On $\mathbb{Q}$-Gorenstein morphisms

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Based on the preprint: arXiv 1612.01690

$\mathbb{Q}$-Gorenstein variety X/k

is a normal variety with (or CM)

rK_X is Cartier for some positive integer r

It is an important object in

1) MMP

2) Moduli theory

We define $\mathbb{Q}$-Gorenstein scheme X

in a more general form.

locally Noetherian scheme

admitting a dualizing complex

locally on X

with Gor in codim 1

Let $X^\circ = \mathrm{Gor}(X)$

dualizing sheaf ω_{X°

$\omega_X^{[m]} := j_*(\omega_{X^\circ}^{\otimes m})$ $j: X^\circ \hookrightarrow X$

is invertible for some positive integer $m = r$

(We do not assume CM-condition because canonical index 1-cover of CM-algebraic variety is not necessary to be CM

Remark

$\omega_X^{[m]}$ satisfies S_2

and reflexive for any $m \in \mathbb{Z}$

Let me discuss '$\mathbb{Q}$-Gor morphism'.

Let $Y \xrightarrow{f} T$ be a morphism of loc finite type between loc Noetherian schemes

F coherent $\mathcal{O}_Y$-module

$Fl(Y/T) = \{y \in Y \mid F_y \text{ is a flat } \mathcal{O}_{T,f(y)}\text{-mod}\}$,

$S_k(F_{(t)}) = \{y \in Y_t \mid F_y \text{ satisfies } S_2 \text{ as } \mathcal{O}_{Y_t,y}\text{-mod}\}$

$F_{(t)} = F \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}$

$S_k(F/T) := Fl(F/T) \cap \bigcup_{t \in T} S_k(F_{(t)})$

Remark

i) $Fl(Y/T)$ open

ii) F flat over T

$\Rightarrow S_k(F/T)$ open

Let $f: Y \to T$ be an S_2-morphism

$(S_2(\mathcal{O}_Y/T) = Y)$

Suppose Y_t is a $\mathbb{Q}$-Gor scheme

for $\forall t \in T$

f is called a $\mathbb{Q}$-Gorenstein morphism

if $\omega_{Y/T}^{[m]} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \omega_{Y_t}^{[m]}$

for any $m \in \mathbb{Z}$

[Kollár condition]

($= \omega_{Y/T}^{[m]}$ satisfies relative S_2 over T

i.e. $S_2(\omega_{Y/T}^{[m]}/T) = Y$

motivation : Develop $\mathbb{Q}$-Gor deformation theory in a general form which is applicable to mixed char.

$\mathbb{Q}$-Gorenstein deformation

deformation of $\mathbb{Q}$-Gor schemes with 'extra condition'

originally considered by Kollár, Shepherd-Barron

- Hacking, Tziolas
  consider $\mathbb{Q}$-Gor deformations of slc surfaces $/\mathbb{C}$
- Abramovich-Hassett
  CM alg schemes over a fixed field
- Kollár, Patakfalvi
  some related works.

main application is to construct compact moduli space of algebraic varieties $/\mathbb{C}$

Also there are some important applications to algebraic surfaces

Example)

[—, Nakayama 2013]

We construct some interesting surfaces of general type in positive char.

by using [—, Park 2007]

and $\mathbb{Q}$-Gor deformation theory of singularities of class T

Example)

$$F_4 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(4)) \to \mathbb{P}^1$$

$\downarrow \leftarrow$ contract (-4)-curve

$$Y_0 = \mathbb{P}(1,1,4)$$

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}(2) + \mathcal{O}(2) \to \mathcal{O}(4) \to 0$$

gives a flat family

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \\ Y \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

$X_0 = F_4$

$X_t = \mathbb{P}^1 \times \mathbb{P}^1 \quad t \neq 0$

$Y_0 = \mathbb{P}(1,1,4)$

$Y_t = \cancel{X}_t$

$\omega_{Y/\mathbb{A}^1}$ is not $\mathbb{Q}$-Cartier eventhough Y_t is a $\mathbb{Q}$-Gor variety for $\forall t \in \mathbb{A}^1$

more delicate thing can occur in positive char.

[Kollár] $B = \operatorname{Spec} R$

R DVR with maximal ideal m

$R_n = R/m^n$, $B_n = \operatorname{Spec} R_n$

$$\begin{tikzcd} Y_n \arrow[r] \arrow[d] & Y \arrow[d] \\ B_n \arrow[r] & B \end{tikzcd}$$

ω_{Y_n/B_n} is $\mathbb{Q}$-Cartier

($\because \omega_{Y_0}$ $\mathbb{Q}$-Cartier & char $p > 0$

But $\omega_{Y/B}$ is not $\mathbb{Q}$-Cartier

So no infinitesimal criterion for $\omega_{Y/B}$ to be $\mathbb{Q}$-Cartier

relative Gor index goes to ∞ when $n \to \infty$

(There is an infinitesimal criterion for $\omega_{Y/B}$ to be $\mathbb{Q}$-Cartier if we assume bounded relative Gorenstein index

Introduce two weak forms
'naively $\mathbb{Q}$-Gor morphism'
'virtually $\mathbb{Q}$-Gor morphism'

<u>naively $\mathbb{Q}$-Gorenstein morphism</u>

$Y \xrightarrow{f} T$ flat

Y_t $\mathbb{Q}$-Gor scheme for $\forall t \in T$

$\omega_{Y/T}$ is $\mathbb{Q}$-Cartier

<u>virtually $\mathbb{Q}$-Gorenstein morphism</u>

(related with canonical index one cover)

$Y \xrightarrow{f} T$ $y \in Y$ $0 = f(y)$

f is virtually $\mathbb{Q}$-Gor at y

iff i) f is flat at y

ii) Y_0 is $\mathbb{Q}$-Gor scheme at y

iii) $\exists$ open nbd U of y in Y and a reflexive $\mathcal{O}_U$-mod L

s.t. i) $L \otimes_{\mathcal{O}_U} \mathcal{O}_{U_0} \simeq \omega_{U_0/k(0)}$

ii) $L^{[m]} := (L^{\otimes m})^{\vee\vee}$ satisfies relative S_2 over T at y

$\mathbb{Q}$-Gor morphism

$\Rightarrow$ naively $\mathbb{Q}$-Gor morphism

virtually $\mathbb{Q}$-Gor morphism

naively $\mathbb{Q}$-Gor $\not\Rightarrow$ $\mathbb{Q}$-Gor morphism

[Patakfalvi] $\exists$ example $Y \to T$ naively $\mathbb{Q}$-Gor morphism in char 0

But $\omega_{Y/T} \otimes \mathcal{O}_{Y_0} \not\simeq \omega_{Y_0}$

[—, Nakayama]

For any $m \in \mathbb{Z}_{\geq 2}$

$\exists$ example in char 0 $Y \to T$ naively $\mathbb{Q}$-Gor morphism

But $\omega_{Y/T}^{[m]} \otimes \mathcal{O}_{Y_0} \not\simeq \omega_{Y_0}^{[m]}$

virtually $\mathbb{Q}$-Gor morphism

$\not\Rightarrow$ $\mathbb{Q}$-Gor morphism

[—, Nakayama]

$$\begin{array}{c} V \\ \downarrow \text{ canonical index one cover} \\ X \end{array} \qquad \mathrm{Def}_X^{\mathbb{Q}\mathrm{Gor}} \to \mathrm{Def}_V^{\mu_r}$$

$\cong$ ..

Let $Y \to T$ be a virtually $\mathbb{Q}$-Gor ~~$\mathbb{Q}$-Gor morphism~~ only virtually $\mathbb{Q}$-Gor morphism

Each Y_t satisfies S_3

or $\exists$ positive integer r (coprime to char $k(t)$)

s.t. $\omega_{Y/T}^{[r]}$ is invertible along Y_t

$\Rightarrow$ $Y \to T$ is a $\mathbb{Q}$-Gor morphism.

<u>Criteria for $\mathbb{Q}$-Gor morphisms</u>

1) Infinitesimal criterion

2) valuative criterion

3) $f : Y \to T$ is a $\mathbb{Q}$-Gor morphism along a fiber $Y_t = f^{-1}(t)$ iff

i) Y_t is $\mathbb{Q}$-Gor scheme

ii) Y_t is Gor in codim 2

iii) $\omega_{Y_t}^{[m]}$ satisfies S_3 for any $m \in \mathbb{Z}$

(canonical index 1-cover satisfies S_3)

<u>Cor</u> X $\mathbb{Q}$-Gor scheme/k
k is alg closed field in char 0

Let $Y \to T$ be a k-deformation of X.

Assume Y has only l.t

Then $Y \to T$ is a $\mathbb{Q}$-Gor morphism