Poincaré Problem and Birational Invariants of a Differential Equation

Sheng-Li Tan

(East China Normal University, Shanghai)

Workshop on Higher Dimensional Algebraic Geometry, Holomorphic Dynamics and Their Applications

Jan. 17 , 2017, IMS/NUS

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1. Differential Equations and Holomorphic Foliations

1.1. Differential Equations

Darboux's Method (1878)

- Differential equation on $\mathbb{R}^2$

$$\frac{dy}{dx} = \frac{q(x,y)}{p(x,y)}\ , \tag{1}$$

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- Differential equation on $\mathbb{CP}^2$

$$\alpha = 0 \tag{2}$$

α is a nonzero rational 1-form on $\mathbb{CP}^2$,

$$\alpha\big|_{\mathbb{C}^2} = p(x,y)dy - q(x,y)dx$$

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- The poles and 1-dimensional zeros of α are canceled

$$\begin{aligned} \alpha &= \left\{\, a(x,y)\,dy - b(x,y)\,dx \,\right\}, \\ &= \left\{\, s(x,y) \cdot \Big(p(x,y)dy - q(x,y)dx \Big) \,\right\}, \\ &= s \cdot \omega, \end{aligned}$$

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- ω is holomorphic with at worst isolated zero points.

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  $N_{\mathcal{F}} = \mathcal{O}(-D(\alpha))$: Normal bundle of $\mathcal{F}$.

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- $\varphi = \varphi(x, y) : X \dashrightarrow \mathbb{P}^1$: non-constant rational function.

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$$\mathcal{F}(d\varphi) = \Lambda(\varphi).$$

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- **Definition:**

1) φ is called the rational first integral of $\alpha = 0$ or $\mathcal{F}$.

2) $\varphi : X \dashrightarrow \mathbb{P}^1$ is birational to some fibration $f : X \to B$ of genus g.

$$g(\varphi) = g(\mathcal{F}) = g.$$

2. Poincaré Problem and Painlevé Problem

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  Part 1: Find a height inequality for $\varphi = f/g$,

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  $H = H(d\varphi)$ depends only on the differential equation $d\varphi = 0$.

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Part 2: Algebraic computation.

2.1. Poincaré Problem

- **Poincaré's Height Formula**: If $C_t = \sum_i n_i(t) C_{t,i}$, then

$$2h(\varphi) - 2 = d + \sum_t \sum_i (n_i(t) - 1) \deg C_{t,i}$$

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- **Poincaré's Genus Formula**: If the DE $d\varphi = 0$ has $d^2 + d + 1$ distinct singularities and $d \geq 4$, then

$$g(\varphi) = \frac{d-4}{4} h(\varphi) + 1.$$

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- **Theorem:** *If $\mathcal{F}$ contains at least N algebraic curves, then all of the curves in $\mathcal{F}$ are algebraic.*

  1. *Darboux (1878):* $N = \frac{1}{2}d(d+1) + 2$
  2. *Jouanolou (1979):* $N = h^0(X, K_{\mathcal{F}}) + h^{1,1}(X) - h^{1,0}(X) + 2$

2.2. Painlevé's Problems

[1] *"Sur les intégrales algébriques des équations différentielles du premier ordre"* and *"Mémoire sur les équations différentielles du premier ordre"*, 1890s.

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- **Problem 2:** Is it possible to decide if $\alpha = 0$ admits a rational first integral φ which is an elliptic function? $(g = 1)$
- **Problem 3:** Is it possible to decide if $\alpha = 0$ admits a rational first integral φ which is a hyperelliptic function? $(g \geq 2)$

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- **Painlevé's Problem:** For a given g, is it possible to decide if $\alpha = 0$ admits a rational first integral φ of genus g?
- **Equivalent Problem:** Find an inequality

$$g(\varphi) \leq G = G(d\varphi),$$

G depends only on the CDE $d\varphi = 0$.

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(B) The study of one dimensional families of curves $\Lambda(\varphi)$.

Poincaré's suggestion:

1) Study the properties of families of algebraic curves $\Lambda(\varphi)$.

2) Check if these are the properties of the CDE $d\varphi = 0$.

3) Generalize them to arbitrary differential equations $\alpha = 0$.

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5) Apply to some problems on real differential equations.

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$$\begin{cases} \dfrac{dx}{dt} = p(x,y), \\ \dfrac{dy}{dt} = q(x,y). \end{cases} \tag{4}$$

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AG: Singular points of $\Lambda(\varphi) = \mathcal{F}(d\varphi)$.

DE: Singular points of $\mathcal{F}(\omega)$, $\big(\omega = f(x,y)dy + g(x,y)dx\big)$

4. Singularities of a Foliation

AG: Singular points of $\Lambda(\varphi) = \mathcal{F}(d\varphi)$.

1) $\frac{\partial\varphi}{\partial x}(p) = \frac{\partial\varphi}{\partial y}(p) = 0.$ $\left(d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy\right)$

DE: Singular points of $\mathcal{F}(\omega)$, $\left(\omega = f(x,y)dy + g(x,y)dx\right)$

1) $f(p) = g(p) = 0.$ **(Darboux 1878)**

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2) Base points of $\Lambda(\varphi)$.

DE: Singular points of $\mathcal{F}(\omega)$, $\left(\omega = f(x,y)dy + g(x,y)dx\right)$

1) $f(p) = g(p) = 0.$ **(Darboux 1878)**
2) Dicritical points p of $\mathcal{F}(\omega)$: There are infinite many curves in $\mathcal{F}(\omega)$ passing through p. **(Poincaré 1891)**

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1) $f(p) = g(p) = 0.$ **(Darboux 1878)**
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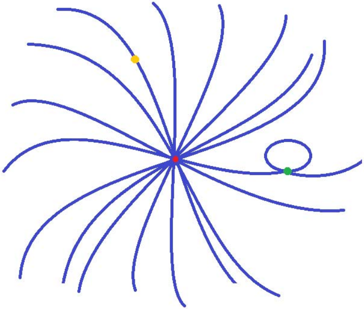
3) Non-dicritical singular points of $\mathcal{F}(\omega)$.

4. Singularities of a Foliation

- Base point of $\Lambda(\varphi)$ or dicritical singular point of $\mathcal{F}$.

$X = \bigcup_{C \in \mathcal{F}(\alpha)} C,$



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AG: **Resolution of base points of** $\Lambda(\varphi)$

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4. Singularities of a Foliation

AG: Resolution of base points of $\Lambda(\varphi)$

- There is a resolution $\sigma : S \to X$ such that $\Lambda(\sigma^*(\varphi))$ is a fibration $f : S \to B$.

DE: Resolution of dicritical singular points of $\mathcal{F}(\alpha)$

- There is a resolution $\sigma : S \to X$ such that $\mathcal{F}(\sigma^*\alpha)$ has no dicritical singular points. **(Seidenberg 1968)**

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$$\left(d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy\right)$$

- Milnor number:

$$\mu_p = I_p\left(\frac{\partial\varphi}{\partial x}, \ \frac{\partial\varphi}{\partial y}\right)$$

DE: Singular points of $\mathcal{F}(\omega)$, $\left(\omega = f\,dy + g\,dx\right)$

- Multiplicity (Darboux 1878):

$$m_p(\omega) = I_p(f, g)$$

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- Assume $p = (0,0)$

$$\begin{cases} \dot{x} = p(x,y) = ax + by + P_2(x,y) \\ \dot{y} = q(x,y) = cx + dy + Q_2(x,y) \end{cases}$$

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- If $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$, then the eigenvalue of α at p is defined by

$$\lambda_p = \frac{\lambda_2}{\lambda_1} \quad \text{or} \quad \lambda_p = \frac{\lambda_1}{\lambda_2}$$

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2) λ_p is not a positive rational number.

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AG: Resolution of Singularities of $\Lambda(\varphi)$

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- The invariant $I(\alpha)$ is said to be a birational invariant if its definition is independent of the choice of the minimal models.

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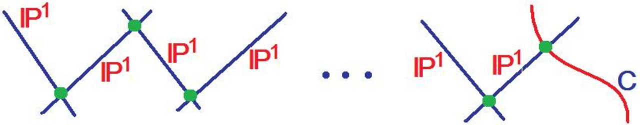

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$$p_a(E) = 0, \quad \text{or} \quad 1.$$

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3) Kodaira dimension $\kappa(\alpha)$ and numerical Kodaira dimension $\nu(\alpha)$.

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- For a complex number $u \in \mathbb{C}$, we define

$$\beta(u) = \begin{cases} \frac{\gcd(a,b)^2}{ab}, & \text{if } u = \dfrac{a}{b} \in \mathbb{Q} \setminus \{0\}, \\ 0, & \text{otherwise}. \end{cases}$$

$$\chi(u) = \frac{1}{12}\left(u + \frac{1}{u} + \beta(u)\right) - \frac{1}{4}, \quad (u \neq 0).$$

- $\chi(u)$ is called the Dedekind number of u
- Let $\varphi = x^a y^b$. Then $d\varphi = 0$ is $aydx + bxdy = 0$, the eigenvalue of the singular point $p = (0,0)$ is

$$\lambda_p = -\frac{a}{b}, \quad \text{or} \quad \lambda_p = -\frac{b}{a}$$

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AG: Local invariants

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DE: Local invariants

- For a reduced singular point p of $\alpha = 0$,

$$\begin{aligned} \beta_p(\alpha) &:= \beta(-\lambda_p), \\ \chi_p(\alpha) &:= -\frac{1}{12}\Big(BB_p(\alpha) + m_p(\alpha) - \beta_p(\alpha)\Big), \end{aligned}$$

where $BB_p(\alpha)$ is the Baum-Bott index.

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2) The modular invariants can be recognized by their differential equations. So we can define the Chern numbers of $\alpha = 0$. (Tan, 2015)

$$c_1^2(\alpha), \qquad c_2(\alpha), \qquad \chi(\alpha).$$

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- If $\mathcal{F}$ has a rational first integral of genus g, then we have

$$\frac{4g-4}{g}\chi(\alpha) \leq c_1^2(\alpha) \leq 12\chi(\alpha).$$

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- **Rationality (Miyaoka, 1985):** $K(\alpha)$ is not pseudo-effective $\iff$ $\mathcal{F}(\alpha)$ is a rational fibration.
- **Zariski's decomposition:** Suppose $K(\alpha)$ is pseudo-effective and $K(\alpha) = P + N$ is the Zariski's decomposition. Then

$$c_1^2(\alpha) = P^2 \geq 0.$$

$c_1^2(\alpha) > 0$ iff $\alpha = 0$ is of general type, i.e., $\kappa(\alpha) = 2$.

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1) Chern numbers: $c_1^2(\alpha),\ c_2(\alpha),\ \chi(\alpha)$.
2) $c_1^2(X),\ c_2(X),\ K(\alpha)^2,\ K(\alpha)D(\alpha),\ D(\alpha)^2$.
3) $\Sigma(\alpha) = \left\{ \left(\lambda_p(\alpha),\ m_p(\alpha) \right) \,\middle|\, p \in \mathrm{Sing}(\alpha) \right\}$

6. New Birational Invariants

- **Theorem:** Suppose $c_1^2(\alpha) < 4\chi(\alpha)$. If $\alpha = 0$ is algebraically integrable with a rational first integral φ of genus g, then

$$g \leq \frac{4\chi(\alpha)}{4\chi(\alpha) - c_1^2(\alpha)}$$

In particular, if $c_1^2(\alpha) < \frac{4a}{a+1}\chi(\alpha)$, then $g \leq a$.

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- **Theorem:** A CDE $\alpha = 0$ satisfying $0 < c_1^2(\alpha) < 2\chi(\alpha)$ is not algebraically integrable.

7. BMMM Classification (Brunella-McQuillan-Mendes-Miyaoka)

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- $\alpha = 0$ is induced by a vector field if and only if

$$K(\alpha) \overset{\text{num.}}{\sim} 0$$

7. Classification

- A rational foliation is a rational fibration $f : X \to C$, i.e., the genus $g = 0$.
- A Hilbert modular foliation is induced by one factor $\mathbb{H}$ of a Hilbert modular surface $X = \mathbb{H} \times \mathbb{H}/\Gamma$.
- A Riccati equation is an equation of the following type on some ruled surface

$$\frac{dy}{dx} = a_0(x)y^2 + a_1(x)y + a_2(x).$$

- $\alpha = 0$ is induced by a vector field if and only if

$$K(\alpha) \overset{\text{num.}}{\sim} 0$$

- A Turbulent foliation is the foliation of a Turbulent equation.

8. Effective Behavior

Theorem. Let $c_1^2(\alpha) > 0$, let $K(\alpha) = P + N$, and let r be the index. Suppose $\tau_0(\alpha) = 2$ or ∞, and m is divided by r. Then we have

1) If $m \geq r \cdot (\eta_0(\alpha) + 1)$, then $H^1(mP) = H^2(mP) = 0$.

   $$P_m(\alpha) = \dim H^0(mP) = \chi(\mathcal{O}_X) + \frac{1}{2}m(m-1)c_1^2(\alpha) - \frac{1}{2}mP \cdot D(\alpha).$$

2) If $m \geq r \cdot (\eta_0(\alpha) + 2)$, then $|mP|$ is base point free and $|mK(\alpha)| = |mP| + mN$.

3) If $m \geq r \cdot (\eta_0(\alpha) + 3)$, then the m-canonical map $\Phi_m : X \to \mathbb{P}^n$, $n = P_m(\alpha) - 1$ is just the contraction morphism of the exceptional set of $\alpha = 0$.

Thank you very much!