on τ, nor on the limit.

One can develop a calculus on theses coh. class.

Applications.

1. Discrete dynamics.

$f : \mathbb{C}^k \to \mathbb{C}^k$ polyn. automorphism algebraic degree $d > 1$.

~~not~~ linear in $\mathbb{P}^k$. ext. as birational map.

Assume $$I_+ \cap I_- = \phi.$$

Then $\exists\ p$ $\quad \dim I_+ = k-p-1 \quad \dim I_- = p-1.$

Then

$$\lim \frac{1}{d^{pn}}\left([\Gamma_n] \wedge \Delta\right) =: \mu$$

$$= \left(\lim \frac{[\Gamma_n]}{d^{pn}}\right) \wedge [\Delta].$$

Regularity of $\mathbb{T} := \lim \frac{[\Gamma_n]}{d^{pn}}$

$\to$ geometry of μ. $\qquad \hat{\Gamma}_n = (x, [v])$

$\hat{\mathbb{T}} = \lim \frac{[\hat{\Gamma}_n]}{d^{pn}}$ $\qquad$ on $Gr_k(\mathbb{P}^k \times \mathbb{P}^k)$.

A <u>consequence</u>

<u>Th.</u> ($k=2$ Bedford-Lyubich Smillie; $k \geq 2$ Dinh-S).

The saddle periodic points δ_{a_n}

$$\frac{1}{d^{pn}} \sum \delta_{a_n} \to \mu \text{ measure of maximal entropy.}$$

key point. $\tilde{\Gamma}_n := \{(x, v)$ $\quad$ v vector non transverse to $\Gamma_n\}$

$\frac{\tilde{\Gamma}_n}{d^{pn}} \to \tilde{\mathbb{T}}_+$ has density 0 along $\hat{\Delta}$.

2. Foliation in $\mathbb{P}^2$.

$$Z = P\frac{\partial}{\partial z} + Q\frac{\partial}{\partial w} \quad \text{in } \mathbb{C}^2$$

P, Q polynomials.

extend to foliation in $\mathbb{P}^2$. degree $d \geq 2$.

Assume 1. All sing. points are hyperbolic. $\frac{\lambda_1}{\lambda_2} \notin \mathbb{R}$.

2. There is a unic. inv. curve C.

known. All leaves are uniformized by unit disk.

$$\phi_a : \mathbb{D} \to L \qquad \phi_a(0) = a.$$

Thm (Dinh-S) $\frac{1}{T(r,\phi)} \phi_*\left(\log^+ \frac{r}{|\zeta|}\right) \longrightarrow \frac{[C]}{\|C\|}$.

$$\| $$

$$\frac{1}{T(r,\phi)} \int_0^r \frac{dt}{t} \phi_*(\mathbb{D}_t)$$

connection All the mass concentrate near C.

tool. Density of ≥ 0 $\partial\bar{\partial}$-closed currents near C.