Theory of Stochastic Differential Equations
- An Overview and Examples -

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0 Introduction.

We consider Itô's stochastic differential equation (SDE). First, we would review a *standard* theory under *standard* assumptions. Then we would see how such a standard theory should be modified under different and more general assumptions and how and what new notions need be introduced to discuss such modifications.

Consider the following SDE on $\mathbf{R}^d$

$$dX^i(t) = \sigma^i_k(X(t))dB^k(t) + b^i(X(t))dt, \quad X(0) = x \tag{1}$$

where $X(t) = (X^1(t), \cdots, X^d(t))$ is a d-dimensional continuous process and $x = (x^1, \cdots, x^d) \in \mathbf{R}^d$. $\sigma^i_k(x)$ and $b^i(x)$, $i = 1, \cdots, d$, $k = 1, \cdots, r$, are real Borel functions on $\mathbf{R}^d$. We allow the case $r = \infty$ and then we always assume that $||\sigma(x)|| := \sqrt{\sum_{i=1}^d \sum_{k=1}^r \sigma^i_k(x)^2} < \infty$ for each x. Set $|b(x)| = \sqrt{\sum_{i=1}^d b^i(x)^2}$. $B(t) = (B^k(t))$ is an r-dimensional Wiener process with $B(0) = 0$.

The assumption Itô posed in his first work on SDE ([I 2], 1942; [I-Sel.], p.63) is the following *Lipschitz condition* of coefficients:

Assumption 0.1 *There exist positive constants A and B such that*

$$||\sigma(x) - \sigma(y)|| \le A|x-y| \quad \text{and} \quad |b(x) - b(y)| \le B|x-y| \quad \text{for all} \quad x, y \in \mathbf{R}^d. \tag{2}$$

Itô established the existence and uniqueness of solutions $X^x = (X^x(t))$ for SDE (1) and, furthermore, the fact that the solution defines a diffusion process (i.e. a strong Markov process with continuous paths) such that $u(t,x) = E[f(X^x(t))]$ satisfies the Kolmogorov differential equation

$$\frac{\partial u}{\partial t}(t,x) = Lu(t,x) \quad \text{with the intial condition} \quad u(0+,x) = f(x) \tag{3}$$

where L is a second order semi-elliptic differential operator on $\mathbf{R}^d$ given by

$$L = \frac{1}{2}\sum_{i,j=1}^d a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b^i(x)\frac{\partial}{\partial x^i} \quad \text{with} \quad a^{ij}(x) = \sum_{k=1}^r \sigma^i_k(x)\sigma^j_k(x). \tag{4}$$

Now we would discuss various aspects of developments in Itô's theory. We would start by making precise the notion of solutions and their uniqueness

1 Solutions and uniqueness. Strong and non-strong(weak) solutions

Definition 1.1. *Let $x \in \mathbf{R}^d$ be fixed. A solution $X^x = (X^x(t))$ of SDE (1) is a continuous $\mathbf{R}^d$-valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$, which satisfies the following conditions:*

(i) X^x *is* $\mathbf{F}$*-adapted; i.e.,* $X^x(t)$ *is* $\mathcal{F}_t/\mathcal{B}(\mathbf{R}^d)$*-measurable for each* $t \in [0, \infty)$*.*

(ii)

$$\int_0^t \left[||\sigma(X^x(s))||^2 + |b(X^x(s))| \right] ds < \infty \quad \text{for each} \quad t \in [0, \infty) \quad a.s..$$

(iii) $B(t) = (B^k(t))$ *is an* r*-dimensional* $\mathbf{F}$*-Wiener process with* $B(0) = 0$*, i.e.,* $\mathbf{F}$*-adapted and the increment* $B(t) - B(s)$ *is independent of* $\mathcal{F}_s$ *for every* $s < t$*.*

(iv) The following holds for every t*, a.s.:*

$$(X^x(t))^i = x^i + \sum_{k=1}^r \int_0^t \sigma_k^i(X(s))dB^k(s) + \int_0^t b^i(X(s))ds, \quad i = 1, \cdots, d,$$

where the integral by $dB^k(s)$ *is understood in the sense of Itô's stochastic integral.*

In this definition, the initial value x may be replaced by any d-dimensional $\mathcal{F}_0$-measurable random variable.

Let $\mathbf{W}^d$ be the space of all continuous functions $w : [0, \infty) \ni t \mapsto w(t) \in \mathbf{R}^d$ with the topology of uniform convergence on finite intervals. *From now on, we assume that a solution* X^x *of SDE (1) exists for every* $x \in \mathbf{R}^d$*.*

Definition 1.2. *(i) We say that* the uniqueness in law *of the solution of (1) holds if the law of a solution* X^x *of (1) on* $\mathbf{W}^d$ *is uniquely determined by* x *for every* x*.*

(ii) We say that the pathwise uniqueness *of the solution of (1) holds if, for any fixed* $x \in \mathbf{R}^d$*,* X^x *and* $\tilde{X}^x$ *are two solutions of (1) on a same probability space with a same filtration* $\mathbf{F}$ *and with respect to a same* r*-dimensional* $\mathbf{F}$*-Wiener process* $(B(t))$*, then it holds* $X^x(t) = \tilde{X}^x(t)$ *for all* t*, a.s..*

Definition 1.3. *A solution* X^x *of (1) with the accompanying Wiener process* $B = (B^k(t))$ *is called* a strong solution *if it is adapted to the natural filtration* $\mathbf{F}^B$ *of* B*. Otherwise, it is called* a non-strong solution *or* a weak solution.

We say that SDE (1) has a unique strong solution *if there exists a function*

$$F : \mathbf{R}^d \times \mathbf{W}_0^r \ni (x, w) \mapsto F(x, w) \in \mathbf{W}^d,$$

where $\mathbf{W}_0^r = \{w \in \mathbf{W}^r | w(0) = 0\}$*, with the following properties:*

(i) F *is Borel-measurable and, for each* $t \geq 0$ *and* $x \in \mathbf{R}^d$*, the map* $\mathbf{W}_0^r \ni w \mapsto F(x, w)(t) \in \mathbf{R}^d$ *is* $\mathcal{B}_t(\mathbf{W}_0^r)/\mathcal{B}(\mathbf{R}^d)$*-measurable.*

(ii) *For any $x \in \mathbf{R}^d$ and an r-dimensional Wiener process $B = (B^k(t))$, $X^x = F(x, B)$ is a solution of (1),*

(iii) *Conversely, for any solution X^x with respect to B, it holds that $X^x = F(x, B)$, a.s..*

If a unique strong solution $F(x, w)$ exists, then $X^x(w) = F(x, w)$ itself is a strong solution of SDE (1) realized on the r-dimensional Wiener space $(\mathbf{W}_0^r, \mathcal{B}(\mathbf{W}_0^r), P^W)$ with the initial value x and with respect to the canonical Wiener process $W(t) = (w^k(t))$. It is obvious that the existence of a unique strong solution implies the pathwise uniqueness of solutions. Its converse is also true. Namely we have the following

Theorem 1.1. *([YW], [ZK], [Ka]; also standard texts, [IW], [KS], [RW], [RY])*
(i) *The pathwise uniqueness of solutions implies the uniqueness in law of solutions.*
(ii) *If the pathwise uniqueness for (1) holds and if a solution of (1) exists for any given initial value, then there exists a unique strong solution of (1).*

A standard sufficient condition for the pathwise uniqueness of solutions is the Lipschitz condition (2). In this case, Itô constructed a unique strong solution directly by using Picard's method of successive approximations ([I 2], 1942; [I-Sel.], pp. 63-70, also [I 3], 1951; [I-Sel.], pp.117-167) . G. Maruyama ([Mar], 1955) applied Cauchy's method of polygonal approximations, which is now well-known as *Euler-Maruyama's scheme* in numerical analysis of SDE's.

The Lipschitz condition has been improved by later works; particularly in the one-dimensional case of $d = 1$, sharp improvements have been obtained. We mention here the work of J. F. Le Gall ([L], 1981), who, following E. Perkins's idea, recovered Yamada-Watanabe's condition ([YW], 1971) in terms of the modulus of continuity of coefficients, and also, in the case $r = 1$, improved Nakao's condition ([N], 1972) of the (local) uniform positivity and of bounded variation property of the diffusion coefficients to the second-order bounded variation property. Le Gall's method is based on the stochastic calculus for continuous semimartingales under a non-smooth transformation like $x \to |x|$, of which main ingredients are Itô-Tanaka's formula and local times (cf. [RW], [RY]). We can see a sharpness of his result by a famous counter-example of the pathwise uniqueness given by M. Barlow ([B], 1982).

Example 1.1. (Tanaka's Equation) *This is a simplest example of SDE for which the uniqueness in law holds but the pathwise uniqueness does not hold. Consider the case $d = 1$ and $r = 1$ with the coefficients given by*

$$\sigma_1^1(x) = \mathbf{1}_{[0,\infty)}(x) + (-1)\mathbf{1}_{(-\infty,0)}(x) = \frac{x}{|x|}\mathbf{1}_{[x\neq 0]}(x) + \mathbf{1}_{[x=0]}(x), \quad b^1(x) = 0,$$

so that SDE is given by

$$dX(t) = \sigma_1^1(X(t))dB(t), \quad X(0) = x. \tag{5}$$

Since $|\sigma_1^1(x)| \equiv 1$, a solution $X^x(t) = x + \int_0^t \sigma_1^1(X^x(s))dB(s)$ is a Brownian motion starting at x and hence, the uniqueness in law holds. However, the Itô-Tanaka formula applied to $(X^x(t))$, we have

$$B(t) = \int_0^t \sigma_1^1(X^x(s))dX^x(s) = |X^x(t)| - |x| - \lim_{\epsilon \to 0}(2\epsilon)^{-1}\int_0^t \mathbf{1}_{[0,\epsilon)}(|X^x(s)|)ds$$

and hence, the natural filtration $\mathbf{F}^{|X^x|}$ of $\{|X^x(t)|\}$ satisfies that $\mathbf{F}^B \subset \mathbf{F}^{|X^x|}$ so that $\mathbf{F}^{X^x} \subset \mathbf{F}^B$ is impossible.

Remark 1.1. *If we consider the case $\sigma_1^1(x) = \mathbf{1}_{\{x \neq 0\}}(x) \cdot x/|x|$ so that SDE (5) is replaced by the following*

$$dX(t) = \mathbf{1}_{\{X(t)\neq 0\}}\frac{X(t)}{|X(t)|}dB(t), \quad X(0) = x,$$

then, even the uniqueness in law of solutions no longer holds. Indeed, if $x = 0$, for example, $X(t) \equiv 0$ is a solution and there exist many nonzero solutions: A solution $X = \{X(t)\}$ with $x = 0$, having the strong Markov property in the sense of Def.3.1 in Section 3, is parametrized uniquely (up to the equivalence in law) by a constant $m \in [0, \infty]$. If $m \in (0, \infty)$, the solution X_m corresponding to m is characterized by the property that the local time L_t^0 at 0 of X_m (cf. [RY], p.207) satisfies $L_t^0 = m^{-1}\int_0^t \mathbf{1}_{\{0\}}(X_m(s))ds$. X_m is constructed from a one-dim. Brownian motion $\beta(t)$ as $X_m(t) = \beta(A^{-1}(t))$ where $A(t) = t + ml_t^0$, l_t^0 being the local tome at 0 of $\beta(t)$. If $m = \infty$, then $X_m \equiv 0$. If $m = 0$, then X_m is the law unique solution of SDE (5) with $x = 0$, so that $\int_0^t \mathbf{1}_{\{0\}}(X_m(s))ds = 0$ in this case.

Example 1.2. (SDE for Brownian motion with sticky boundary) *Let θ be a given positive constant. Consider the case $d = 1$ and $r = 1$ with the discontinuous coefficients given by*

$$\sigma_1^1(x) = \mathbf{1}_{(0,\infty)}(x), \quad \text{and} \quad b^1(x) = \frac{\theta}{2}\mathbf{1}_{[x=0]}(x),$$

so that the SDE is given by

$$dX(t) = \mathbf{1}_{\{X(t)>0\}}dB(t) + \frac{\theta}{2}\mathbf{1}_{\{X(t)=0\}}dt, \quad X(0) = x. \tag{6}$$

If $x < 0$, then the solution $X^x(t) \equiv x$ so that this is essentially SDE on the half line $\mathbf{R}_+ = [0, \infty)$. We can show that a solution exists for every $x \geq 0$ uniquely in law: We can show more precisely that the joint law (X^x, B) on $\mathbf{W}^1 \times \mathbf{W}^1$ is unique. However, the pathwise uniqueness does not hold; it holds $0 \leq X^0(t) \leq r(t)$, a.s., where $r(t) = B(t) - \min_{0 \leq s \leq t} B(s)$ and

$$P(X^0(t) \leq y|\mathcal{F}_t^B) \cdot \mathbf{1}_{\{0 \leq y \leq r(t)\}} = \exp\{-\theta(r(t) - y)\} \cdot \mathbf{1}_{\{0 \leq y \leq r(t)\}}$$

([War.1], [Wat. 2]). Hence, X^0 cannot be a strong solution. If $x > 0$, then setting $\tau = \inf\{t > 0 | x + B(t) = 0\}$, $X^x(t) = x + B(t)$ for $t < \tau$. Also, $\{X^x(\tau + t); t \geq 0\}$

is independent of $\mathcal{F}_\tau$ and is identically distributed as X^0, so that X^x cannot be a strong solution as well.

When $\theta = \infty$, the equation (6) is to be understood by the Skorohod equation *on $\mathbf{R}_+ = [0, \infty)$:*

$$dX(t) = dB(t) + d\phi(t), \quad X(0) = x \in \mathbf{R}_+, \tag{7}$$

$\phi(t)$ being a continuous $\mathbf{F}$-adapted increasing process such that $\int_0^t \mathbf{1}_{\{0\}}(X(s))d\phi(s) = \phi(t)$. In this case, the pathwise uniqueness holds: If $x = 0$, then the solution $X^0(t)$ is given by $X^0(t) = r(t) := B(t) - \min_{0 \le s \le t} B(s)$. If $x > 0$, then $X^x(t) = x + B(t)$ when $t < \tau := \inf\{t > 0 | x + B(t) = 0\}$ and $X^x(t) = B(t) - \min_{\tau \le s \le t} B(s)$ when $t \ge \tau$. Thus, a unique strong solution can be given directly and explicitly.

Example 1.3. (SDE's generating Harris's stochastic flows) *Consider the case $d = 1$ and $1 \le r \le \infty$ and so, we are dealing with one-dimensional SDE's, although a similar equation may be defined in the higher dimensional case. Let $b(x)$ be a continuous real positive definite function on $\mathbf{R}$ such that $b(0) = 1$. Note that $b(x) = b(-x)$. Let $H(\subset \mathbf{C}_b(\mathbf{R} \to \mathbf{R}))$ be the (real) reproducing kernel Hilbert space associated with $b(x)$ so that, defining $f_x \in H$ by $f_x(y) = b(x - y)$, linear conbinations $\sum c_i f_{x_i}$ are dense in H and $(f_x, f_y)_H = b(x - y)$. Let $\dim H = r$ and choose an orthonormal basis (ONB) $\{e_k\}_{1 \le k \le r}$ in H. Set*

$$\sigma^1_k(x) = e_k(x), \quad k = 1, \cdots, r, \qquad b^1(x) = 0,$$

so that SDE is given by

$$dX(t) = \sum_{k=1}^{r} e_k(X(t))dB^k(t), \quad X(0) = x. \tag{8}$$

Note that we have $\sum_k e_k(x)e_k(y) = b(x - y)$. Hence

$$||\sigma(x) - \sigma(y)||^2 = \sum_k |e_k(x) - e_k(y)|^2 = 2(1 - b(x - y)).$$

Theorem 1.2. For SDE (8), the pathwise uniqueness of solutions holds if and only if

$$\int_{0+} \frac{1}{1 - b(x)} dx = \infty. \tag{9}$$

Here is an outline of the proof: the condition (9) is just Yamada-Watanabe's condition for the diffusion coefficients σ so that the 'if part' follows fromYamada-Watanabe's theorem. The 'only if part' follows from the following couplng argument: We assume that (9) fails; namely, we assume

$$\int_{0+} \frac{1}{1 - b(x)} dx < \infty. \tag{10}$$

*Let $0 \le \rho \le 1$. We mean, by a ρ-*coupling of the solutions of (8)*, a pair $(X^x, B), (\tilde{X}^x, \tilde{B})$ of solutions of (8) given on a same probability space with a same filtration $\mathbf{F}$ such that $\mathbf{F}$-Wiener processes are ρ-correlated; i.e., $E(dB^k(t)d\tilde{B}^l(t)) =$*

$\rho\delta_{k,l} \cdot dt$. A ρ-coupling always exists and, if $\rho < 1$, its joint law is unique. Furthermore, if we set $\xi_\rho(t) = X^x(t) - \tilde{X}^x(t)$, then the process $\xi_\rho = \{\xi_\rho(t)\}$ is a Feller diffusion on $\mathbf{R}$ with the canonical scale $s(x) = x$ and the speed measure $(1 - \rho b(x))^{-1}dx$ which starts at the origin (cf. [IM] for a general theory of one-dimensional Feller diffusions). Then ξ_ρ can be represented, by a one-dimensional Brownian motion $\beta(t)$ with $\beta(0) = 0$, as

$$\xi_\rho(t) = \beta(A^{-1}(t)), \quad A(t) = \frac{1}{2}\int_0^t \frac{1}{1 - \rho b(\beta(s))} ds,$$

($t \to A^{-1}(t)$ is the inverse function of $t \to A(t)$.) Thus, $E[\xi_\rho(t)^2] = E[A^{-1}(t)]$. Under the assumption (10), it holds that $\lim_{\rho\uparrow 1} E[A^{-1}(t)] > 0$ so that we have $\lim_{\rho\uparrow 1} E[\xi(t)^2] > 0$. On the other hand, if the pathwise uniqueness holds, this limit must vanish and hence, the pathwise uniqueness fails under (10). Cf. [WW 2004], for details.

2 Stochastic flows associated with SDE's

Under the standard assumption of Lipschitz condition (2), the unique strong solution $F(x, w)$ (cf. Def. 1.3) satisfies that $x \in \mathbf{R}^d \mapsto F(x, w) \in \mathbf{W}_0^r$ is continuous, a.a. w (P^W) (P^W being the r-dimensional Wiener measure), so that the map $X_t : x \in \mathbf{R}^d \mapsto X_t(x) := F(x, w)(t) \in \mathbf{R}^d$ is continuous for every t, a.a. w (P^W). Under a more stringent assumption on the smoothness of coefficients, this map can be shown to be invertible and X_t is homeomorphism or diffeomorphism of $\mathbf{R}^d$. Spaces of homeomorphisms or diffeomorphisms can be given a topology by a family of Sobolev norms and, denoting such a space as $\mathbf{G}(\mathbf{R}^d)$, we have a continuous map $t \in [0, \infty) \mapsto X_t \in \mathbf{G}(\mathbf{R}^d)$. This continuous process with values in $\mathbf{G}(\mathbf{R}^d)$ is called a *stochastic flow of homeomorphisms* or *diffeomorphisms* on $\mathbf{R}^d$ because it has the *flow property*: $X_{t+s} = X_t \circ X_s, t, s \in [0, \infty)$, a.s., (cf. [Ku] for details, also standard texts, e.g., [IW], [RW]).

If we relax the smoothness of coefficients, it may happen that a unique strong solution does not exist; it is even unclear if we can realize whole family of solutions with different initial values on a *same* probability space with a *same* filtration. In order to discuss stochastic flows generated by solutions of SDE's in general, it seems natural and useful to introduce the following notion and assumption.

Definition 2.1. *By a* system of coalescing solutions *of SDE (1), we mean a family $\{X^x; x \in \mathbf{R}^d\}$ of d-dimensional continuous processes defined on a probability space with a filtration $\mathbf{F}$ such that, for an r-dimensional $\mathbf{F}$-Wiener process $B = \{B(t)\}$ with $B(0) = 0$, (X^x, B) is a solution of (1) for each $x \in \mathbf{R}^d$ and, further more, for every $x, y \in \mathbf{R}^d$, $x \neq y$, X^x and X^y have the following* coalescing property*: if $X^x(t) = X^y(t)$ for some $t > 0$, then $X^x(s) = X^y(s)$ for all $s \geq t$, a.s..*

Assumption 2.1 *A system $\{X^x; x \in \mathbf{R}^d\}$ of coalescing solutions exists and its joint probability law is unique.*

Assumption 2.1 is obviously satisfied if a unique strong solution $F(x, w)$ exists: Indeed, setting up an r-dimensional $\mathbf{F}$-Wiener process B on a probability space with an filtration $\mathbf{F}$, we define $X^x = F(x, B)$, $x \in \mathbf{R}^d$. Then, $\{X^x; x \in \mathbf{R}^d\}$ is a system of coalescing solutions. Its law uniqueness is obvious as well.

Example 2.1. (Tanaka's Equation) *We consider the SDE (5) in Ex.1.1. Then Assumption 2.1 is satisfied. To see this, we first set up a solution (X^0, B) of (5) with the intial value 0 and define X^x, for $x \neq 0$, by setting*

$$X^x(t) = x + \frac{x}{|x|}B(t) \ \text{ for } \ t < \tau \quad \text{and} \quad X^x(t) = X^0(t) \ \text{ for } t \geq \tau$$

where $\tau = \inf\{t | x + (x/|x|)B(t) = 0\}$. It is easy to see that the family $\{X^x\}$ is a system of coalescing solutions of (5) which is unique in law.

Example 2.2. (SDE for Brownian motion with sticky boundary) *We consider the SDE (6) in Ex.1.2. Then Assumption 2.1 is satisfied. This can be shown in a similar way as in Ex.2.1*

Example 2.3. *We consider here the case of SDE (8) in Ex.1.3.; In this case, Assumption 2.1 holds if we assume further that the positive definite function $b(x)$ satisfies, either that it is $\mathcal{C}^2$ on $\mathbf{R} \setminus \{0\}$, or that, for any finite different points $x_1, \cdots, x_n$ in $\mathbf{R}$, the matrix $(b(x_i - x_j))$ is strictly positive definite. (We omit the details, cf. Harris [H].) Then we have the system $\{X^x\}$ of coalescing solutions of (8) and, for fixed t, the map $X_t : x \mapsto X^x(t)$ is increasing because of the coalescing property of solutions. Concerning this map, we have the following (cf. Harris [H], Matsumoto [M]):*

(i) If
$$\int_{0+} \frac{x}{1 - b(x)} dx = \infty,$$
then $X_t : x \in \mathbf{R} \mapsto X_t(x) := X^x(t)$ is a homeomorphism of $\mathbf{R}$ for every t, a.s.

(ii) If
$$\int_{0+} \frac{x}{1 - b(x)} dx < \infty,$$
then, for every $t > 0$, X_t is not invertible and, under a slight additional assumption on $b(x)$, $X_t(x)$ is an increasing step function of x for each $t > 0$, a.s.. (In fact, it holds that $E[\sharp\{X_t(x); x \in I\}] < \infty$ for every $t > 0$ and every bounded interval $I \subset \mathbf{R}$.)

(iii) If $b(x)$ is $\mathcal{C}^\infty$, then $x \mapsto X_t(x)$ is a diffeomorphism of $\mathbf{R}$ for every t, a.s..

We give here a general definition of stochastic flows on a general topological space S. Let $\mathbf{G}(S)$ be a space formed of mappings $\phi : S \to S$ which is closed under the composition, be given a topology under which it is a standard Borel space (in the sense of [P]; in the terminology of Bourbaki, it is a Lusin space) such that the evaluation map at $x \in S$; $\phi \in \mathbf{G}(S) \mapsto \phi(x) \in S$ and the composition ($(\phi \circ \psi)(x) := \phi[\psi(x)]$, $\phi, \psi \in \mathbf{G}(S)$) are all Borel operations. Furthermore, we assume that id(the identity map)$\in$ $\mathbf{G}(S)$ and satisfies the following continuity property:

$$\phi_n \to \mathrm{id} \ \text{ and } \ \psi_n \to \mathrm{id} \ \text{ imply that } \ \psi_n \circ \rho \circ \phi_n \to \rho \ \text{ for all } \ \rho \in \mathbf{G}(S).$$

Definition 2.2. *By a $\mathbf{G}(S)$-stochastic flow on S, we mean a family $\{X_{s,t}\}_{s,t\in\mathbf{R},s\le t}$ of $\mathbf{G}(S)$-valued random variables which possesses the following properties:*

(i) (flow property) If $s \le t \le u$, then $X_{s,u} = X_{t,u} \circ X_{s,t}$ and $X_{s,s} = \mathrm{id}$.

(ii) (independence property) For any $t_0 < t_1 < \cdots < t_n$, the random variables $\{X_{t_{i-1},t_i}\}$ are mutually independent.

(iii) (stationarity) For $s \le t$ and $h \in \mathbf{R}$, $X_{s,t} \overset{d}{=} X_{s+h,t+h}$.

(iv) (continuity) $X_{0,h} \to \mathrm{id}$ in prob. as $h \to 0$.

Now, assuming that Assumption 2.1 is satisfied, we would associate with SDE (1) a $\mathbf{G}(\mathbf{R}^d)$-stochastis flow on $\mathbf{R}^d$, for a suitable space $\mathbf{G}(\mathbf{R}^d)$ of mappings with the properties mentioned above. A difficulty in this problem is how to choose the space $\mathbf{G}(\mathbf{R}^d)$ properly. We would not be much involved in this problem and, instead, would assume that SDE(1) satisfies the following

Assumption 2.2 *Assuming that SDE satisfies Assumption 2.1, we assume further that there exists a space $\mathbf{G}(\mathbf{R}^d)$ of mapping on $\mathbf{R}^d$ with the properties mentioned above such that the following holds:*

$$\text{the map } \ X_t := [x \mapsto X^x(t)] \in \mathbf{G}(\mathbf{R}^d), \ \text{a.s.,} \ \ \text{for every } \ t \ge 0. \tag{11}$$

In one-dimensional case of $d = 1$, there is no problem concerning Assumption 2.2; we may take as $\mathbf{G}(\mathbf{R})$ the canonical space formed of all right continuous non-decreasing functions with the topology of convergence at all continuity points of the limit function. However, there are many cases in which we may choose, as $\mathbf{G}(\mathbf{R})$, a space much smaller than the canonical one.

Example 2.4. *(1) In the case of SDE (5) in Ex.1.1 and Ex.2.1, Assumption 2.2 is satisfied and (11) holds if we take as $\mathbf{G}(\mathbf{R})$ the following space of mapping on $\mathbf{R}$, which forms a finite dimensional semigroup under the composition:*

$$\mathbf{G}(\mathbf{R}) = \{\phi_{a,b,c}; a, b \in \mathbf{R}, c \in \{1, -1\}, \ \textit{such that } b \ge 0 \textit{ and } a; b \ge 0\}$$

where $\phi_{a,b,c} := [x \in \mathbf{R} \mapsto \phi_{a,b,c}(x) \in \mathbf{R}]$ is given by

$$\phi_{a,b,c}(x) = (x-a)\mathbf{1}_{(-\infty,-b)}(x) + c(a+b)\mathbf{1}_{[-b,b)}(x) + (x+a)\mathbf{1}_{[b,\infty)}(x).$$

Indeed, $X^x(t) = \phi_{a,b,c}(x)$ with

$$a = B(t), \ b = -\min_{0\le s\le t} B(s) \ \textit{ and } \ c = \mathbf{1}_{\{X^x(0)\ge 0\}} - \mathbf{1}_{\{X^x(0)<0\}}.$$

(2) *In the case of SDE (6) in Ex.1.2 and Ex.2.2, Assumption 2.2 is satisfied. Note that this is a case of SDE on* $\mathbf{R}_+ = [0, \infty)$ *and (11) holds if we take as* $\mathbf{G}(\mathbf{R}_+)$ *the following space of mappings on* $\mathbf{R}_+$*, which forms a finite dimensional semigroup under the composition:*

$\mathbf{G}(\mathbf{R}_+) = \{\psi_{a,b,c}; a, b, c \in \mathbf{R}$ *such that* $b \geq 0, 0 \leq c \leq a + b$ *and* $c = a$ *if* $b = 0\}$

where $\psi_{a,b,c} := [x \in \mathbf{R}_+ \mapsto \psi_{a,b,c}(x) \in \mathbf{R}_+]$ *is given by*

$$\psi_{a,b,c}(x) = c\mathbf{1}_{[0,b)}(x) + (x + a)\mathbf{1}_{[b,\infty)}(x), \quad x \in \mathbf{R}_+.$$

Indeed, $X^x(t) = \psi_{a,b,c}(x)$ *with*

$$a = B(t), \; b = -\min_{0 \leq s \leq t} B(s) \; \text{ and } \; c = X^x(0).$$

Now, we have the following theorem on the existence and uniqueness of stochastic flows associated with SDE (1).

Theorem 2.1. *If SDE satisfies Assumptions 2.1 and 2.2 with a space* $\mathbf{G}(\mathbf{R}^d)$ *of mappings on* $\mathbf{R}^d$*, there exists a* $\mathbf{G}(\mathbf{R}^d)$*-stochastic flow* $\{X_{s,t}\}_{s \leq t}$ *on* $\mathbf{R}^d$ *such that*

$$X_{s,t} \overset{d}{=} X_{(t-s)} := \left[x \ni \mathbf{R}^d \mapsto X^x(t - s) \in \mathbf{R}^d\right] \quad \text{for all} \;\; s \leq t.$$

Furthermore, the law of $\{X_{s,t}\}_{s \leq t}$ *is uniquely determined.*

This theorem covers SDE (5) in Ex.1.1, SDE (6) in Ex.1.2 and SDE (8) in Ex.1.3, so that we have stochastic flows in each cases; these are called *splitting flow*, *sticky flow* and a *Harris flow*, respectively.

If a unique strong solution $F(x, w)$ exists and there is a nice space $\mathbf{G}(\mathbf{R}^d)$ of mappings on $\mathbf{R}^d$ such that $[x \mapsto F(x, w)] \in \mathbf{G}(\mathbf{R}^d)$, then a $\mathbf{G}(\mathbf{R}^d)$-stochastic flow can be obtained directly by setting $X_{s,t} = [x \mapsto F(x, W(t) - W(s))]$, where $\{W(t); t \in \mathbf{R}\}$ is an r-dim. Wiener process. In the case of an invertible flow (such as flow of homeomorphisms or diffeomorphisms), we have $X_{s,t} = X_{0,t} \circ X_{0,s}^{-1}$ so that it is determined by $X_t := X_{0,t}$ and $X_t(x)$ is essentially a solution of SDE with the initial value x.

3 Examples of two-dimensional SDE's

First, we consider two-dimensional version of SDE's in Ex. 1.1 and Rem. 1.1. Its solution is a well-known *Walsh's Brownian motion*. We identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$ and denote its point $z \in \mathbf{C}$ as $z = re^{i\theta}$, $r = |z| \in [0, \infty)$, $\theta = \arg z \in [0, 2\pi)$.

Example 3.1. *Let*

$$\sigma(z) = \mathbf{1}_{\{z \neq 0\}}(z) \frac{z}{|z|}.$$

Let $B = \{B(t)\}$ *be a one-dimensional Wiener process with* $B(0) = 0$ *and consider the following SDE for continuous process* $Z = \{Z(t)\}$ *on* $\mathbf{C}$*:*

$$dZ(t) = \mathbf{1}_{\{Z(t) \neq 0\}} \cdot \frac{Z(t)}{|Z(t)|} dB(t), \quad Z(0) = z. \tag{12}$$

We consider the case $z = 0$ exclusively; if $z = re^{i\theta} \neq 0$, then setting $\tau = \inf\{t; r + B(t) = 0\}$, the solution $Z^z = \{Z^z(t)\}$ of (12) is given, for $0 \leq t \leq \tau$, by $Z^z(t) = e^{i\theta}(r + B(t))$, so that the whole solution Z^z is obtained by extending this to the time interval $[\tau, \infty)$ with an independent solution starting from 0 at time τ.

So we consider a solution $Z = \{Z(t)\}$ of (12) with $z = 0$. Obviously, the uniqueness of solutions does not hold; $Z(t) \equiv 0$ is a solution and also there are many non zero solutions. We are only interested in the solutions having the strong Markov property:

Definition 3.1. *A solution $Z = \{Z(t)\}$ of (12) with $z = 0$ is said to have the* strong Markov property *if, for any $\mathbf{F}$-stopping time ρ such that $P(\rho < \infty, Z(\rho) = 0) > 0$ and any $A \in \mathcal{F}_\rho$ such that $P(\{\rho < \infty, Z(\rho) = 0\} \cap A) > 0$, the process $\{Z(\rho + t); t \geq 0\}$, under the conditional probability $P(\, * \,|\{\rho < \infty, Z(\rho) = 0\} \cap A)$, has the same law as Z.*

Theorem 3.1. *A solutions $Z = \{Z(t)\}$ of (12) with $z = 0$ having the strong Markov property is parametrized uniquely (up to the equivalence in law) by a constant $m \in [0, \infty]$ and a Borel probability measure $\mu(d\theta)$ on $[0, 2\pi)$ such that $\int_{[0,2\pi)} e^{i\theta}\mu(d\theta) = 0$.*

Here we give a construction of the solution $Z := Z_{m,\mu}$ corresponding to a given pair (m, μ). A basic tool in this construction is a *Posson point process of Brownian excursions*, a notion introduced by Itô ([I 4], 1970); [It-Sel.], pp 543-557). Similar constructions have been discussed in the case of Brownian motions in [IW], in the case of diffusions in a domain with Wentzell's boundary conditions in [Wat 1] and [TW].

If $m = \infty$, then $Z_{m,\mu}(t) \equiv 0$. So we assume $m \in [0, \infty)$.

We introduce the following path space $\mathcal{W}^+$ which we call the space of excursions on $[0, \infty)$ away from 0;

$$\begin{aligned}\mathcal{W}^+ \;=\; & \{w \in \mathcal{C}([0,\infty) \to [0,\infty));\;\; \exists 0 < \sigma(w) < \infty \;\text{ such that} \\ & w(0) = 0, \;\; w(t) > 0 \text{ if } t \in (0, \sigma(w)), \text{ and } \; w(t) = 0 \text{ if } t \geq \sigma(w)\}.\end{aligned}$$

Let $\mathbf{n}^+$ be a σ-finite measure on $(\mathcal{W}^+, \mathcal{B}(\mathcal{W}^+))$, called the Itô excursion measure, which is a Markovian measure associated with the absorbing Brownian motion on $[0, \infty)$ with an entrance law at 0: (cf. [IW], p. 125; and a beautiful description by D. Williams using dim-3 Bessel processes, p. 144).

Let $\mathcal{W} = \mathcal{W}^+ \times [0, 2\pi)$ and define a σ-finite measure $\mathbf{n}$ on $\mathcal{W}$ by $\mathbf{n} = \mathbf{n}^+ \otimes \mu$.

Let $p = (p(t))$ be a stationary Poisson point process on $\mathcal{W}$ with the characteristic measure $\mathbf{n}$. So each sample p is a map

$$p : t \in D_p \subset (0, \infty) \mapsto p(t) \in \mathcal{W}$$

from a countable subset D_p of $(0, \infty)$ to $\mathcal{W}$. Note that the counting measure

$$N_p(dt, dw, d\theta) := \sharp\{ \; t \mid t \in dt, \; p(t) \in dw \times d\theta \; \}$$

is a Poisson random measure on $[0, \infty) \times \mathcal{W}$ with intensity $dt, \mathbf{n}(dw, d\theta)$.

We write $p(t) \in \mathcal{W} = \mathcal{W}^+ \times [0, 2\pi)$ as $p(t) = (p_1(t), p_2(t))$ so that $p_1(t) \in \mathcal{W}^+$ and $p_2(t) \in [0, 2\pi)$.

Being set up such a point process $p = (p_1(t), p_2(t))$, we deine a $\mathbf{C}$-valued continous process $\{Z(t)\}$ and $\mathbf{R}$-valued continous process $\{B(t)\}$as follows:

First, define an increasing process $t \mapsto A(t)$ by

$$A(t) = \sum_{s \in D_p, s \leq t} \sigma(p_1(s)) + mt.$$

Given $t \in [0, \infty)$, we can find unique $s := \phi(t)$ such that $A(s-) \leq t \leq A(s)$.

If $A(s-) = A(s)$, then we set $Z(t) = 0$ and $B(t) = 0$.

If $A(s-) < A(s)$, then this implies that $s \in D_p$ and we set

$$Z(t) = e^{ip_2(s)} \cdot [p_1(s)](t - A(s-)), \quad \text{and} \quad B(t) = [p_1(s)](t - A(s-)) - \phi(t).$$

Then, we can show as in [IW] that $\{B(t)\}$ is one-dim. Brownian motion with $B(0) = 0$ and $\{Z(t), B(t)\}$ is a solution of SDE (12) with $Z(0) = 0$ with respect to a suitably chosen filtration $\mathbf{F}$.

Example 3.2. *Let*

$$\sigma(z) = \mathbf{1}_{\{z \neq 0\}}(z) \frac{z}{|z|} \quad \text{and} \quad b(z) = \mathbf{1}_{\{z \neq 0\}}(z) \frac{z}{|z|^3},$$

so that SDE is given by

$$dZ(t) = \mathbf{1}_{\{Z(t) \neq 0\}} \cdot \frac{Z(t)}{|Z(t)|} dB(t) + \mathbf{1}_{\{Z(t) \neq 0\}} \cdot \frac{Z(t)}{|Z(t)|^3} dt, \quad Z(0) = z. \tag{13}$$

If $z = re^{i\theta} \neq 0$, then setting $\tau = \inf\{t; r + B(t) = 0\}$, the solution $Z^z = \{Z^z(t)\}$ of (13) is given, for $0 \leq t \leq \tau$, by $Z^z(t) = \exp\{i \int_0^t (r + B(s))^{-3}\}(r + B(t))$. So, as in the previous example, we consider the case of $z = 0$ only.

Definition 3.2. *Let* $Z = \{Z(t)\}$ *by a* $\mathbf{C}$*-valued continuous process with* $Z(0) = 0$. Z *is called* rotationally invariant *if* $\{e^{i\theta} Z(t); t \geq 0\} \stackrel{d}{=} \{Z(t); t \geq 0\}$ *for every* $\theta \in [0, 2\pi)$.

Theorem 3.2. *A solutions* $Z = \{Z(t)\}$ *of (13) with* $z = 0$*, having the strong Markov property and is rotationally invariant, is parametrized uniquely (up to the equivalence in law) by a constant* $m \in [0, \infty]$.

If $m = \infty$, then $Z(t) \equiv 0$. So we assume $m \in [0, \infty)$. We give its construction only: It can be given in a similar way as in the previous example.

We note that for almost all $w \in \mathcal{W}^+$ with respect to $\mathbf{n}^+(dw)$, there exists unique $m(w) \in (0, \sigma(w))$ such that $\max_{0 \leq s \leq \sigma(w)} w(s) = w(m(w))$.

Let $p = (p(t))$ be a stationary Poisson point process on $\mathcal{W} = \mathcal{W}^+ \times [0, 2\pi)$ with the characteristic measure $\mathbf{n} = \mathbf{n}^+ \otimes \mu_0$, μ_0 being the uniform measure on $[0, 2\pi)$: i.e., $\mu_0 = (2\pi)^{-1} d\theta$. We write $p(t) = (p_1(t), p_2(t))$ so that $p_1(t) \in \mathcal{W}^+$ and $p_2(t) \in [0, 2\pi)$.

We define the increasing process $A(t)$ in the same way as in the previous example and define a $\mathbf{C}$-valued continuous process $Z = (Z(t)$ with $Z(0) = 0$ in a similar way; only we we modify the definition of $Z(t)$ in the case $A(s-) \leq t \leq A(s)$, $A(s-) < A(s)$, as

$$Z(t) = \exp\left[i\left(p_2(s) + \int_{m(p_1(s))}^{t-A(s-)} \frac{1}{[p_1(s)](u)^3} du\right)\right] \cdot [p_1(s)](t - A(s-)).$$

(Note that $\sigma(p_1(s)) = A(s) - A(s-)$ when $s \in D_p$.

For both Examples 3.1 and 3.2, solutions satisfy Assumptions 2.1 and 2.2 and hence, by Th.2.1, we can have stochastic flows associated with them. On the other hand, a stochastic flow generates a *noise*, a notion introduced by Tsirelson ([T 1], [T 2], [T 3]) and a noise defines a *continuous product of Hilbert spaces*, equivalently, an E_0*-semigroup* or an *Arveson system*, a notion attracting much attention and being studied much in the fields of operator algebras and quantum dynamics. Noises associated with SDE's in Examples 3.1 and 3.2 are *nonclassical*, (equivalently, associated continuous products are *non-Fock*). Tsirelson ([T 4], 2004) made, from a viewpoint in Arveson systems, a deep study on a striking difference of noises associated with SDE's in Examples 3.1 and 3.2, respectively.

Classical noises are those familiar noises generated by Wiener processes (called Gaussian or white noises) and by stationary Poisson point processes (called Poisson noises). I believe that Itô always considered these classical noises as a most fundamental basis on which stochastic analysis can be developed; we can appreciate this idea in his very first work on Lévy-Itô theorem concerning the structure of paths of Lévy processes ([I 1], 1942) and in his many later works on SDE's and Poisson point processes formed of excursions of Markov process, and so on.

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