On the Eisenbud-Goto conjecture

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Outline

1. Motivation and basic background
   - Castelnuovo's problem
   - Hilbert-Gotzman regularity theorem
   - Eisenbud-Goto regularity conjecture
2. Known results
3. Counter-examples due to Mccullough-Peeva
4. $\mathcal{O}_X$-regularity and boundary cases for smooth varieties

Castelnuovo's problem

- $X^n \subset \mathbb{P}^{n+e}$: a non-degenerate projective variety(irreducible, reduced) of dim n, codim e, and degree d defined over $\mathbb{C}$.
- $S := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(\mathbb{P}^{n+e}, \mathcal{O}_{\mathbb{P}^{n+e}}(m))$ be the homogeneous coordinate ring of $\mathbb{P}^{n+e}$.
- **The homogeneous coordinate ring**: $S_X := S/I_{X|\mathbb{P}^{n+e}}$ where $I_{X|\mathbb{P}^{n+e}} := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(\mathbb{P}^{n+e}, \mathcal{I}_{X|\mathbb{P}^{n+e}}(m))$ is the saturated ideal.
- **The section ring**: $R_X := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}_X(m))$.

$$0 \to S_X \to R_X \to \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^1(\mathbb{P}^{n+e}, \mathcal{I}_{X|\mathbb{P}^{n+e}}(m)) \to 0.$$

[Castelnuovo, version I]

Give a bound for m_0 in terms of d, e, n such that $H^1(\mathbb{P}^{n+e}, \mathcal{I}_{X|\mathbb{P}^{n+e}}(m)) = 0$ for all $m \geq m_0$.

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Definition

- X is called m-regular (due to Mumford) if the following two conditions hold:
  1. (Castelnuovo normality) $H^0(\mathcal{O}_{\mathbb{P}^{n+e}}(m-1)) \twoheadrightarrow H^0(\mathcal{O}_X(m-1))$ is surjective, i.e. X is $(m-1)$-normal;
  2. ($\mathcal{O}_X$-regularity) $H^i(\mathcal{O}_X(m-1-i) = H^i(\mathcal{L}^{\otimes(m-1-i)}) = 0$ for all $i \geq 1$, i.e. $\mathcal{O}_X$ is $(m-1)$-regular with respect to $\mathcal{L} \simeq \mathcal{O}_X(1)$.
- $\mathrm{reg}(X) := \min\{m \mid X \text{ is } m\text{-regular}\}$.

More generally,

- Let $\mathcal{L}$ be an ample and globally generated line bundle on X. A coherent sheaf $\mathcal{F}$ on X is m-regular with respect to $\mathcal{L}$ if $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes(m-i)}) = 0$ for $i \geq 1$.
- $\mathrm{reg}_{\mathcal{L}}(\mathcal{F})$ is the minimum of m such that $\mathcal{F}$ is m-regular w.r.t. $\mathcal{L}$
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- Castelnuovo normality heavily depends on the embedding of X in $\mathbb{P}^{n+e}$ but $\mathcal{O}_X$-regularity, i.e., the vanishing $H^i(X, \mathcal{L}^{\otimes(m-1-i)}) = 0$ is intrinsic.

The m-regularity of $\mathcal{F}$ with respect to $\mathcal{L}$ has nice properties as follows:

Mumford's Regularity Theorem

- $\mathcal{F} \otimes \mathcal{L}^{\otimes(m+k)}$ is generated by its global sections for all $k \geq 0$;
- The natural maps are surjective for all $k \geq 0$; $H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \otimes H^0(X, \mathcal{L}^{\otimes k}) \twoheadrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes(m+k)})$.
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Syzygies and Regularity

Let M be a finitely generated graded S-module.
By Hilbert's syzygy theorem, we have the **minimal free resolution**:

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_t \leftarrow 0$$

where $F_i = \bigoplus_j S(-i-j)^{\beta_{i,j}(M)}$ and $\beta_{i,j}(M) = \dim_{\mathbb{C}} \operatorname{Tor}_i^S(M, \mathbb{C})_{i+j}$.
Then we obtain the **Betti table** of M:

$$\begin{matrix} \beta_{0,0}(M) & \beta_{1,0}(M) & \cdots & \beta_{t,0}(M) \\ \beta_{0,1}(M) & \beta_{1,1}(M) & \cdots & \beta_{t,1}(M) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{0,s}(M) & \beta_{1,s}(M) & \cdots & \beta_{t,s}(M) \end{matrix}$$

The **Castelnuovo-Mumford regularity** of M is the height of the betti table, i.e., $\mathrm{reg}(M) = s$, and the projective dimension of M is the width of the Betti table, i.e., $\mathrm{pd}(M) = t$.

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Syzygies and regularity of S_X and R_X

- $\mathrm{reg}(R_X) = \mathrm{reg}(\mathcal{O}_X)$.
- $\mathrm{reg}(S_X) = \mathrm{reg}(\mathcal{I}_{X|\mathbb{P}^{n+e}}) - 1 = \mathrm{reg}(X) - 1$.
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For a rational curve C of $d \geq e+2$ with a $(d-e+1)$-secant line, $\mathrm{reg}(R_C) = \mathrm{reg}(\mathcal{O}_C) = 1$ and $\mathrm{reg}(S_C) = \mathrm{reg}(C) - 1 = d - e \geq 2$.

Let $a(I_{X|\mathbb{P}^{n+e}})$ be the maximal degree of minimal generators. Then,

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[Castelnuovo, version II]

Give a bound for m_0 in terms of d, e, n such that $\mathrm{reg}(X) = \mathrm{reg}(\mathcal{I}_{X|\mathbb{P}^{n+e}}) \leq m_0$.
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$\mathrm{reg}(X) = \mathrm{reg}(\mathcal{I}_{X|\mathbb{P}^{n+e}}) \leq m_0$.
Note again that m-regularity of X implies $(m+1)$-regularity of X.

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Syzygies and regularity of S_X and R_X

- $\mathrm{reg}(R_X) = \mathrm{reg}(\mathcal{O}_X)$.
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- $\mathrm{reg}(R_X) \leq \mathrm{reg}(S_X)$.

For a rational curve C of $d \geq e+2$ with a $(d-e+1)$-secant line, $\mathrm{reg}(R_C) = \mathrm{reg}(\mathcal{O}_C) = 1$ and $\mathrm{reg}(S_C) = \mathrm{reg}(C) - 1 = d - e \geq 2$.

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Uniform regularity bound

Gotzmann Regularity Theorem (Gotzmann,1978)

If $X \subset \mathbb{P}^r$ is a projective scheme with Hilbert polynomial P, then there is an integer $s \geq 0$ (called the "Gotzmann number"), depending only on P, such that the ideal sheaf $\mathcal{I}_X$ is s-regular.

Thus, with the same hypothesis above, by Mumford regularity theorem, X is completely determined by $(I_X)_s$, a $(\binom{r+s}{s} - P(s))$-dimensional subspace of $H^0(\mathbb{P}^r, O_{\mathbb{P}^r}(s))$. Using this fact, Gotzmann explicitly constructed the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^r)$ (as an subscheme of a Grassmannian) of subschemes in $\mathbb{P}^r$ having P as their Hilbert polynomial:

$$\mathrm{Hilb}_P(\mathbb{P}^r) \hookrightarrow \text{Grassmannian}\left(\binom{r+s}{s} - P(s), H^0(\mathbb{P}^r, O_{\mathbb{P}^r}(s))\right)$$

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Regularity Conjecture

$X^n \subset \mathbb{P}^{n+e}$: irreducible and reduced of codim e and degree d.

Regularity Conjecture (Eisenbud-Goto, 1984)

- $\mathrm{reg}(X) \leq d - e + 1$ namely,
  1. (Castelnuovo normality conjecture) X is m-normal for all $m \geq d - e$, i.e. $H^0(\mathcal{O}_{\mathbb{P}^{n+e}}(m)) \to H^0(\mathcal{O}_X(m))$ is surjective for all $m \geq d - e$;
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- An interesting problem is to classify varieties of maximal regularity with geometric meanings.

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Known results-Curves

Theorem (Castelnuovo 1893)

Let $C \subset \mathbb{P}^3$ be a non-degenerate smooth projective curve of degree d. Then $\mathrm{reg}(C) \leq d - 1$.

Theorem (Gruson-Lazarsfeld-Peskine 1983)

Let $C \subset \mathbb{P}^r$ be a projective curve (not necessarily smooth) of degree d and codimension e.

- $\mathrm{reg}(C) \leq d - e + 1$.
- *the equality holds $\Leftrightarrow$ $C \subset \mathbb{P}^r$ is a plane curve, an elliptic normal curve, a rational normal curve, a rational curve with $d = e + 2$, or a smooth rational curve having a $(d - e + 1)$-secant line.*

- Eisenbud-Goto conjecture for curves was completely solved with classification on extremal cases.

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Surfaces

Theorem (Pinkham, Lazarsfeld 1986)

Let $S \subset \mathbb{P}^r$ be a projective smooth surface of degree d and codimension e.

- $\mathrm{reg}(S) \leq d - e + 1$.
- *No classification on the boundary cases yet!*

For singular surfaces, it is easy to show the following by using the known regularity bound of hyperplane section curves:

- $H^1(O_S(j)) = 0$ for every $j \geq \mathrm{d} - e - 1$;
- $H^2(O_S(j)) = 0$ for every $j \geq \lceil \frac{d-1}{e} \rceil - 2$;
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Threefolds

Theorem (K-, 1998)

Let X be a smooth threefold of degree d in $\mathbb{P}^5$.

- *X is m-normal for all $m \geq d - 4$;*
- *$\mathrm{reg}(X) \leq d - 1$;*
- *The bound $d - 4$ is sharp(e.g., the Palatini scroll in $\mathbb{P}^5$ of degree 7).*

A proof used the following three facts for a smooth $X^3 \subset \mathbb{P}^5$:

- X is linearly normal due to F. Zak;
- (Barth theorem) $H^1(\mathcal{O}_X) = 0$; and
- the $(\dim X + 2)$-secant lemma due to [Z. Ran,1991] which implies

$$\dim \overline{\cup\{\text{line } \ell \mid \#(X \cap \ell) \geq 5)\}} \leq 4,$$

that is, 5-secant lines to X do not fill up the whole space $\mathbb{P}^5$.

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- X is linearly normal due to F. Zak;
- (Barth theorem) $H^1(O_X) = 0$; and
- the $(\dim X + 2)$-secant lemma due to [Z. Ran,1991] which implies

$$\dim \overline{\cup\{\text{line } \ell \mid \#(X \cap \ell) \geq 5)\}} \leq 4,$$

that is, 5-secant lines to X do not fill up the whole space $\mathbb{P}^5$.

A singular threefold in $\mathbb{P}^5$ as a counterexample

Using "Rees algebra", Mccullough and Peeva(preprint, 2016) constructed a counterexample to the Regularity Conjecture which is "computable" with `Macaulay2`.

Construction

- $I = (a^{11}, b^{11}, p = a^2c^9 + b^2d^9 + abce^8 + abdf^8) \subset R = k[a, ..., f]$
- (Rees algebra) $F : R[x, y, z] \to R[It]$ by $F(x) = a^{11}t, F(y) = b^{11}t, F(z) = pt$.
  Note that $deg(x) = deg(y) = deg(z) = 12$.
- $Q = \ker F$ is a prime ideal.
- (Homogenization) $G : R[x, y, z] \to S = R[X, Y, Z, A, B, C]$ by $G(x) = XA^{11}, G(y) = YB^{11}, G(z) = ZC^{11}$.

Let $P \subset S$ be the ideal generated by the image $G(Q)$.

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Then, Mccullough-Peeva showed that P is a prime ideal in $S = k[a, b, c, d, e, f, X, Y, Z, A, B, C]$ which is homogeneous with respect to the usual grading and is non-degenerate.

Over the base field $k = \mathbb{Z}/2$ (or its algebraic closure), we have a prime ideal P with 924 minimal generators in S:

$$P = (a^2c^9YB^{11} + b^2d^9YB^{11} + abce^8YB^{11} + abdf^8YB^{11} + b^{11}ZC^{11},$$

$$\ldots,$$

$$bc^{85}e^{320}XA^{11} + bc^{84}de^{312}f^8XA^{11} + \cdots + a^{10}d^4f^{392}ZC^{11}).$$

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The following are invariants of $P \subset S$ computed with `Macaulay2`.

Invariants

- The number of minimal generators of P is 924.
- The maximal/minimal degree of minimal generators of P is $418/23$.
- The degree of S/P is 375.
- The codimension of P is 2.
- The projective dimension of S/P is 5 and the depth of S/P is 7.

Let $X^9 \subset \mathbb{P}^{11}$ be the variety defined by P. Then X is a 9-dimensional variety of codimension 2. Since the depth is ≥ 2, we have

$$H^1(\mathcal{I}_{X|\mathbb{P}^{11}}(m)) = 0, \quad \text{for all } m \in \mathbb{Z}.$$

But, the regularity of X is much bigger than conjectured, i.e.,

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Now, taking general linear sections 6 times on $X \subset \mathbb{P}^{11}$, we get a singular three fold $Y \subset \mathbb{P}^5$ with $\mathrm{reg}(X) = \mathrm{reg}(Y)$ because the depth of S/P is 7. Therefore,

$$\operatorname{reg} Y \geq 418 >> \deg Y - \operatorname{codim} Y + 1 = 374.$$

But $Y \subset \mathbb{P}^5$ is not arithmetically normal any more because the depth of Y is 1.

This "singular" 3-fold Y in $\mathbb{P}^5$ also gives us a counter-example to the Regularity Conjecture.
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Higher dimensional smooth varieties

Theorem (K-, 1998)

Let X be a smooth threefold of degree d and codimension e in $\mathbb{P}^r$.

- $\mathrm{reg}(X) \leq (d - e + 1) + 1$;

Note that if **dim** $\overline{\cup\{\textbf{line } \ell \mid \#(X \cap \ell) \geq 4)\}} \leq 4$, we can obtain the conjectured bound $d - e + 1$ for smooth threefolds in general [Lazarsfeld, Ran].

Theorem (Mumford, Bertram-Ein-Lazarsfeld)

Let X be a smooth variety of dimension n, degree d and codimension e in $\mathbb{P}^{n+e}$. we have $\mathrm{reg}(X) \leq \min\{n+1, e\}(d)$.

We believe that there is a good regularity bound for smooth case.

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A sharp bound for $\mathcal{O}_X$-regularity for smooth varieties

Theorem (K- and J. Park, peprint)

Let $X^n \subset \mathbb{P}^{n+e}$ be a non-degenerate smooth projective variety of degree d, codimension e. Then

(1) $\mathrm{reg}(\mathcal{O}_X) \leq d - e$.

(2) $\mathrm{reg}(\mathcal{O}_X) = d - e$ *if and only if* $X \subset \mathbb{P}^r$ *is linearly normal and* $d \leq e + 2$ *or a hypersurface.*

(3) $\mathrm{reg}(\mathcal{O}_X) = d - e - 1$ *if and only if* $X \subset \mathbb{P}^r$ *is an isomorphic projection of (2) at one point, linearly normal and* $d = e + 3$, *or a complete intersection of type* $(2, 3)$ $(d = e + 4)$.

Sketch of Proof

Use the semiampleness of double point divisor $D_{inn} := -K_X + (d - e - n - 1)H$ and apply Castelnuovo's genus bound for a general curve section for classification.

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Ideas of a proof

Proposition (Birational double point formula)
Let $\varphi\colon V^n \to M^{n+1}$ be a morphism of smooth projective varieties such that $\varphi\colon V \twoheadrightarrow W := \varphi(V) \subset M$ is birational.
Then, $\varphi^*(K_M + W) - K_V \sim D - E$ where D and E are effective divisors on V such that E is φ-exceptional. Moreover, if φ is isomorphic at $x \in V$, then $x \notin \mathrm{Supp}(D - E)$.

Proof. see Lemma 10.2.8(Positivity in Algebraic Geometry II).

Let $x_1, \ldots, x_{e-1} \in X$ be general points, and let $\Lambda := \langle x_1, \ldots, x_{e-1} \rangle$. Consider the inner projection at Λ and the blow-up $\widetilde{X}$ of X at $x_1, \ldots, x_{e-1}$ with the following diagram:

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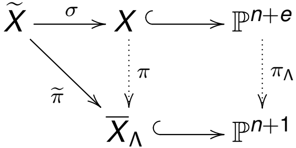

From the morphism $\widetilde{\pi} : \widetilde{X} \twoheadrightarrow \overline{X}_\Lambda \subset \mathbb{P}^{n+1}$ and $\deg(\overline{X}_\Lambda) = d - (e-1)$, the birational double point formula implies that

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If we assume that $\widetilde{\pi} : \widetilde{X} \twoheadrightarrow \overline{X}_\Lambda$ has no exceptional divisor, i.e. $\widetilde{E} = \emptyset$, then, the non-isomorphic double point locus $D(\widetilde{\pi})$ of $\widetilde{\pi}$ is equivalent to $(d-(e-1)-n-2)\widetilde{H} - K_{\widetilde{X}}$. Define $D(\pi) := \overline{\sigma(D(\widetilde{\pi})|_{\widetilde{X}\setminus E_1\cup\cdots\cup E_{e-1}})}$ which is called the double point divisor from inner projection π_Λ and linearly equivalent to

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Suppose that X is not a scroll over a smooth projective curve, the Veronese surface in P5, or a Roth variety. Then, due to Noma, D_{inn} is semiample (so, nef).

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If we assume that $\widetilde{\pi} : \widetilde{X} \twoheadrightarrow \overline{X}_\Lambda$ has no exceptional divisor, i.e. $\widetilde{E} = \emptyset$, then, the non-isomorphic double point locus $D(\widetilde{\pi})$ of $\widetilde{\pi}$ is equivalent to $(d-(e-1)-n-2)\widetilde{H} - K_{\widetilde{X}}$. Define $D(\pi) := \overline{\sigma(D(\widetilde{\pi})|_{\widetilde{X}\setminus E_1\cup\cdots\cup E_{e-1}})}$ which is called the double point divisor from inner projection π_Λ and linearly equivalent to

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Note that $(d-e-i)H = K_X + (n+1-i)H + D_{inn}$ where $D_{inn} = (d-e-n-1)H - K_X$.

Proposition (K-J. Park)

On the other hand, we have the following $\mathcal{O}_X$-regularity computation;

- $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^4$ *or in* $\mathbb{P}^5 \Rightarrow \operatorname{reg}(\mathcal{O}_X) = 1$;
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Counter-example to $\mathcal{O}_X$-regularity for singular varieties

J. Mccullough and I. Peeva constructed homogeneous prime ideals using "Rees-like algebra" and "Homogenization" methods with arbitrary homogeneous ideals.

Theorem (J. Mccullough-I. Peeva,2016(preprint))

Let I be a homogeneous ideal of a standard polynomial ring R with n variables, minimally generated by $f_1, ..., f_m$, $m \geq 2$. Then, one can construct a homogeneous non-degenerate "prime" ideal P of a standard polynomial ring $S = R[y_1, ..., y_m, t_1, ..., t_m, z, s]$ such that:

- $\operatorname{reg} S/P = \operatorname{reg} R/I + 2 + \sum_{i=1}^{m} \deg f_i$;
- $\deg S/P = 2\prod_{i=1}^{m}(\deg f_i + 1)$;
- $\operatorname{codim} P = m$.

Furthermore, $\operatorname{depth} S/P = \operatorname{depth} R/I + m + 3 (\geq 2)$.

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Counterexample II

By Koh(1998), for each integer $r \geq 1$, there is an ideal I generated by $(22r-3)$ quadrics and one linear form in $R = k[x_1, ..., x_{22r-1}]$ with

$$\text{maxdeg } \mathrm{Syz}_1^R(I) \geq 2^{2^{r-1}}.$$

By the M-P's construction above, we obtain a prime ideal P of $S = R[y_1, ..., y_{22r-2}, t_1, ..., t_{22r-2}, z, s]$ with $66r-3$ variables such that:

- reg $S/P \geq 2^{2^{r-1}} + 44r - 4$;
- deg $S/P = 4 \times 3^{22r-3}$;
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So, deg S/P − codim P − reg $S/P \leq -2^{2^{r-1}} + 4 \times 3^{22r-3} - 66r + 6$. Since depth S/P = depth $R/I + m + 3 \geq 2$ holds, the variety defined by P is arithmetically normal, and therefore,

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