

ON ITÔ'S ONE POINT EXTENSIONS OF MARKOV PROCESSES

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In [3], Kiyosi Itô considered a general Markov process X for which a point a in the state space S is regular for itself and recurrent. He associated a Poisson point process \mathbf{p} taking values in the space of excursion paths around a using the inverse local time at a and showed that X is determined by \mathbf{p} and the stopped process X^0 of X after hitting a . He also proved that the characteristic measure \mathbf{n} of \mathbf{p} is determined by the transition function $\{p_t^0, t \geq 0\}$ of X^0 and a $\{p_t^0\}$ -entrance law $\{\mu_t; t > 0\}$.

[3] was an outgrowth of Itô's joint work [1] with Henry McKean which described the most general extensions of the Brownian motion on a half line. Hiroshi Tanaka witnessed how a draft of [1] was energetically produced while he was staying in MIT. In a lecture [2] delivered in 1969, Itô determined and constructed most general Markovian extensions of X^0 with discontinuous entry from a , in which case $\mu_t = kp_t^0$ for a σ -finite measure k on $S \setminus \{a\}$.

In the same year, I constructed in [4] a conservative diffusion extension of the absorbing Brownian motion X^0 on a bounded domain $D \subset \mathbb{R}^d$ to its one point compactification $D \cup \{\infty\}$ using a regular Dirichlet form. This extension is simpler than the reflecting Brownian motion on \overline{D} and sometimes useful. Recently Tanaka asked me its relation to [3] so that we wrote a joint paper [5].

All possible Markovian extensions of X^0 need to be described in terms of quantities intrinsic to X^0 . The dependence of the entrance law on X^0 for continuous entry was not fully analyzed in [3]. In [5], [6], we restrict ourselves to the cases where X^0 is symmetric or has a weak dual with respect to a fixed excessive measure m and we decide all possible one-point extensions of X^0 that preserve symmetry or duality. A key observation is in that the entrance law is then uniquely determined by m together with the approaching probability to a of the dual process of X^0 . An application of Itô's extension to a time changed reflecting Brownian motion on the closure of a unbounded domain will also be discussed.

References

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