

# On $\mathbb{Q}$ -Gorenstein morphisms

Yongnam Lee (KAIST)

(joint work with Noboru Nakayama)

Based on the preprint: arXiv 1612.01690

$\mathbb{Q}$ -Gorenstein variety  $X/\mathbb{Q}$

is a normal variety with  
(or CM)

$rK_X$  is Cartier for some positive integer  $r$

It is an important object in

1) MMP

2) Moduli theory

We define  $\mathbb{Q}$ -Gorenstein scheme  $X$

in a more general form.

locally Noetherian scheme  
admitting a dualizing complex  
locally on  $X$

with Gor in codim 1

Let  $X^\circ = \text{Gor}(X)$

dualizing sheaf  $\omega_{X^\circ}$

$\omega_X^{[m]} := j_* (\omega_{X^\circ}^{\otimes m})$   $j: X^\circ \hookrightarrow X$

is invertible for some positive  
integer  $m=r$

We do not assume CM-condition  
because canonical index 1-cover  
of CM-algebraic variety is not  
necessary to be CM

Remark

$\omega_X^{[m]}$  satisfies  $S_2$   
and reflexive for any  $m \in \mathbb{Z}$

Let me discuss ' $\mathbb{Q}$ -Gor morphism'

Let  $Y \xrightarrow{f} T$  be a morphism of loc  
finite type between loc Noetherian  
schemes

$F$  coherent  $\mathcal{O}_Y$ -module

$\text{Fl}(Y/T) = \{y \in Y \mid F_y \text{ is a flat } \mathcal{O}_{T, f(y)}\text{-mod}\}$

$S_2(F_{\text{fl}}) = \{y \in Y \mid F_y \text{ satisfies } S_2 \text{ as } \mathcal{O}_{T, y}\text{-mod}\}$

$S_2(F/T) = \text{Fl}(F/T) \cap \bigcup_{t \in T} S_2(F_{t,1})$

Remark

i)  $\text{Fl}(Y/T)$  open

ii)  $F$  flat over  $T$   
 $\Rightarrow S_2(F/T)$  open

Let  $f: Y \rightarrow T$  be an  $S_2$ -morphism  
( $S_2(\mathcal{O}_{Y/T}) = Y$ )

Suppose  $Y_t$  is a  $\mathbb{Q}$ -Gor scheme  
for  $\forall t \in T$

$f$  is called a  $\mathbb{Q}$ -Gorenstein morphism

if  $\omega_{Y/T}^{[m]} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \omega_{Y_t}^{[m]}$   
for any  $m \in \mathbb{Z}$

[Kollár condition]

$\omega_{Y/T}^{[m]}$  satisfies relative  $S_2$  over  $T$   
i.e.  $S_2(\omega_{Y/T}^{[m]}/T) = Y$

motivation : Develop  $\mathbb{Q}$ -Gor deformation theory in a general form which is applicable to mixed char.

$\mathbb{Q}$ -Gorenstein deformation

deformation of  $\mathbb{Q}$ -Gor schemes with 'extra condition'

originally considered by Kollár, Shepherd-Barron

- Hacking, Tziolas consider  $\mathbb{Q}$ -Gor deformations of slc surfaces /  $\mathbb{C}$
- Abramovich-Hassett CM alg schemes over a fixed field
- Kollár, Patakfalvi some related works.

main application is to construct compact moduli space of algebraic varieties /  $\mathbb{C}$

Also there are some important applications to algebraic surfaces

Example)

[ —, Nakayama 2013]

We construct some interesting surfaces of general type in positive char.

by using [ —, Park 2007] and  $\mathbb{Q}$ -Gor deformation theory of singularities of class T

Example)

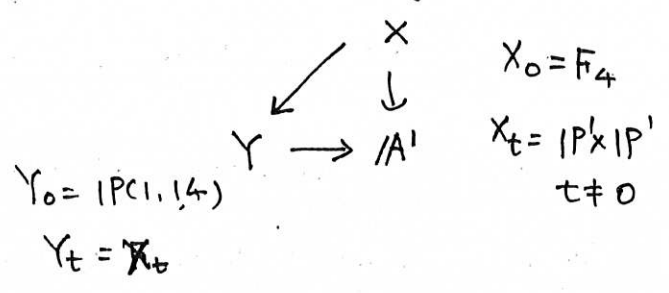
$$F_4 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(4)) \rightarrow \mathbb{P}^1$$

$$\downarrow \leftarrow \text{contract } (-4)\text{-curve}$$

$$Y_0 = \mathbb{P}(1, 1, 4)$$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}(2) \oplus \mathcal{O}(2) \rightarrow \mathcal{O}(4) \rightarrow 0$$

gives a flat family



$\omega_{Y/A^1}$  is not  $\mathbb{Q}$ -Cartier even though  $Y_t$  is a  $\mathbb{Q}$ -Gor variety for  $\forall t \in A^1$

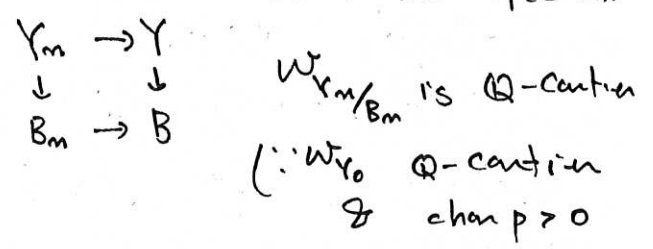
more delicate thing can occur in positive char.

[Kollár]

$$B = \text{Spec } R$$

$$R \text{ DVR with maximal ideal } \mathfrak{m}$$

$$R_{\mathfrak{m}} = R/\mathfrak{m}^n, B_{\mathfrak{m}} = \text{Spec } R_{\mathfrak{m}}$$



But  $W_{Y/B}$  is not  $\mathbb{Q}$ -Cartier so no infinitesimal criterion for  $W_{Y/B}$  to be  $\mathbb{Q}$ -Cartier relative Gor index goes to  $\infty$  when  $n \rightarrow \infty$

There is an infinitesimal criterion for  $W_{Y/B}$  to be  $\mathbb{Q}$ -Cartier if we assume bounded relative Gorenstein index

Introduce two weak forms  
 'naively  $\mathbb{Q}$ -Gor morphism'  
 'virtually  $\mathbb{Q}$ -Gor morphism'

virtually  $\mathbb{Q}$ -Gor morphism  
 $\Rightarrow$   $\mathbb{Q}$ -Gor morphism  
 [ —, Nakayama]

naively  $\mathbb{Q}$ -Gorenstein morphism

$Y \xrightarrow{f} T$  flat  
 $Y_t$   $\mathbb{Q}$ -Gor scheme for  $\forall t \in T$   
 $\omega_{Y/T}$  is  $\mathbb{Q}$ -Cartier

$\forall$   
 $\downarrow$  canonical index one cover  
 $X$   
 $\text{Def}_X^{\mathbb{Q}\text{-Gor}} \rightarrow \text{Def}_Y^{\mathcal{M}_r}$   
 $= \dots$   
 only virtually

Let  $Y \rightarrow T$  be a virtually  $\mathbb{Q}$ -Gor  $\mathbb{Q}$ -Gor morphism  
 Each  $Y_t$  satisfies  $S_3$   
 or  $\exists$  positive integer  $r$  (coprime to  $\text{char}(k)$ )  
 s.t.  $\omega_{Y/T}^{[r]}$  is invertible along  $Y_t$   
 $\Rightarrow Y \rightarrow T$  is a  $\mathbb{Q}$ -Gor morphism.

Virtually  $\mathbb{Q}$ -Gorenstein morphism

(related with canonical index one cover)

$Y \xrightarrow{f} T$   $y \in Y$   $0 = f(y)$   
 $f$  is virtually  $\mathbb{Q}$ -Gor at  $y$   
 if 1)  $f$  is flat at  $y$   
 ii)  $Y_0$  is  $\mathbb{Q}$ -Gor scheme at  $y$   
 iii)  $\exists$  open nbd  $U$  of  $y$  in  $Y$   
 and a reflexive  $\mathcal{O}_U$ -mod  $L$   
 s.t. i)  $L \otimes_{\mathcal{O}_U} \mathcal{O}_{U_0} \simeq \omega_{U_0/\mathbb{A}^1_0}$   
 ii)  $L^{[m]} := (L^{\otimes m})^{\vee}$  satisfies  
 relative  $S_2$  over  $T$  at  $y$

Criteria for  $\mathbb{Q}$ -Gor morphisms

- 1) infinitesimal criterion
- 2) valuative criterion
- 3)  $f: Y \rightarrow T$  is a  $\mathbb{Q}$ -Gor morphism along a fibra  $Y_t = f^{-1}(t)$  if
  - i)  $Y_t$  is  $\mathbb{Q}$ -Gor scheme
  - ii)  $Y_t$  is Gor in codim 2
  - iii)  $\omega_{Y_t}^{[m]}$  satisfies  $S_3$  for any  $m \in \mathbb{Z}$
 (canonical index 1-cover) satisfies  $S_3$

$\mathbb{Q}$ -Gor morphism  
 $\Rightarrow$  naively  $\mathbb{Q}$ -Gor morphism  
 virtually  $\mathbb{Q}$ -Gor morphism

naively  $\mathbb{Q}$ -Gor  $\not\Rightarrow$   $\mathbb{Q}$ -Gor morphism  
 $\exists$  example  
 [Patakfalvi]  $Y \rightarrow T$  naively  $\mathbb{Q}$ -Gor morphism in char 0  
 But  $\omega_{Y/T} \otimes_{\mathcal{O}_{Y_0}} \neq \omega_{Y_0}$

Cor  $X$   $\mathbb{Q}$ -Gor scheme/ $k$   
 $k$  is alg closed field in char 0  
 Let  $Y \rightarrow T$  be a  $k$ -deformation of  $X$ .

Assume  $Y$  has only i.t.  
 Then  $Y \rightarrow T$  is a  $\mathbb{Q}$ -Gor morphism

[ —, Nakayama]  
 For any  $m \in \mathbb{Z}_{\geq 2}$   
 $\exists$  example  $Y \rightarrow T$  naively  $\mathbb{Q}$ -Gor morphism in char 0  
 But  $\omega_{Y/T}^{[m]} \otimes_{\mathcal{O}_{Y_0}} \neq \omega_{Y_0}^{[m]}$