Differential Forms and the Abundance Conjecture

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Nonvanishing Conjecture (NC)

X normal projective variety $/\mathbb{C}$, canonical singularities, (terminal)

K_X pseudo effective $\Rightarrow \kappa(X) \geq 0$.

MMP predicts: $X \overset{\text{birat}}{\dashrightarrow} X'$, X' canonical singularities, (terminal)

$K_{X'}$ nef.

Abundance Conjecture (AC) $K_{X'}$ is semiample.

(NC) and (AC) known in dimension ≤ 3

One shortly says that X has a "good(minimal)model".

Important: Version for klt pairs (X, Δ).

Report on joint work with V. Lazić. (on arXiv)

Theorem 1 X normal projective variety with only terminal singularities, terminal variety for short.

Assume K_X pseudoeffective.

Assume furthermore $(n = \dim X)$

(a) Existence of good models for klt pairs in dimension $\leq n-1$ (o.k. if $\dim X = 4$) or

(b) K_X nef, $\nu(X) = 1$ (so $K_X \not\equiv 0$, $K_X^2 = 0$).

Assume finally: $\exists \pi : Y \to X$ desingularization, $\exists q > 0$, such that

$$H^0(Y, \Omega^q_Y \otimes \pi^*(mK_X)) \neq 0$$

for ∞ many m (s. that mK_X is Cartier).

Then $\kappa(X) \geq 0$.

<u>Theorem 2</u> X terminal variety, K_X pseudoeffective, $\chi(\mathcal{O}_X) \neq 0$, $n = \dim X$.

(a) Assume existence of good models for klt pairs in dimension $\leq n-1$. If K_X has a (singular) metric with algebraic singularities and semipositive curvature (current), then $\kappa(X) \geq 0$.

(b) If K_X nef, $\nu(X) = 1 \Rightarrow \kappa(X) \geq 0$.

Remark (1) If K_X is hermitian semipositive, then in Thm 2 (a), $\kappa(X) \geq 0$. Even more: K_X is semiample.

(2) Assume $\dim X = 4$, $\chi(\mathcal{O}_X) \neq 0$.

May assume $q(X) = 0$ (Kawamata, Lai, ...), hence $H^3(\mathcal{O}_X) \neq 0$, so $\exists$ holomorphic 3-form.

$\rightsquigarrow$ foliation by curves. Useful?

Idea of proof of Thm 1, (a)

Say X smooth. Get ∞ many m: $-mK_X \hookrightarrow \Omega_X^q$ for some q. $\mathcal{F} := \mathrm{Im}\left(\bigoplus_m \mathcal{O}(-mK_X) \to \Omega_X^q \right)$

$\rightsquigarrow \det \mathcal{F} \subset \Lambda^r \Omega_X^q$, saturated wlog.

$Q := \Lambda^r \Omega_X^q / \det \mathcal{F}$; then $\det Q$ is pseudo-effective; and $\exists\, \ell \in \mathbb{N}$:

$$\ell K_X = \det \mathcal{F} + \det Q$$

Now $H^0(mK_X \otimes \det \mathcal{F}^*) \neq 0$, ∞ many m

$\rightsquigarrow N_m \sim mK_X + \det \mathcal{F}^*$, N_m effective.

$$\Rightarrow \quad \underbrace{N_m + \mathrm{div}\, Q}_{\text{indep of. } m} \sim \underbrace{(m+\ell)\, K_X}_{\text{indep. of } m}.$$

Now apply:

<u>Theorem 3</u> Assume existence of good models for klt pairs in dim $\leq n-1$. Let (X,Δ) be a $\mathbb{Q}$-factorial klt pair of dimension n, $K_X+\Delta$ pseudoeffective. Assume $\exists$ pseudoeffective $\mathbb{Q}$-divisor F on X, $\exists$ $S \subset \mathbb{N}$ infinite, such that

$$N_m + F \sim_{\mathbb{Q}} m(K_X+\Delta),$$

N_m effective integral. Then

$$\kappa(K_X+\Delta) \geq \max\{\kappa(X, N_m) \mid m \in S\} \geq 0.$$

<u>Idea of proof of Theorem 2, (a)</u>

Choose desingularization $\pi: Y \to X$, choose ℓ such that ℓK_X Cartier.

By assumption obtain singular metric h on $\pi^*(\ell K_X)$:

$$\varphi = \sum_{i=1}^{r} \alpha_j \log |g_j| + \text{loc. bounded},$$

$\alpha_j \in \mathbb{Q}^+ \rightsquigarrow$ multiplier ideal sheaves

$$\mathcal{J}(h^{\otimes m}) = \mathcal{O}_Y\Big(-\sum_{j=1}^{r} \lfloor m\alpha_j \rfloor D_j\Big) \qquad (*)$$

Assume $\kappa(X) = -\infty$. Thm 1 gives:

$$H^0(Y, \Omega_Y^q \otimes \pi^*(\ell m K_X)) = 0, \quad \forall q, \ \forall m \gg 0.$$

$$\Rightarrow H^0(Y, \Omega_Y^q \otimes \pi^*(\ell m K_X) \otimes \mathcal{J}(h^m)) = 0$$

$$\Big\Downarrow \wedge \omega^{n-q} \qquad \text{Hard Lefschetz}$$

$$H^{n-q}(Y, K_Y \otimes \pi^*(\ell m K_X) \otimes \mathcal{J}(h^m)).$$

So $H^{n-q} = 0$.

$(*)$ plus Serre duality $\Rightarrow$

$$\chi\Big(\mathcal{O}_Y\Big(\sum_{j=1}^{r} \lfloor m\alpha_j \rfloor D_j - m\ell\, \pi^*(K_X)\Big)\Big) = 0, \quad m \gg 0.$$

Set $\alpha_j = p_j/q_j$, $D := \sum \frac{q\, p_j}{q_j} D - q\ell\pi^* K_X$

$$\Rightarrow \chi(\mathcal{O}_Y(mD)) = 0 \quad \forall m \text{ suff. divisible}$$

$$\Rightarrow \chi(\mathcal{O}_Y) = 0 \quad \text{\Large ↯}.$$

What is better if $\nu(X) = 1$? Key :

<u>Theorem 4</u> X_n projective, $\mathbb{Q}$-factorial, L nef, $\nu(L) = 1$. Assume $\exists$ F pseudo effective $\mathbb{Q}$-divisor and a $\mathbb{Q}$-divisor $D \neq 0$, effective, such that $D + F \sim_{\mathbb{Q}} L$. Then $\exists$ effective $\mathbb{Q}$-divisor E, s. that $L \equiv E$ and $\kappa(E) \geq \kappa(D)$.

Reason : $\nu(L) = 1$ allows to reduce to surface and to apply Hodge Index Theorem.

<u>Corollary</u> Assume existence of good models for klt pairs in dimension $\leq n-1$. Let X_n be terminal variety. Assume $\exists$ $\pi : Y \to X$ desingularization, $\exists$ ℓ, such that $\pi^*(\ell K_X)$ has singular metric with semipositive curvature and vanishing Lelong #. If the polynomial $\chi(X, \mathcal{O}_X(m\ell K_X))$ is not identically 0, then K_X is semiample.

Proof was result of Gongyo – Matsumura.

Remark Actually more is true in Corollary:
if K_X is not semi ample, then $H^q(X, \mathcal{O}_X(-m K_X)) = 0$
$\forall q$ $\forall m$ sufficiently divisible.

Concerning (AC):

Theorem 5 X_n terminal variety. Assume either

(a) existence of good models for klt pairs in dimension $\leq n-1$, $|\chi(\mathcal{O}_X)| > 2^{n-1}$, $\kappa(X) \geq 0$

(b) K_X nef, $\chi(\mathcal{O}_X) \neq 0$, $\nu(X) = 1$.

(c) K_X nef, $\nu(X) = n-1$, $\exists\, \pi : Y \to X$ desingularization, s. that

$\pi^* K_X^{n-2} \cdot c_2(Y) \neq 0$

or $\pi^* K_X^{n-2} \cdot c_2(Y) = 0$ and $\pi^* K_X^{n-3} \cdot c_2(Y) \cdot K_Y \neq 0$

Then K_X is semi ample, in cases (b), (c) resp. X has a good model in (a).

Remark ~~Most~~ Several results extend to klt pairs (X, Δ).

<u>Remarks</u> (1) In (a) of Thm 5, if X does not have a good model, then

$\forall q \geq 0$: $h^q(X, \mathcal{O}_X) \leq \binom{n}{q}$.

(2) Strengthening of (c) in case $n = 4$:

suffices $\exists \pi : Y \to X$ desingularization, s. that

$\pi^* K_X^2 c_2(Y) \neq 0$ or $\pi^* K_X^2 c_2(Y) = 0$ and

$\chi(\mathcal{O}_X) > 0$.