Mixed Spin Field theory on the quintic.

$\mathbb{C}^5$, $W = X_1^5 + \cdots + X_5^5$, we get a CY 3-fold $X = \{W=0\} \subset \mathbb{P}^4$, called a quintic.

We have GW theory and mirror symmetry.

Gromov-Witten theory:

$$M_{g,n}(X,\beta) = \left\{ f : \Sigma \to X \;\middle|\; \begin{array}{l} \Sigma : \text{genus-}g \text{ Riemann surface}, \\ n : \text{marked points}: P_1, \ldots, P_n \\ H_2(X,\mathbb{Z}) \ni \beta = f_*(\Sigma) \end{array} \right\}$$

f holomorphic.

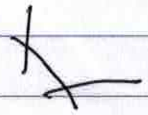
$\downarrow ev_i$

$f(P_i) \in X$.

GW invariants: $\displaystyle\int_{M_{g,n}(X,\beta)} \prod_{i=1}^{n} ev_i^* \alpha_i$

= Counting the number of curves Σ in X intersecting $A_i = PD(\alpha_i)$.

- $M_{g,n}(X,\beta)$ is not compact $\Rightarrow$ Kontsevich compactification $\overline{M}_{g,n}(X,\beta)$

stable maps:  $\xrightarrow{f} X$, $|\mathrm{Aut}(f)| < +\infty$.

- $\dim \overline{M}_{g,n}(X,\beta)$ is not the correct dim $\Rightarrow$

Li-Tian, ~~Bef~~ Behrend-Fantechi's virtual cycle technique

$\Rightarrow [\overline{M}_{g,n}(X,\beta)]^{vir}$.

Known: genus zero: GW-invariants $\xleftrightarrow{\text{Mirror Symmetry}}$ Period integral

quintic case: Candelas et. al 1991 $X \longleftrightarrow \hat{X}$ on $\hat{X}$

Givental, Lian-Liu-Yau.

genus 1: J. Li-Zinger, Zinger.

higher genus: only physical conjectures

Maulik-Pandharipande.

$$N_{g,d} = \int_{[\overline{M}_g(X,d)]^{vir}} 1$$

H.L. Chang-J. Li's P-fields theory:

Moduli of stable maps to $\mathbb{P}^4$ with fields

$$\overline{M}_g(\mathbb{P}^4,d)^p = \left\{ [f,C,p] \;\middle|\; \begin{array}{l}[f,C]\in \overline{M}_g(\mathbb{P}^4,d)\\ p\in\Gamma(C, f^*\mathcal{O}_{\mathbb{P}^4}(-5)\otimes\omega_C)\end{array}\right\} \Big/ \sim$$

cosection $\sigma: Ob_{\overline{M}_g(\mathbb{P}^4,d)^p} \longrightarrow \mathcal{O}_{\overline{M}_g(\mathbb{P}^4,d)^g}$ $\quad \overline{M}_g(\mathbb{P}^4,d)^p \supset$

[Kiem-Li]-cosection localization theory

the degenerate locus of $\sigma \triangleq \{\sigma=0\} = \overline{M}_g(X,d)$

the invariants defined by the virtual cycle from the cosection = GW-virtual cycle up to a sign.

Regarding W as a map $\mathbb{C}^5 \xrightarrow{W} \mathbb{C}$, we have

- Singularity theory
- LG-theory: physical theory.
  - $\Rightarrow$ Witten conjecture: intersection numbers on moduli space of spin curves & KdV hierarchies.
  - $\Rightarrow$ CY-LG correspondence

The mathematical theory is FJRW-theory
Fan-Jarvis-Ruan-Witten

The algebraic geometric construction of FJRW (narrow sector) is by Chiodo, ~~Polo~~ Polishchuk, Chang-J. Li-W.P.

[CLL]: $M_g^p(\mathbb{Z}_5)=\{[C,L_j,p_j]_{j=1}^5 \mid [C,L_j]\in M_g(\mathbb{Z}_5),\ L_j^{\otimes 5}=\omega_C$, $p_j\in\Gamma(L_j)\}$, C twisted curve (orbifold curve)

$\overline{M}_g(\mathbb{Z}_5)=\{[C,L] \mid C\in\overline{M}_g,\ L_j^{\otimes5}=\omega_C\}$ There is a cosection of the obstruction sheaf $\xrightarrow{\text{Kiem-Li}} [\overline{M}_g(\mathbb{Z}_5)^p]^{vir}\in A_*(\overline{M}_g(\mathbb{Z}_5))$

This invariant $=$ FJRW-invariants.

P3

Mixed Spin P-fields (MSP-fields)

$\mathbb{C}^*$ acts on $\mathbb{C}^6_{1,\cdots,1,-5}$, $t\cdot(a_1,\cdots,a_5,b)=(ta_1,\dots,ta_5,t^{-5}b)$

GIT quotients: $\left(\mathbb{C}^6-\{(0,\cdots,0,b)\}\right)/\mathbb{C}^*=K_{\mathbb{P}^4}$

$\left(\mathbb{C}^6-\{(a_1,\dots,a_5,0)\}\right)/\mathbb{C}^*=\mathbb{C}^5/\mathbb{Z}_5$

Consider maps from Riemann surfaces to $[\mathbb{C}^6/\mathbb{C}^*]$

The maps to $K_{\mathbb{P}^4}\Rightarrow$ GW-theory with p-fields

The maps to $\mathbb{C}^5/\mathbb{Z}_5\Rightarrow$ FJRW-theory.

The problem is: there is an Artin point $[\vec{0}/\mathbb{C}^*]$

Enlarge the space (Master Space Technique)

$\cancel{[\mathbb{C}^7/\mathbb{C}^*]_{1,\cdots,1,-5,1} \quad (a_1,\cdots,a_5,b,c)}$

~~MSP field~~ $\cancel{\xi=(C,\Sigma^C,\mathcal{L},\mathcal{N},}$

$\cancel{\mathbb{C}^6\times\mathbb{C}^2/\mathbb{C}^*\times\mathbb{C}^*}$

$\mathbb{C}^5\times\mathbb{C}\times\mathbb{P}^1/\mathbb{C}^*$: $t\cdot(a_1,\cdots,a_5,b,[u_1,u_2])=(ta_1,\dots ta_5,t^{-5}b,[tu_1,u_2])$

It has a GIT quotient: $S=(a_i=0=u_1)\cup(p=0=u_2)$

$Z=[\mathbb{C}^5\times\mathbb{C}\times\mathbb{P}^1-S]/\mathbb{C}^*$

Fields of Riemann surfaces mapped to $([\mathbb{C}^5\times\mathbb{C}\times\mathbb{P}^1-S]/\mathbb{C}^*)$

Z has a T-action ($\mathbb{C}^*$-action)

$t(a_1,\cdots a_5,p,[u_1,u_2])=(a_1,\cdots a_5,p,[tu_1,u_2])$

$Z^T=K_{\mathbb{P}^4}\times\{0\}\sqcup\vec{0}\times(\mathbb{P}^1-\{0,\infty\})/\mathbb{C}^*\sqcup(\mathbb{C}^5/\mathbb{Z}_5)\times\{\infty\}$.

is MSP fields: $\xi = (C, \Sigma, \mathcal{L}, N, \varphi, \rho, \nu)$

$\mathcal{L}, N$: line bundles on C.

$\varphi = (\varphi_1, \dots, \varphi_5) \in H^0(\mathcal{L}^{\oplus 5})$

$\rho \in H^0(\mathcal{L}^{\otimes -5} \otimes \omega^{\log})$

$(\nu_1, \nu_2) \in H^0(\mathcal{L} \otimes N, N))$

Conditions: (ν_1, ν_2) nowhere vanishing

(φ, ν_1) nowhere vanishing

(ρ, ν_2) nowhere vanishing

Ex: $\nu_1 = 0 \Rightarrow N \cong \mathcal{O}_C$. $\nu_2 =$ constant,

$\varphi \Rightarrow C \longrightarrow \mathbb{P}^4$ with a p-field $\rho \in H^0(f^*\mathcal{O}_{\mathbb{P}^4}(-5)$

(stable maps with p-fields)

Ex: $\nu_2 = 0$; $\nu_1 =$ const. $\mathcal{L} \otimes N \cong \mathcal{O}_C$. $N = \mathcal{L}^{-1}$.

$\rho =$ constant $\Rightarrow \mathcal{L}^{\otimes 5} \cong \omega_C^{\log}$, five p-fields $\varphi_i \in H^0(\mathcal{L}$

~~Cosection localization~~:

The obstruction sheaf has a cosection σ

Cosection localization gives the degenerate locus of σ ~~is~~ $W^- = \{ \xi \mid (\varphi = 0) \cup (\rho = 0, \varphi_1^5 + \dots + \varphi_5^5 = 0) \}$

Theorem. W^- is proper. ~~Wg~~ $W^-_{g,d}$

Moduli of MSP-fields has a $\mathbb{C}^*$-action, called T-action,

$t \in T = \mathbb{C}^*$, ~~$(\varphi_1, \dots, \varphi_5)$~~

$t(C, \Sigma, \mathcal{L}, N, \varphi, \rho, [\nu_1, \nu_2]) = (C, \Sigma, \mathcal{L}, N, \varphi, \rho, [t\nu_1, \nu_2])$

W^- has T-action.

$d = (d_0, d_\infty) = (\deg(L\otimes N), \deg N)$

$\overline{W}_{g,\vec{d}}$ = moduli of stable MSP fields of curve genus $= g$ P_ρ

Subject : Date:

Γ: a graph associated to fixed points of T-action on $W_{g,d}$, and F_Γ: connected component of $W^T_{g,d}$ of the graph type. Cosection-localization $\Rightarrow$ $[W_{g,\vec d}]^{vir}_{loc} \in A^{\mathbb{C}^*}_*(W_{g,\vec d})$

u : parameter for $H^*_T(pt)$

(*) Then $\left[u^{d_0 + d_\infty + 1 - g} [W_{g,d}]^{vir}_{loc} \right]_0 = 0$

$[\ \ ]_0 \Rightarrow$ degree zero term in variable u.

(*) $\Rightarrow$ identities

Thm: Let $d_\infty = 0$. (*) provides an effective algorithm to evaluate GW invariants $N_{g,d}$ provided the following are known

(1) $N_{g',d'}$ for (g', d') such that $g' < g$, $d' \le d$

(2) $N_{g,d'}$ for $d' < g$.

(3) $\Theta_{g',k}$ for $g' \le g-1$ and $k \le 2g-4$

(4) $\Theta_{g,k}$ for $k \le 2g-2$.

$$\Theta_{g,k} = \int_{[\overline{M}_{g,(r_j)}(\mathbb{Z}_5)^p]^{vir}_{loc}} 1 \in \mathbb{Q} \quad \text{for } k+2-2g \equiv 0 \mod 5$$

$r_j = \zeta^2$, $-\frac{2}{5}$ insertion

~~determines d~~ $1 \le j \le k$, $\zeta = \exp(2\pi i/5)$

If C_0 has k markings