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Higgs sheaves on singular spaces,
the nonabelian Hodge correspondence,
and the Miyaoka-Yau inequality for minimal varieties of
general type

Workshop talk, NUS Singapore, 19 Jan 2017

arXiv: 1511.08822

1608.06644 / $\mathbb{C}$

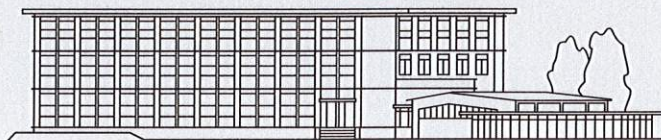
joint w/ Grab, Peternell, Taji



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§ 1 Main result

Thm X minimal of g.c. type (= terminal, K_X nef). Then

$$\left(2 \cdot (n+1) c_2(\mathcal{I}_X) - n \cdot c_1(\mathcal{I}_X)^2 \right) [K_X]^{n-2} \geq 0, \text{ where } n = \dim X.$$

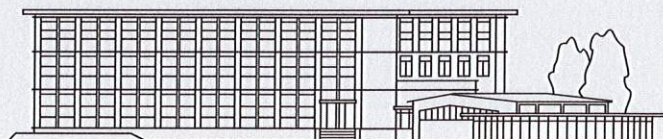
In case of equality, canonical model X_{can} is smooth in codim. two, and in fact a singular ball quotient.

Rem terminal $\Rightarrow$ smooth in codim. two, so c_1 & c_2 make sense.

Prop & Def X normal, compact. Then equivalent

- 1) $X = \mathbb{B}^n / \Gamma$ where $\Gamma \subset \text{Aut}(\mathbb{B}^n)$ discrete, cocompact, acts freely in codim two.
- 2) $X = Y/G$ where Y smooth ball quotient, G finite, acts freely in codim. two.

Rem More is true: have results for klt spaces, get complete char. of sing. ball quotients. Guenancia-Taji found formulation for min. models of orbifold Kodaira-dimension.



§ 2 Recap: nonabelian Hodge theory

Thm (Simpson) X a proj. manifold, $H \in \text{Div}(X)$ ample. Equivalent

$$\left\{ \begin{array}{l} H\text{-semi-stable Higgs} \\ \text{bundles w/ vanishing} \\ \text{Chern classes} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Representations of} \\ \pi_1(X) \end{array} \right\}.$$

Stable Higgs bundles satisfy Bogomolov-Gieseker inequality $\square$

Recall Higgs bundle = locally free $\mathcal{E}$ w/ $\mathcal{O}_X$ -linear map $\theta: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$

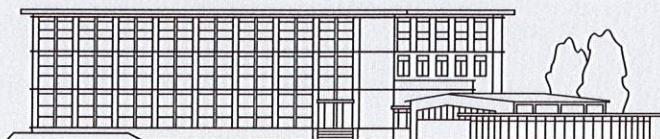
$$\text{s.t. } \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_X^1 \longrightarrow \mathcal{E} \otimes \Omega_X^1 \otimes \Omega_X^1 \text{ vanishes.}$$

Subbundle $\mathcal{F} \subset \mathcal{E}$ is Higgs subbundle if $\theta(\mathcal{F}) \subset \mathcal{F} \otimes \Omega_X^1 \subset \mathcal{E} \otimes \Omega_X^1$
 „stability“ is defined wrt. Higgs subbundles.

Example:

$$\begin{aligned} (\Omega_X^1 \oplus \mathcal{O}_X) &\longrightarrow (\Omega_X^1 \oplus \mathcal{O}_X) \otimes \Omega_X^1 \\ a + b &\longmapsto (0 + 1) \otimes a \end{aligned}$$

Example / Thm Given a complex, pol. VHS on X , the Gauß-Manin conn. defines a ~~VHS~~ rep. of $\pi_1(X)$, thus a Higgs bundle. The Higgs bundle appearing in this way are prec. those whose isom. class is $\mathbb{C}^*$ -invariant: $(\mathcal{E}, \theta) \simeq (\mathcal{E}, \lambda \cdot \theta) \quad \forall \lambda \in \mathbb{C}^* \quad \square$



§ 3 Other ingredients

§ 3.1 Pull-back map for refl. differentials

Thm (GKKP, K) Given a morphism $f: X \rightarrow Y$ of klt spaces, then exists pull-back of reflexive forms. $\square$

Reflexive form = section in $(\Omega_X^p)^{**} =$ differential form on X_{reg} .

Pull-back = map $f^* \Omega_Y^{[1]} \rightarrow \Omega_X^{[1]}$

Explain...

Main application here: can define a notion of Higgs sheaf on sing. spaces.

Def: X klt. A Higgs sheaf is a coherent $\mathcal{E}$, together w/ $\mathcal{O}_X$ -linear

$\Theta: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^{[1]}$ s.t. $\Theta \wedge \Theta = 0$. Subsheaf $\mathcal{F} \subseteq \mathcal{E}$ is generically

Higgs invariant if $\mathcal{F}|_{X_{\text{reg}}}$ is Higgs subsheaf. Stability looks at generically invariant sheaves.

Warning - Not an Abelian category! No HNF!

But We do have pull-back.

$$f^* \mathcal{E} \rightarrow f^* \mathcal{E} \otimes f^* \Omega_X^{[1]} \rightarrow f^* \mathcal{E} \otimes \Omega_Y^{[1]}$$

$$\text{If } Y \text{ is smooth, then } (f^* \mathcal{E})^{**} \rightarrow (f^* \mathcal{E})^{**} \otimes \Omega_Y^1$$



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Application

- Can compare stability on X and Y
- get Mordell-Weil theorem

Warning: Reflexive pull-back is not functorial.

§ 3.2 Fundamental groups

Thm (GKP 16) X klt. Then, $\exists$ a quasi-étale $Y \rightarrow X$ s.t.

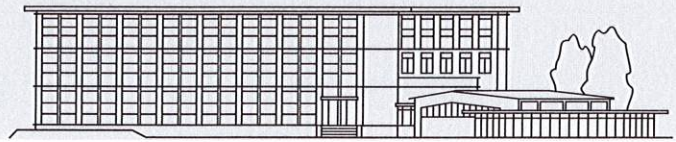
$\hat{\pi}_1(Y_{\text{reg}}) \rightarrow \hat{\pi}_1(Y)$ is isomorphic. In particular, any repr. of

$\pi_1(Y_{\text{reg}})$ comes from a representation of $\pi_1(Y)$. $\square$



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§ 4 Idea of proof in the simplest case

Simplifying assumptions: $\hat{\pi}_1(X_{\text{reg}}) = \hat{\pi}_1(X)$ and $X = X_{\text{con}}$.

Look at: $\mathcal{E} := \Omega_x^{[1]} \oplus \mathcal{O}_x$, Higgs field as in example.

Enoki / Guenancia This is stable!

Take genl. hyperplanes $H_1, \dots, H_{n-1} \in |mK_X|$ for $m \gg 0$

$S := H_1 \cap \dots \cap H_{n-1}$ is smooth, contained in X_{reg} .

Mitcha-Romanthan: $(\Omega_x^{[1]} \oplus \mathcal{O}_x, \Theta)|_S$ is stable, satisfies BG inequalities, get MY-inequality for J_X

What about uniformisation. Look at $\text{End}(\Omega_x^{[1]} \oplus \mathcal{O}_x)|_S$

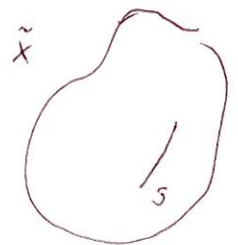
this carries a Higgs field, is semistable, c_1 & c_2 vanish

$\leadsto$ get representation of $\pi_1(S)$

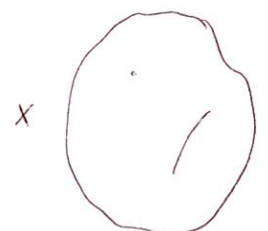
Goresky - MacPherson: $\pi_1(S) = \pi_1(X_{\text{reg}})$

Takayama: $\pi_1(X) = \pi_1(\tilde{X})$

$\leadsto$ get repr. of $\pi_1(X) = \pi_1(\tilde{X})$



↓

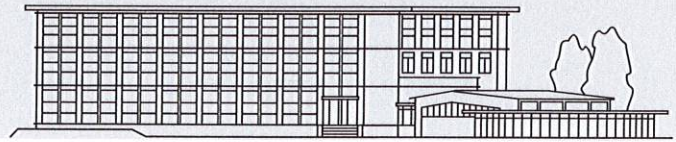


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Simpson: Get a Higgs-bundle on $\tilde{X}$ that comes from this repr.

Prove: this bundle comes from X , in fact equals $\text{End}(\Omega_X^{[1]} \oplus \mathcal{O}_X)$

$\Rightarrow \Omega_X^{[1]}$ is locally free $\Rightarrow X$ smooth, so is a sing. ball
quotient by classical results.

□

§ 5 Where next

- Study sing. ball quotients (hyperbolicity, Mostow-Rigidity, ...)
- Nonabelian Hodge corr. for singular spaces.
- More uniformisation