Globally F-regular type of moduli spaces of parabolic sheaves

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Motivation:Jacobian variety and theta functions

- Let C be a smooth projective curve of genus g,

$$J_C^d = \{\text{vector bundles } E \text{ of } \mathrm{rk}(E) = 1,\ \deg(E) = d \text{ on } C\}$$

- Let $\mathcal{E}$ be a universal line bundle on $C \times J_C^d \xrightarrow{\pi} J_C^d$, and

$$\Theta_{J_C^d} := \det R\pi(\mathcal{E})^{-k} \otimes \det(\mathcal{E}_y)^{k(d+1-g)}$$

- $H^0(J_C^d, \Theta_{J_C^d})$ is the so called space of **theta functions of order** k

$$\dim H^0(J_C^d, \Theta_{J_C^d}) = k^g$$

- A. Weil (1938) (Généralisation des fonctions abéliennes) suggested to **generalize the theory to higher rank** $r > 1$.

Motivation: Generalized theta functions

- (Mumford, Narasimhan-Seshadri): There exist **moduli spaces**

$$\mathcal{U}_C = \{\text{s.s. bundles } E \text{ of } \mathrm{rk}(E) = r,\ \deg(E) = d \text{ on } C\}$$

  and **theta line bundles** $\Theta_{\mathcal{U}_C}$ **on** $\mathcal{U}_C$.

- $H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$: space of **generalized theta functions of order** k

$$\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = ?$$

- A formula was predicted by **Conformal Field Theory**, when $r = 2$,

$$\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = \left(\frac{k}{2}\right)^g \left(\frac{k+2}{2}\right)^{g-1} \sum_{i=0}^{k} \frac{(-1)^{id}}{(\sin \frac{(i+1)\pi}{k+2})^{2g-2}}$$

Motivation: Degeneration method

- Degenerate $C_t \rightsquigarrow C_0$ to an irreducible curve C_0 with one node $x_0 \in C_0$.
- Need to prove: $\dim H^0(\mathcal{U}_{C_t}, \Theta_{\mathcal{U}_{C_t}}) = \dim H^0(\mathcal{U}_{C_0}, \Theta_{\mathcal{U}_{C_0}})$.
- Let $\pi : \widetilde{C_0} \to C_0$ be the normalization, $\pi^{-1}(x_0) = \{x_1, x_2\}$, then $\widetilde{C_0}$ has genus $\tilde{g} = g - 1$.

Theorem 1 (Sun, 2000)

$$H^0(\mathcal{U}_{C_0}, \Theta_{\mathcal{U}_{C_0}}) \cong \bigoplus_{\mu} H^0(\mathcal{U}^{\mu}_{\widetilde{C_0}}, \Theta_{\mathcal{U}^{\mu}_{\widetilde{C_0}}})$$

where $\mu = (\mu_1, \cdots, \mu_r)$ *runs through* $0 \le \mu_r \le \cdots \le \mu_1 \le k - 1$.

Motivation: known result

Theorem 2 (Sun, 2000)

Let C be a projective curve of genus $g \geq 3$ with at most one node. Then

$$\mathrm{H}^1(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = 0.$$

- $\mathcal{R} \supset \mathcal{R}_\omega^{ss} \xrightarrow{\psi} \mathcal{R}_\omega^{ss}//G := \mathcal{U}_C$, $H^1(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = H^1(\mathcal{R}_\omega^{ss}, \Theta_\omega)^G$.
- $\Theta_\omega = \omega_{\mathcal{R}} \otimes \Theta_{\omega'}$, $\quad \mathcal{R} \supset \mathcal{R}_{\omega'}^{ss} \xrightarrow{\varphi} \mathcal{R}_{\omega'}^{ss}//G := \mathcal{U}_{\omega'}$.
- $H^1(\mathcal{R}_\omega^{ss}, \Theta_\omega)^G = H^1(\mathcal{R}, \Theta_\omega)^G = H^1(\mathcal{R}_{\omega'}^{ss}, \Theta_\omega)^G$ if codimension ...
- $H^1(\mathcal{R}_\omega^{ss}, \Theta_\omega)^G = H^1(\mathcal{R}_{\omega'}^{ss}, \Theta_\omega)^G = H^1(\mathcal{U}_{\omega'}, (\varphi_*\omega_{\mathcal{R}_{\omega'}^{ss}})^G \otimes \Theta_{\mathcal{U}_{\omega'}})$.
- If $\omega_{\mathcal{U}_{\omega'}} = (\varphi_*\omega_{\mathcal{R}_{\omega'}^{ss}})^G$, $H^1(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = H^1(\mathcal{U}_{\omega'}, \omega_{\mathcal{U}_{\omega'}} \otimes \Theta_{\mathcal{U}_{\omega'}}) = 0$.

Globally F-regular varieties

Let X be a variety over a perfect field k of $char(k) = p > 0$,

$$F : X \to X$$

be the Frobenius map, $F^e : X \to X$ be its e-th iterate.

Definition 1

A normal variety X over a perfect field is called stably Frobenius D-split if

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(D)$$

is split for some $e > 0$.

Definition 2

A normal variety X over a perfect field is called globally F-regular if X is stably Frobenius D-split for any effective divisor D.

Globally F-regular varieties: Projective case

Proposition 1

Let X be a projective variety over a perfect field. Then the following statements are equivalent.

1. *X is normal and is stably Frobenius D-split for any effective D;*
2. *X is stably Frobenius D-split for any effective Cartier D;*
3. *For any ample line bundle $\mathcal{L}$, the section ring of X*

$$R(X,\mathcal{L}) = \bigoplus_{n=0}^{\infty} H^0(X,\mathcal{L}^n)$$

is strongly F-regular.

Globally F-regular varieties: Theory of Characteristic 0

F or any scheme X of finite type over a field K of characteristic zero, there is a finitely generated $\mathbb{Z}$-algebra $A \subset K$ and an A-flat scheme

$$X_A \to S = \mathrm{Spec}(A)$$

such that $X_K = X_A \times_S \mathrm{Spec}(K) \cong X$. $X_A \to S = \mathrm{Spec}(A)$ is called an integral model of X/K, and a closed fiber $X_s = X_A \times_S \mathrm{Spec}(\overline{k(s)})$ is called "**modulo p reduction of X**" where $p = \mathrm{char}(k(s)) > 0$.

Definition 3

*A variety X over a field of characteristic zero is said to be of globally F-regular type if its "***modulo p reduction of X***" are globally F-regular for a dense set of p.*

A normal projective variety X is called a *Fano variety* if

$$\omega_X^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X)$$

is an ample line bundle.

Varieties of globally F-regular type

Proposition 2 (Hara, Mehta-Srinivas, 1997-1998)

A Fano variety (over a field of characteristic zero) with at most rational singularities is of globally F-regular type.

Proposition 3 (K. E. Smith)

Let X be a projective variety over a field of characteristic zero. If X is of globally F-regular type, then we have

(1) *X is normal, Cohen-Macaulay with rational singularities. If X is $\mathbb{Q}$-Gorenstein, then X has log terminal singularities.*

(2) *For any nef line bundle $\mathcal{L}$ on X, we have $H^i(X, \mathcal{L}) = 0$ when $i > 0$. In particular, $H^i(X, \mathcal{O}_X) = 0$ whenever $i > 0$.*

Choose ample effective divisor D such that $\mathcal{L}^n(D)$ is ample for all $n \geq 0$. Thus $H^i(X, \mathcal{L}^n(D)) = 0$ for all $n \geq 0$, which and split $\mathcal{L} \to F_*^e(\mathcal{L}^{p^e}(D))$ imply $H^i(X, \mathcal{L}) = 0$ since $H^i(X, \mathcal{L}) \hookrightarrow H^i(X, \mathcal{L}^{p^e}(D)) = 0$.

Moduli spaces: globally F-regular type

Definition 4

Let C be a smooth projective curve, $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x\in I})$ and

$$\det : \mathcal{U}_{C,\omega} \to J_X^d, \quad E \mapsto \det(E).$$

For any $L \in J_X^d$, the fiber $\mathcal{U}_{C,\omega}^L := \det^{-1}(L)$ is called **moduli spaces of semi-stable parabolic bundles with fixed determinant**, *which is normal with at most rational singularities.*

Theorem 3 (Sun-Zhou, 2016)

For any data ω, $\mathcal{U}_{C,\omega}^L$ is **of globally F-regular type**. *In particular,*

$$H^i(\mathcal{U}_{C,\omega}, \mathcal{L}) = 0 \quad \forall\, i > 0$$

for any ample line bundle $\mathcal{L}$.

Globally F-regular varieties: useful observations

- If X is globally F-regular, so is any open subschemes $U \subset X$.

Lemma 1

Let $f : X \to Y$ be a morphism of varieties over a perfect field k of $char(k) = p > 0$. If $\mathcal{O}_Y = f_*\mathcal{O}_X$ and X is globally F-regular, then Y is stably Frobenius D-split for any effective Cartier divisor D, and it is globally F-regular when Y is normal.

Proof.

For any Cartier divisor $D \in Div(Y)$, let $H = f^*D$. Then

$$(F_X)^e_*\mathcal{O}_X(H) \xrightarrow{h} \mathcal{O}_X \;\Rightarrow\; f_*(F_X)^e_*\mathcal{O}_X(H) \xrightarrow{f_*h} f_*\mathcal{O}_X = \mathcal{O}_Y$$

$$f_*(F_X)^e_*\mathcal{O}_X(H) = (F_Y^e)_*f_*\mathcal{O}_X(H) = (F_Y^e)_*\mathcal{O}_Y(D).$$

□

Varieties of globally F-regular type: A question

Question 1

Let $f : X \to Y$ be a morphism of varieties over a field of characteristic zero with $f_*\mathcal{O}_X = \mathcal{O}_Y$ and X be of globally F-regular type. Is Y a variety of globally F-regular type ?

- Choose an integral model $f_A : X_A \to Y_A$ of $f : X \to Y$ such that

$$f_{A*}\mathcal{O}_{X_A} = \mathcal{O}_{Y_A}.$$

- For any closed point $\mathrm{Spec}(\overline{k(s)}) \to S = \mathrm{Spec}(A)$, let

$$Y_s = Y_A \times_A \overline{k(s)} \xrightarrow{i_s} Y_A, \quad X_s = X_A \times_A \overline{k(s)} \xrightarrow{j_s} X_A.$$

- If $i_s^* f_{A*}\mathcal{O}_{X_A} = f_{s*} j_s^* \mathcal{O}_{X_A}$ holds for a dense set of $s \in S$, Question will have an affirmative answer.

Varieties of globally F-regular type: A criterion

Definition 5

A morphism $X \xrightarrow{f} Y$ *of varieties over a field of characteristic zero is called* p*-compatible if there is an integral model* $X_A \xrightarrow{f_A} Y_A$ *such that*

$$i_s^* f_{A*} \mathcal{O}_{X_A} = f_{s*} j_s^* \mathcal{O}_{X_A}$$

for $s \in \mathrm{Spec}(A)$.

Lemma 2

Let $X \xrightarrow{f} Y$ *be a* p*-compatible morphism. If the natural map*

$$\mathcal{O}_Y \to f_* \mathcal{O}_X$$

splits and X *is of globally F-regular type, then, for a dense set of* p*, the modulo* p *reduction of* Y *is stably Frobenius* D*-split for any effective Cartier divisor* D*, and* Y *is of globally F-regular type if it is projective.*

Varieties of globally F-regular type: GIT quotient

Proposition 4

Let $\hat{Y}^{ss}(L) \xrightarrow{\psi} Y$, $\hat{Z}^{ss}(L') \xrightarrow{\varphi} Z$ *be the GIT quotients. Assume*

(1) *there are* G*-invariant normal open subschemes* $\mathcal{R} \subset \hat{Y}$, $\mathcal{R}' \subset \hat{Z}$ *such that* $\hat{Y}^{ss}(L) \subset \mathcal{R}$, $\hat{Z}^{ss}(L') \subset \mathcal{R}'$;

(2) *there is a* G*-invariant* p*-compatible morphism* $\mathcal{R}' \xrightarrow{\hat{f}} \mathcal{R}$ *such that* $\hat{f}_*\mathcal{O}_{\mathcal{R}'} = \mathcal{O}_{\mathcal{R}}$;

(3) *there is an* G*-invariant open set* $W \subset \hat{Z}^{ss}(L')$ *such that*

$$\mathrm{Codim}(\mathcal{R}' \setminus W) \geq 2, \quad \hat{X} = \varphi^{-1}\varphi(\hat{X})$$

where $\hat{X} = W \cap \hat{f}^{-1}(\hat{Y}^{ss}(L))$.

If Z *is of globally F-regular type. Then so is* Y.

Moduli spaces: parabolic bundles

- Let C be a smooth projective curve of genus $g \geq 0$, E a vector bundle on C of rank r.
- E has a parabolic structure of type $\vec{n}(x)$ and weights $\vec{a}(x)$ at a smooth point $x \in C$: we mean a choice of flag of quotients

$$E_x = Q_{l_x+1}(E)_x \twoheadrightarrow \cdots\cdots \twoheadrightarrow Q_1(E)_x \twoheadrightarrow Q_0(E)_x = 0$$

of fibre E_x with $n_i(x) = \dim(ker\{Q_i(E)_x \twoheadrightarrow Q_{i-1}(E)_x\})$ and a sequence of integers $0 \leq a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) < k$,

$$\vec{n}(x) := (n_1(x), n_2(x), \cdots, n_{l_x+1}(x))$$

$$\vec{a}(x) := (a_1(x), a_2(x), \cdots, a_{l_x+1}(x))$$

- For any $F \subset E$, let $Q_i(E)_x^F \subset Q_i(E)_x$ be the image of F,

$$n_i^F = \dim(ker\{Q_i(E)_x^F \twoheadrightarrow Q_{i-1}(E)_x^F\})$$

$$\mathrm{par}\chi(F) := \chi(F) + \frac{1}{k}\sum_{x\in I}\sum_{i=1}^{l_x+1} a_i(x) n_i^F(x).$$

Moduli spaces: semistable parabolic bundles

- E is called **semistable** (resp., **stable**) for $\frac{\vec{a}}{k}$ if for any nontrivial subsheaf $E' \subset E$ such that E/E' is torsion free, one has
$$\mathrm{par}\chi(E') \leq \frac{\mathrm{par}\chi(E)}{r} \cdot r(E') \text{ (resp., } <).$$
- For any $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x\in I})$, there exists coarse moduli space
$$\mathcal{U}_{C,\omega} := \mathcal{U}_C(r, d, \omega)$$
of s-equivalence classes of semistable parabolic bundles E on C of rank r and $\deg(E) = d$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$.
- $\mathcal{U}_{C,\omega}$ is a normal projective variety with only rational singularities.

Proof of the Theorem: Recall the construction

- For any data r, d, I and $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x\in I})$, let $\mathbf{Q}$ be the quotient scheme of locally free quotients with fixed determinant L,

$$\mathcal{R}_I = \underset{x\in I}{\times_{\mathbf{Q}}} Flag_{\vec{n}(x)}(\mathcal{F}_x) \to \mathbf{Q}$$

be the flag bundle (determined by $(r, d, I, \{\vec{n}(x)\}_{x\in I})$) and

$$\mathcal{R}_I \supset \mathcal{R}^{ss}_{I,\omega} \to \mathcal{U}^L_{C,\omega} = \mathcal{R}^{ss}_{I,\omega}//SL(V)$$

where $\mathcal{R}^{ss}_{I,\omega} \subset \mathcal{R}_I$ is determined by $(\{\vec{a}(x)\}_{x\in I}, k)$.

- Let $\mathcal{R}_I \supset \mathcal{R}^{ss}_{I,\omega} \xrightarrow{\psi} Y := \mathcal{U}^L_{C,\omega}$ be the GIT quotient.

Proof of the Theorem: Some known results

Proposition 5 (Sun, 2000)

For any $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x\in I})$, *we have*

(1) $\operatorname{codim}(\mathcal{R}^{ss}_{I,\omega} \setminus \mathcal{R}^{s}_{I,\omega}) \geq (r-1)(g-1) + \frac{|\mathrm{I}|}{k}$,

(2) $\operatorname{codim}(\mathcal{R}_{I} \setminus \mathcal{R}^{ss}_{I,\omega}) > (r-1)(g-1) + \frac{|\mathrm{I}|}{k}$.

Proposition 6 (Sun, 2000)

For any given $\mathcal{R}_I$ *(i.e.* $(r, d, I, \{\vec{n}(x)\}_{x\in I})$*), there exists* **canonical weight**

$$\omega_c = (2r, \{\vec{n}(x), \vec{a}_c(x)\}_{x\in I})$$

such that $\mathcal{U}^L_{C,\omega_c}$ *is* **a Fano variety with at most rational singularities** *if*

$$(r-1)(g-1) + \frac{|\mathrm{I}|}{2r} \geq 2.$$

Proof of the Theorem: Increase the number $|I|$

- Add extra parabolic points $x \in J \subset C$, the projection

$$\hat{f} : \mathcal{R}_{I \cup J} = \underset{x \in I \cup J}{\times_{\mathbf{Q}}} Flag_{\vec{n}(x)}(\mathcal{F}_x) \to \mathcal{R}_I = \underset{x \in I}{\times_{\mathbf{Q}}} Flag_{\vec{n}(x)}(\mathcal{F}_x)$$

  is $\mathrm{SL}(V)$-invariant. Choose $|J|$ such that

$$(r-1)(g-1) + \frac{|I \cup J|}{k+2r} \geq 2.$$

- Choose the canonical weight ω_c on $\mathcal{R}_{I \cup J}$, let

$$\mathcal{R}_{I \cup J} \supset \mathcal{R}^{ss}_{I \cup J,\,\omega_c} \xrightarrow{\varphi} Z := \mathcal{U}^L_{C,\,\omega_c}$$

  be the GIT quotient and $W = \mathcal{R}^s_{I \cup J,\,\omega_c} \subset \mathcal{R}^{ss}_{I \cup J,\,\omega_c}$, then

$$\mathrm{Codim}(\mathcal{R}_{I \cup J} \setminus W) \geq 2, \quad \hat{X} = \varphi^{-1}\varphi(\hat{X})$$

  where $\hat{X} = W \cap \hat{f}^{-1}(\mathcal{R}^{ss}_{I,\omega})$.

Proof of the Theorem: The final proof

- the $\mathrm{SL}(V)$-invariant morphism $\mathcal{R}_{I\cup J} \xrightarrow{\hat{f}} \mathcal{R}_I$ is a p-compatible morphism satisfying

$$\mathcal{O}_{\mathcal{R}_I} = \hat{f}_*\mathcal{O}_{\mathcal{R}_{I\cup J}}.$$

- the $\mathrm{SL}(V)$-invariant open set $W = \mathcal{R}^s_{I\cup J,\,\omega_c} \subset \mathcal{R}^{ss}_{I\cup J,\,\omega_c}$ satisfies

$$\mathrm{Codim}(\mathcal{R}_{I\cup J} \setminus W) \geq 2, \quad \hat{X} = \varphi^{-1}\varphi(\hat{X})$$

where $\hat{X} = W \cap \hat{f}^{-1}(\mathcal{R}^{ss}_{I,\,\omega})$.

- $Z = \mathcal{U}^L_{C,\,\omega_c}$ is of globally F-regular type. Thus

$$Y = \mathcal{U}^L_{C,\omega}$$

is of globally F-regular type by Proposition 4.

Comment and conjecture

Remark. The moduli spaces of semi-stable generalized parabolic sheaves on a smooth curve is also of globally F-regular type. ■

Theorem 4

Let C be a projective curve with at most one node and $\mathcal{U}_{C,\omega}$ be the moduli space of parabolic sheaves on C with any given data ω. Then

$$H^1(\mathcal{U}_{C,\omega}, \Theta_{\mathcal{U}_{C,\,\omega}}) = 0.$$

Conjecture 1

Let G be a semi-simple algebraic group over $\mathbb{C}$ and $\mathcal{U}_C^G$ be the semi-stable parabolic principal G-bundles on a smooth projective curve C. Then $\mathcal{U}_C^G$ is of globally F-regular type.

Thank You Very Much !