Infinite Dimensional Affine Processes.

Dedicated to Professor Ito

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Introduction

Affine processes contain the standard linear diffusion equations with constant volatility, like the Ornstein-Uhlenbeck process, but also some nonlinear diffusion equations, like the Feller process [4], used later in many applications, [4], [1]. Basically, an affine process has a characteristic function which is affine with respect to the initial condition. *We want to consider infinite dimensional affine processes*. We use a general framework to study linear infinite dimensional stochastic equations. We consider next a class of non-linear problems. We also show that the framework is adequate to studying infinite-dimensional affine processes.

Feller Process

The Feller process is defined by the equation

$$dy = k(\theta - y)dt + c\sqrt{y}dw$$

If we consider the Riccati equation

$$-q' + kq = \frac{1}{2}c^2q^2, \quad q(T) = \lambda$$

Provided this equation has a global solution on $0, T$ we have the formula

$$E \exp y(T)\lambda = E \exp y(0)q(0) \exp k\theta \int_0^T q(t)dt$$

Linear Random Functionals

Let Φ be a Hilbert space, Φ' its dual.
A 2nd order L.R.F. on Φ' is a family $\zeta_{\phi_*}(\omega)$ such that

$$\phi_* \longrightarrow \zeta_{\phi_*}(\omega) \in \mathcal{L}(\Phi'; L^2(\Omega, \mathcal{A}, P)).$$

We write

$$E\zeta_{\phi_*} =< m, \phi_* >$$

$$E\zeta_{\phi_*}\zeta_{\tilde{\phi}_*} - E\zeta_{\phi_*}E\zeta_{\tilde{\phi}_*} =< \Gamma\phi_*, \tilde{\phi}_* > .$$

m mathematical expectation, Γ covariance operator.

If Γ is nuclear

$$\zeta_{\phi_*} =< \phi_*, \zeta(\omega) >$$

$$\zeta(\omega) = \sum_i \phi_i \zeta_{J\phi_i}$$

where J is the isometry from Φ to Φ', and ϕ_i is an orthonormal basis of Φ; $J\phi_i$ is an orthonormal basis of Φ'. The random variable ζ satisfies

$$E\zeta^2 = \sum_i < J^*\Gamma J\phi_i, \phi_i >$$

Assume

$$\Phi = L^2(0,T;E)$$

where E is a Hilbert space, whose dual is denoted by E'. So

$$\Phi' = L^2(0,T;E')$$

A LRF on Φ' is denoted by $\xi_{e_*(.)}(\omega)$, where $e_*(.)$ is an element of Φ'. We shall be particularly interested in LRFs on Φ' with a covariance operator given by

$$< \Gamma e_*(.), \tilde{e}_*(.) > = \int_0^T < Q(t)e_*(t), \tilde{e}_*(t) > dt \quad (1)$$

where $Q(.) \in L^\infty(0,T;\mathcal{L}(E';E))$.

Gaussian LRF

A gaussian LRF is such that

$$\zeta_{\phi_*} \text{ is gaussian } \forall \phi_*$$

We shall in particular concentrate on the case

$$\Phi' = L^2(0,T;E')$$

with the covariance operator defined by 1, and 0 mean. We then consider a filtration $\mathcal{E}^t$ and assume that

$$\begin{aligned} &\xi_{e_* \mathbb{1}_{0,s}} \text{ is } \mathcal{E}^t \text{ measurable }, \forall e_* \in E', \forall s \leq t \\ &\xi_{e_* \mathbb{1}_{s,t}} \text{ is independant of } \mathcal{E}^s, \forall e_* \in E', \forall s \leq t \end{aligned} \tag{2}$$

When

$$\mathcal{E}^t = \sigma(\xi_{e_* 1\!\!\mathrm{I}_{0,s}}, \forall e_* \in E', \forall s \leq t)$$

then the second property 2 follows from the gaussian assumption and the non correlation of $\xi_{e_* 1\!\!\mathrm{I}_{s,t}}$ with the variables generating $\mathcal{E}^s$. The stochastic processes $\xi_{e_* 1\!\!\mathrm{I}_{0,t}}$, up to an equivalence, form a family of Wiener processes, indexed by e_*. Moreover

$$E\xi_{e_* 1\!\!\mathrm{I}_{0,t}}\xi_{\tilde{e}_* 1\!\!\mathrm{I}_{0,s}} =< \int_0^{\min(t,s)} Q(\tau) d\tau e_*, \tilde{e}_* >$$

Consider a basis $e_{*,i}$ of E', call

$$w_i(t) = \xi_{e_{*,i}} \mathbb{1}_{(0,t)}$$

Then, it is easy to check that

$$\xi_{e_*} \mathbb{1}_{0,t} = \sum_i w_i(t) < e_*, e_i >$$

the convergence taking place in $L^2(\Omega, \mathcal{A}, P)$,for any t.

Set

$$e_i = J^{-1} e_{*,i}$$

where J denotes the isomorphism from E to E'. Next define

$$w(t) = \sum_i w_i(t) e_i$$

This is a formal sum in E.

Note that

$$E||\sum_{i=1}^{N} w_i(t)e_i||^2 = \sum_{i=1}^{N} E(w_i(t))^2 =$$

$$= \sum_{i=1}^{N} \int_0^t < Q(\tau)e_{*,i}, e_{*,i} > d\tau$$

So the formal limit converges in $L^2(\Omega, \mathcal{A}, P; E)$, if one has the nuclear property

$$\sum_{i=1}^{\infty} \int_0^t < Q(\tau)e_{*,i}, e_{*,i} > d\tau < \infty$$

Identification

In the case when the assumptions (2) hold, we have constructed a family of Wiener processes $w_i(t)$. Now if $e_*(.) \in L^2(0,T;E')$ we have the identification

PROPOSITION 1.

$$\xi_{e_*(.)} = \sum_i \int_0^T < e_*(t), e_i > dw_i(t) \tag{3}$$

Generalizing Da Prato-Zabczyk [1], we call the formal sum

$$w(t) = \sum_i w_i(t) e_i \tag{4}$$

a cylindrical Wiener process, with covariance operator $Q(.)$. We define the generalized stochastic integral

$$\int_0^T < e_*(t), dw(t) >= \sum_i \int_0^T < e_*(t), e_i > dw_i(t) = \xi_{e_*(.)} \tag{5}$$

which is well defined as an element of $L^2(\Omega, \mathcal{A}, P)$.

Generalized stochastic integrals

Now the formal sum $w(t)$ converges in a bigger space than E. Indeed take any sequence $\alpha_i \geq 0$, with $\sum_i \alpha_i = 1$ and let us define

$$E_1 = \{e = \sum_i \lambda_i e_i | \sum_i \lambda_i^2 \alpha_i < \infty\} \qquad (6)$$

then we have the
PROPOSITION 2.

$$w(t) \in L^\infty(0,T;L^2(\Omega,\mathcal{A},P;E_1)) \qquad (7)$$

It follows that the stochastic integral (5) is well defined whenever $e_*(.)$ belongs to $L^2(0, T; E_1^{'})$. Note that

$$E_1^{'} \subset E', \text{ with continuous injection}$$

So the generalized stochastic integral is an extension of the ordinary stochastic integral, with integrands in $L^2(0, T; E')$. However, the choice of E_1 is arbitrary, and does not play any role in the definition of $\xi_{e_*(.)}$.

Moreover

$$E \int_0^T < e_*(t), dw(t) >= 0 \tag{8}$$

$$\begin{aligned} E(\int_0^T < e_*(t), dw(t) >)^2 = \\ E \int_0^T < Q(\tau)e_*(\tau;\omega), e_*(\tau;\omega) > d\tau. \end{aligned} \tag{9}$$

All properties of stochastic integrals extend

PROPOSITION 3.

$$E[\int_0^T < e_*(\tau) \mathbb{I}_{s,t}(\tau), dw(\tau) > |\mathcal{E}^s] = 0 \tag{10}$$

$$E\left[\left(\int_0^T < e_*(\tau) \mathbb{I}_{s,t}(\tau), dw(\tau) >\right)^2 |\mathcal{E}^s\right] = E\left[\int_s^t < Q(\tau) e_*(\tau;\omega), e_*(\tau;\omega) > d\tau |\mathcal{E}^s\right] \tag{11}$$

Martingale

Consider the stochastic process

$$I(t;\omega) = \int_0^T < e_*(\tau) 1\!\!\mathrm{I}_{0,t}(\tau), dw(\tau) >$$

PROPOSITION 4. *Up to an equivalence, $I(t)$ is a continuous process, and an $\mathcal{E}^t$ martingale.*

If τ_1, τ_2 are $\mathcal{E}^t$ stopping times, then

$$E[\int_0^T < e_*(t) 1\!\!I_{\tau_1,\tau_2}(t), dw(t) > |\mathcal{E}^{\tau_1}] = 0$$

$$E[\left(\int_0^T < e_*(t) 1\!\!I_{\tau_1,\tau_2}(t), dw(t) >\right)^2 |\mathcal{E}^{\tau_1}] =$$

$$E[\int_{\tau_1}^{\tau_2} < Q(t)e_*(t), e_*(t) > dt|\mathcal{E}^{\tau_1}]$$

Finally, the stochastic integral can be extended to adapted processes such that

$$\int_0^T ||e_*(t)||^2 dt < +\infty, \text{ a.s. } .$$

We shall note

$$I(t) = \int_0^t < e_*(\tau), dw(\tau) >$$

Ito's Formula

A scalar stochastic process $\beta(t)$, continuous, adapted to $\mathcal{E}^t$ has an Ito differential whenever

$$\beta(t) = \beta_0 + \int_0^t \alpha(\tau) d\tau + \int_0^t < e_*(\tau), dw(\tau) > . \tag{12}$$

$$\beta_0 \ \ \mathcal{E}^0 \text{ measurable}$$
$$E|\beta_0|^2 < +\infty. \tag{13}$$

Ito's Formula

$$\alpha(t) \text{ is adapted and } E\int_0^T |\alpha(t)|^2 dt < +\infty. \qquad (14)$$

$$e_*(t;\omega) \text{ is adapted with values in } E'$$

$$E\int_0^T |e_*(t)|^2 dt < +\infty \qquad (15)$$

THEOREM 1. *Assume (12) to (15).Let $\psi(x,t)$ be a $C^{2,1}$ function, then the process $\psi(\beta(t),t)$ has a Ito differential given by*

$$\psi(\beta(t),t) = \psi(\beta(0),0) + \int_0^t [\frac{\partial \psi(\beta(\tau),\tau)}{\partial \tau}$$

$$+\psi'(\beta(\tau),\tau)\alpha(\tau) + \frac{1}{2}\psi''(\beta(\tau),\tau) < Q(\tau)e_*(\tau), e_*(\tau) >]d\tau$$

$$+ \int_0^t \psi'(\beta(\tau),\tau) < e_*(\tau), dw(\tau) > \tag{16}$$

Extensions are natural for functions $\psi(x,t)$ with $x \in R^n$.

Linear Evolution Equations

Consider the natural framework for linear evolution equations introduced by J.L. LIONS [2]. We start with a triple of Hilbert spaces, with continuous embedding

$$V \subset H \subset V'.$$

Let us consider a family of linear operators

$$A(.) \in L^\infty(0,T;\mathcal{L}(V;V')) \qquad (17)$$

$$< A(t)v, v > \geq \alpha ||v||^2, \alpha > 0. \qquad (18)$$

Let E be another Hilbert space, and

$$B(.) \in L^\infty(0,T;\mathcal{L}(E;V')). \tag{19}$$

We consider two L.R.F. $\zeta_h(\omega)$ on H, and $\xi_{e_*(.)}(\omega)$ on $L^2(0,T;E')$, with covariance operators P_0 and

$$Q(.) \in L^\infty(0,T;\mathcal{L}(E';E))$$

We assume that

$$\xi_{e_*(.)}(\omega) = \int_0^T < e_*(t), dw(t) > \tag{20}$$

Recall that $w(t)$ is the cylindrical Wiener process

$$w(t) = \sum_i w_i(t) e_i$$

and has values in E_1. We want next to define the cylindrical process

$$\int_0^t B(s) dw(s)$$

For any $v(.) \in L^2(0,T;V)$, we define

$$\int_0^T < v(t), B(t)dw(t) >= \int_0^T < B^*(t)v(t), dw(t) >$$

where the right hand side is well defined since

$$B^*(.)v(.) \in L^2(0,T;E')$$

Also

$$\int_0^T < v(t), B(t)dw(t) >= \xi_{B^*(.)v(.)} =$$

$$= \sum_i \int_0^T < v(t), B(t)e_i > dw_i(t)$$

We have the estimate

$$E\left(\int_0^T < v(t), B(t)dw(t) >\right)^2 dt =$$

$$E\int_0^T < B(t)Q(t)B^*(t)v(t), v(t) > dt$$

In particular taking

$$v(.) = v \mathbb{1}_{(0,t)}, v \in V$$

we obtain

$$< v, \int_0^t B(s)dw(s) >= \int_0^T < v \mathbb{1}_{(0,t)}(\tau), B(\tau)dw(\tau) > \tag{21}$$

As usual, $\int_0^t B(s)dw(s)$ can be viewed as an element of

$$V_1 = \{\sum_i \lambda_i v_i | \sum_i \lambda_i^2 \alpha_i < \infty\}$$

where v_i is an orthonormal basis of V. We just write

$$\int_0^t B(s)dw(s) = \sum_i v_i < v_i, \int_0^t B(s)dw(s) >$$

We state the

THEOREM 2. *Assume (20),(17),(18),(19) . There exists a unique L.R.F.* $y_{\phi_*}(\omega)$ *on* $L^2(0,T;V')$ *and a unique family* $y_h(t;\omega)$ *of L.R.F. on* H*, such that*

$$y_h(t) \in L^\infty(0,T;\mathcal{L}(H;L^2(\Omega,\mathcal{A},P))$$

$$\int_0^T y_{\phi_*(t)}(t;\omega)dt = y_{\phi_*}(\omega) \tag{22}$$

with an extension of the left hand side equation from $L^2(0,T;H)$*to* $L^2(0,T;V')$. *Equation (23) holds.*

$$y_h(t) + \int_0^t y_{A^*(\tau)h}(\tau)d\tau = \zeta_h + < h, \int_0^t B(\tau)dw(\tau) >$$

$$\forall h \in V, \forall t, \text{ a.s.} \qquad (23)$$

Moreover, let p and q_t be defined by

$$-p' + A^*(t)p = \phi_*(t), \quad p(T) = 0 \tag{24}$$

$$-q_t' + A^*(\tau)q_t = 0, \quad q_t(t) = h \tag{25}$$

then one has the relations

$$y_h(t) = \zeta_{q_t(0)} + \int_0^t < q_t(\tau), B(\tau)dw(\tau) > \tag{26}$$

$$y_{\phi_*} = \zeta_{p(0)} + \int_0^T < p(\tau), B(\tau)dw(\tau) > \tag{27}$$

Correlation Operator

The correlation operator is defined by

$$(\Pi(t)h, h') = Ey_h(t)y_{h'}(t) \tag{28}$$

It verifies

$$\Pi(.) \in L^\infty(0,T;\mathcal{L}(H;H)), \Pi(t) \geq 0, \text{ self adjoint}$$

THEOREM 3. *and*

If $\theta \in L^2(0,T;V), \theta' \in L^2(0,T;V')$

$-\theta' + A^*\theta \in L^2(0,T;H)$

then $\Pi\theta \in L^2(0,T;V), (\Pi\theta)' \in L^2(0,T;V')$ (29)

$(\Pi\theta)' + \Pi(-\theta' + A^*\theta) + A\Pi\theta = BQB^*\theta$

$\Pi(0) = P_0.$

Nonlinear Evolution Equations

Consider

$$H_1 = \{h = \sum_i \lambda_i h_i | \sum_i (\lambda_i)^2 \alpha_i < \infty\}$$

then the solution of 23 satisfies

$$y(t) = \sum_i y_{h_i}(t) h_i \in L^\infty(0, T; L^2(\Omega, \mathcal{A}, P; H_1)) \tag{30}$$

We can treat some natural nonlinearities. Consider

$$g : H_1 \longrightarrow H, \; B : H_1 \longrightarrow \mathcal{L}(E;H)$$

such that

$$|g(h) - g(h')|_H \leq C|h - h'|_{H_1} \tag{31}$$

$$||B(h) - B(h')||_{\mathcal{L}(E;H)} \leq C|h - h'|_{H_1} \tag{32}$$

We consider the problem

$$y_h(t) + \int_0^t y_{A^*(\tau)h}(\tau)d\tau =$$

$$\zeta_h + \int_0^t (h, g(y(\tau)))d\tau + \int_0^t < h, B(y(\tau))dw(\tau) > \tag{33}$$

$$\forall h \in V, \forall t, \text{ a.s.}$$

where $y(t)$ is given by 30.

Statement of results

THEOREM 4. *We make the assumptions of 2 and 31,32. There exists a unique L.R.F.* $y_{\phi_*}(\omega)$ *on* $L^2(0,T;V')$ *and a unique family* $y_h(t;\omega)$ *of L.R.F. on* H*, such that*

$$y_h(t) \in L^\infty(0,T;\mathcal{L}(H;L^2(\Omega,\mathcal{A},P))$$

solution of 33.

Affine Processes

We take

$$g(h,t) = f(t) + G(t)h, \quad G(.) \in L^\infty(0,T;\mathcal{L}(H;H)) \tag{34}$$

$$B(h,t) \in L^\infty(0,T;\mathcal{L}(E;H))$$

$$B(h,t)Q(t)B^*(h,t) = K_0(t) + \sum_{i=1}^{\infty} K_i(t)(h,h_i)$$

$$K_0(.), K_i(.) \in L^\infty(0,T;\mathcal{L}(H;H)), \text{ self- adjoint} \tag{35}$$

We assume

$$\sum_i \sup_{0 \leq t \leq T} ||K_i(t)|| \leq C$$

We can see that if we take in the definition of H_1, $\alpha_i = \sup_{0 \leq t \leq T} ||K_i(t)||$ then $B(h, t)$ definition extends to H_1. But we do not have the Lipschitz property 32. Note that $B(h, t)$ may not be defined for all h.

Evolution Equation

We want to study the equation

$$y_h(t) + \int_0^t y_{A^*(\tau)h}(\tau)d\tau = (y_0, h) + \zeta_h +$$

$$\int_0^t (f(\tau), h)d\tau + \int_0^t y_{G^*(\tau)h}(\tau)d\tau + \int_0^t < h, B(y(\tau))dw(\tau) \tag{36}$$

$$\forall h \in V, \forall t, \text{ a.s.}$$

We cannot solve (36) in a strong sense.

We will again use the transposition method, but with different $q_{t,h}$, and define the solution of (36) in a weak sense. We define $q_{t,h}(\tau), \tau \leq t$ as follows

$$-q' + A^*(\tau)q = \frac{1}{2}\sum_i (q, K_i(\tau)q)h_i + G^*(\tau)q$$

$$q(t) = h \tag{37}$$

We also define $\alpha_{t,h}(\tau)$ by

$$\alpha_{t,h}(\tau) = \int_\tau^t [(f(s), q_{t,h}(s)) + \frac{1}{2}(q_{t,h}(s), K_0(s)q_{t,h}(s))] \tag{38}$$

Study of the Riccati Equation

To fix the ideas, we take $t = T$ in 37 and we write $q(.) = q_{T,h}(.)$. Consider (see 18)

$$< A(t)v, v > -(G(t)v, v) \geq \alpha ||v||^2 - \sup_{0 \leq t \leq T} ||G(t)|||v|^2$$

Therefore, we may assume that

$$- < A(t)v, v > +(G(t)v, v) \leq \beta |v|^2, \forall v \in H, \beta \in R \tag{39}$$

Define also

$$C = \sup_{0 \leq t \leq T} (\sum_i ||K_i(t)||^2)^{\frac{1}{2}} \tag{40}$$

We shall make the following assumption

$$\text{if } \beta > 0, \text{ then } \frac{\exp \beta T - 1}{\beta} < \frac{1}{|h|C}$$

$$\text{if } \beta \leq 0, \text{ then either } \beta > |h|C \text{ or } T < \frac{1}{|h|C} \tag{41}$$

This assumption is a condition of smallness of T or $|h|$. At any rate it follows that, whatever be the sign of β one has

$$\frac{\exp \beta(T-t) - 1}{\beta} < \frac{1}{C|h|} \tag{42}$$

THEOREM 5. *Assume that the injection of V into H is compact and we make the assumption 41. Then the Riccati equation*

$$-q' + A^*(t)q = \frac{1}{2}\sum_i (q, K_i(t)q)h_i + G^*(t)q \tag{43}$$

$$q(T) = h$$

has a solution q *in* $L^2(0,T;V) \cap C^0([0,T];H)$, $q' \in L^2(0,T;H)$. *Moreover*

$$|q(t)| \leq \exp \beta^+ T \frac{\beta|h|}{\beta - C|h|(\exp \beta T - 1)} \tag{44}$$

Weak solution of 36

THEOREM 6. *We make the assumption 41. We also assume that the assumption is valid with* $|h| = 1.$ *There exists a unique family* $y_h(t;\omega)$ *of L.R.F. on* H*, such that*

$$y_h(t) \in L^\infty(0,T;\mathcal{L}(H;L^2(\Omega,\mathcal{A},P))$$

solution of 36 in a weak sense. The process $y_h(t)$ *is given explicitly by the formula*

$$E\exp y_h(t) = E\exp\zeta_{q_{t,h}(0)}\exp[(y_0,h)+\alpha_{t,h}(0)]$$

$$= \exp[\frac{1}{2}(P_0 q_{t,h}(0),q_{t,h}(0)) + (y_0,h) + \alpha_{t,h}(0)] \quad (45)$$

Example

Assume

$$B(h) = \sqrt{(h, h_1}I$$

defined for h such that $(h, h_1) \geq 0$. Moreover

$$< A(t)h, \tilde{h} >= ((h, \tilde{h})), \forall h, \tilde{h}.$$

We write

$$dw(x,t) = \sum_k h_k(x) dw_k(t).$$

So we get the infinite set of equations

$$dy_i(t) + \lambda_i y_i(t)dt = \sqrt{y_1}dw_i(t)$$

The first equation defines y_1 as a positive process provided the initial condition is positive. For the other equations the stochastic integral is given.

Another Example of Affine Process

We take here

$$H = L^2(0,1), \quad V = H_0^1(0,1).$$

We will consider the stochastic partial differential equation

$$dy - y\text{''}\,dt = \sqrt{y}\sum_k h_k dw_k(t)$$

$$y(0,t) = y(1,t) = 0 \qquad (46)$$

$$y(x,0) = \zeta(x) \geq 0$$

where $y\text{''}$ is the second partial derivative with respect to x.

We can only give a weak meaning to the solution of (46). We check formally that the solution of

$$dy - y"\,dt = \sqrt{y+} \sum_k h_k dw_k(t)$$

is positive, provided that the initial condition is positive. Indeed, consider $\frac{1}{2}(y^-)^2$ then

$$\frac{1}{2} E \frac{d}{dt} \int (y^-)^2(x,t)dx = \int y"\,y^- dx$$

from which it follows easily that $(y^-)^2(x,t) = 0$.

Weak solution

The solution will be defined through the Riccati equation

$$
\begin{aligned}
&-\frac{\partial q}{\partial t} - q" = \frac{1}{2}q^2 \\
&q(0,t) = q(1,t) = 0 \\
&y(x,T) = h(x)
\end{aligned}
\tag{47}
$$

If we perform a formal computation of $d \exp \int y(x,t)q(x,t)dx$ we obtain

$$\frac{\partial}{\partial t} E \exp \int y(x,t)q(x,t)dx = 0$$

which implies

$$E \exp y_h(T) = E \exp \zeta_{q_h(0)} \tag{48}$$

with the usual notation.

Study of the Riccati equation

Testing the Riccati equation with q we obtain

$$-\frac{1}{2}\frac{\partial}{\partial t}|q(t)|_H^2 + \int (q')^2 dx = \frac{1}{2}\int q^3 dx.$$

Next we use

$$\int |q|^3 dx \leq 2||q||_{H_0^1}^2 + \frac{1}{8}|q(t)|_H^4$$

and we obtain easily the inequality

$$\frac{\partial}{\partial t}\frac{1}{|q(t)|^2} \leq \frac{1}{8}$$

Therefore we can assert the estimate

$$|q(t)|^2 \leq \frac{8|h|^2}{8 - T|h|^2}$$

provided that $T|h|^2 < 8$. Under this smallness condition the relation 48 is well defined.

Linear Filtering

We consider the linear evolution equation 23, and the LRF $y_h(t)$ represents the state of a dynamic system at time t. This state does not take values in H, but in H_1. However, coordinates on a basis of H are well defined.
We define the observation process also by a L.R.F. Let $\mathcal{C}(.) \in L^\infty(0, T; \mathcal{L}(V; F))$.

Consider a gaussian L.R.F. on $L^2(0,T;F')$, denoted $\eta_{f_*(.)}(\omega)$, independent from ζ_h and $\xi_{e_*(.)}$, with correlation operator $R(t)$

$$E\eta_{f_*^1(.)}\eta_{f_*^2(.)} = \int_0^T < R(t)f_*^1(t), f_*^2(t) > dt$$

$$R(.) \in L^\infty(0,T;\mathcal{L}(F';F)),\ R^{-1}(.) \in L^\infty(0,T;\mathcal{L}(F;F')) \quad (49)$$

In a way similar to $\xi_{e_*(.)}$ we can write

$$\int_0^T < f_*(t), db(t) >= \sum_i \int_0^T < f_*(t), e_i > db_i(t) = \eta_{f_*} \tag{50}$$

where $b_i(t)$ are Wiener processes defined by

$$b_i(t) = \eta_{f_{*,i}} 1\!\!1_{(0,t)}$$

where the $f_{*,i}$ form am orthonormal basis of F'.

The observation is the L.R.F. defined by

$$Z_{f_*(.)} = \int_0^T y_{\mathcal{C}^*(t)f_*(t)}(t)dt + \int_0^T < f_*(t), db(t) > . \tag{51}$$

Let

$$\mathcal{B} = \sigma(Z_{f_*(.)}, f_* \in L^2(0,T;F')).$$

It is the σ algebra of observations .
We define a L.R.F. on H by

$$\hat{y}_h(T) = E[y_h(T)|\mathcal{B}]. \tag{52}$$

Kalman Filter

THEOREM 7. *The conditional probability of*

$$y_{h_1}(T), \cdots, y_{h_m}(T)$$

given $\mathcal{B}$ *is a gaussian with conditional mean*

$$\hat{y}_{h_1}(T), \cdots, \hat{y}_{h_m}(T)$$

and conditional correlation

$$E[(y_{h_i}(T) - \hat{y}_{h_i}(T))(y_{h_j}(T) - \hat{y}_{h_j}(T))|\mathcal{B}] =$$

$$(P(T)h_i, h_j).$$

$P(t)$ *is deterministic.*

Assume

$$\mathcal{C}(.) \in L^\infty(0,T;\mathcal{L}(H;F)) \tag{53}$$

then $\hat{y}_h(t)$ is solution of the Kalman filter

$$\hat{y}_h(t) + \int_0^t \hat{y}_{(A^*+\mathcal{C}^*R^{-1}\mathcal{C}P)(\tau)h}(\tau)d\tau = \\ = Z_{R^{-1}\mathcal{C}P(.)h 1\!\!1_{(0,t)}} \forall h \in V, \forall t, \text{ a.s.} \tag{54}$$

The operator $P(t)$ is the solution of the Riccati equation

$$P(.) \in L^\infty(0,T;\mathcal{L}(H;H)), \geq 0, \text{ self adjoint}$$

If $\theta \in L^2(0,T;V), \theta' \in L^2(0,T;V')$

$$-\theta' + A^*\theta \in L^2(0,T;H)$$

then $P\theta \in L^2(0,T;V), (P\theta)' \in L^2(0,T;V')$

$$(P\theta)' + P(-\theta' + A^*\theta) + AP\theta + P\mathcal{C}^*R^{-1}\mathcal{C}P\theta = BQB^*\theta$$

$$P(0) = P_0 \tag{55}$$

Innovation

Consider the L.R.F. on $L^2(0,T;F')$

$$I_{f_*(.)}(\omega) = Z_{f_*(.)}(\omega) - \int_0^T \hat{y}_{\mathcal{C}^*(t) f_*(t)}(t) dt$$

It is called the innovation L.R.F. Then one has the following result

THEOREM 8. *$I_{f_*(.)}(\omega)$ is $\mathcal{B}$ measurable, gaussian, with*

$$EI_{f_*(.)} = 0$$

$$EI_{f^1_*(.)} I_{f^2_*(.)} = \int_0^T < R(t) f^1_*(t), f^2_*(t) > dt$$

the same as the noise on the observation.

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