Variational YTD (jt. w/ Berman - Jonsson)

1) Kähler - Einstein metrics

X^n compact Kähler ω Kähler form $[\omega] = \alpha \in H^{1,1}(X,\mathbb{R})$

(KE) $-\mathrm{Ric}(\omega) = \lambda \omega$, $\lambda \in \mathbb{R}$

↓ curvature form of induced metric $\log \omega^n$ on K_X

(normalized so that $[\mathrm{Ric}(\omega)] = c_1(-K_X)$)

$\lambda = 0$: Calabi-Yau case

$\lambda > 0$: $K_X > 0$ can. pol. case

$\lambda < 0$: $(-K_X)$ Fano case

Pick reference Kähler form $\omega \in \alpha$:

$\partial\bar\partial$-lemma $\Rightarrow$ any other Kähler form in α is of the form $\omega_\varphi := \omega + i\partial\bar\partial\varphi > 0$ $\varphi \in C^\infty(X)$

$\varphi \in \mathcal{H} = \{$Kähler potentials$\} \subset C^\infty(X)$ open, convex

$\mathcal{H}/\mathbb{R} \simeq \{$Kähler forms in $\alpha\}$

$-\mathrm{Ric}(\omega_\varphi) = \lambda\omega_\varphi \Leftrightarrow \omega_\varphi^n = e^{\lambda\varphi + c}\mu$ (E. Kähler 1933!)

$\mu = \mu_\omega > 0$ vol. form $c \in \mathbb{R}$

$V = (\alpha^n) = \int \omega_\varphi^n = \int e^{\lambda\varphi + c}\mu$

$$V^{-1}\omega_\varphi^n = \frac{e^{\lambda\varphi}\mu}{\int(\quad)}$$

2) Variational formulation

$$V^{-1}(\omega + i\partial\bar\partial\varphi)^m = \begin{cases} e^{\lambda\varphi}\mu / \int(\quad) & \lambda \neq 0 \\ \mu / \int\mu & \lambda = 0 \end{cases}$$

<u>RHS</u> $L_\lambda(\varphi) = \frac{1}{\lambda}\log \int e^{\lambda\varphi}\mu \qquad \lambda \neq 0$

$L_0(\varphi) = \int \varphi\mu$

$L'_\lambda(\varphi) = e^{\lambda\varphi}\mu / \int(\quad)$

$L_\lambda \nearrow$, $L_\lambda(\varphi + c) = L_\lambda(\varphi) + c \quad c \in \mathbb{R}$

L_λ convex ($\lambda \geq 0$), concave ($\lambda < 0$).

<u>LHS</u> : $E(\varphi) = \frac{1}{m+1}\sum_{j=0}^{m} V^{-1}\int \varphi\, \omega_\varphi^j \wedge \omega^{m-j}$

$E'(\varphi) = V^{-1}(\omega_\varphi^m)$ (Monge-Ampère operator) (also in Ē. Kähler!)

$E \nearrow$, $E(\varphi + c) = E(\varphi) + c$

$E'' \geq 0$, <u>E concave</u>.

<u>Ding functional</u>: $D_\lambda(\varphi) = L_\lambda(\varphi) - E(\varphi)$

$D'_\lambda(\varphi) = 0 \Leftrightarrow -\mathrm{Ric}(\omega_\varphi) = \lambda\omega_\varphi$.

($D_\lambda(\varphi + c) = D_\lambda(\varphi) \rightsquigarrow$ descends to $\mathcal{H}/\mathbb{R}$ = Kähler forms)

Ex: $n = 1$ $\lambda = 0$ $(V = 1)$

solve $\omega + i\partial\bar\partial\varphi = \mu \iff \Delta\varphi = \mu - \omega$

$D_0(\varphi) = \int \varphi (\mu - \omega) + \frac{1}{2} \|\nabla\varphi\|^2$ — Dirichlet norm

Riemann's "Dirichlet pple"

Hilbert ⇝ Sobolev space, weak compactness → weak solution, $\Delta\varphi = \mu - \omega$

C^∞ by ellipt. of Δ

normalize by $\int \varphi\omega = 0 \Rightarrow \|\nabla\varphi\|$ is a norm
cst.

Higher dim: $\lambda \geq 0 \Rightarrow L_\lambda$ convex $\Rightarrow D_\lambda$ convex on $\mathcal{H}$

$\Rightarrow$ crit. pts are minimizers.

Thm (Ding-Tian, ...) Also true when $\lambda < 0$ (Fano case)

[cont. method, or use a more sophist. notion of (weak) geodesics, for which E affine L_λ convex (Berndtsson)]

Existence of minimizer? Tian's programme.

def: D_λ is coercive if $D_\lambda \geq \delta J - C$

$J(\varphi) = V^{-1}\int \varphi\omega^n - E(\varphi)$ (concav. of E)

(Aubin J-func.) J convex, ≥ 0, $= 0 \Leftrightarrow \varphi = $ cst.

$\underline{n=1}$: $J(\varphi) = \frac{1}{2} \|\nabla \varphi\|^2_{L^2}$ | L convex, ↗, $L(\varphi+c) = L(\varphi)+c$ $\Rightarrow |L| \leq J^{1/2}$ on $\{\int \varphi = 0\}$

"uniform" on $\mathcal{E}^1$

"Coercivity" of M wrt J / E

$\mathcal{E}^1(X,\omega)$ (Guedj - Zeriahi)

$\Rightarrow e^{-\varphi} \in L^p$ $\forall p < \infty$

ω-psh functions with finite energy.

$\mathcal{E}^1_0 = \{\int \varphi = 0\}$.

really a coercivity wrt metric d_1!

$D \geq \delta J - C$ | J weakly lsc (Darvas) | D also.

$(J \leq C)$ weakly cpt

↘ $\exists$ min. for D

need to prove $D' = 0$

$(\omega + dd^c\varphi)^n =$

weak soln

MA theory $\Rightarrow \varphi \in C^\infty$.

N.B: coer $\Rightarrow$ Aut(X) finite
$\lambda \neq 0$

Thm (Tian, Darvas-Rubinstein) $\lambda \neq 0$ Aut(X) finite
PRSSW ... KE $\Leftrightarrow$ coercive
Phong Song Sturm Weinkove ... ...