ON ITÔ'S ONE POINT EXTENSIONS OF MARKOV PROCESSES

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§1. Itô's point processes and related problems

[IM] K. Itô and H. P. McKean, Jr., Brownian motions on a half line,
Illinois J. Math. **7**(1963), 181-231

[I-1] K. Itô, Poisson point processes and their application to Markov processes,
Lecture note of Mathematics Department, Kyoto University (unpublished), September 1969

[I-2] K. Itô, Poisson point processes attached to Markov processes,
in: *Proc. Sixth Berkeley Symp. Math. Stat. Probab.* **III**, 1970, pp225-239

In [I-2], Kiyosi Itô showed the following.

$X = (X_t, \mathbf{P}_x)$: a standard Markov process on a state space E.
a: a point of E such that a is regular for itself;

$$\mathbf{P}_a(\sigma_a = 0) = 1, \quad \sigma_a = \inf\{t > 0 : X_t = a\}$$

$\{\ell_t,\ t \geq 0\}$: a local time at a, namely, a PCAF of X with support $\{a\}$.
Its right continous inverse is defined by

$$\tau_t = \inf\{s : \ell_s > t\}, \quad \inf \emptyset = \infty.$$

Define the space W of excursion paths around a by
$W = \{w : [0, \infty) \mapsto E,\ 0 < \sigma_a(w),\ w_t = a,\ \forall t \geq \sigma_a(w)\}$

Define a W-valued point process (*Itô's point process*) $\mathbf{p}$ by

$$\mathcal{D}(\mathbf{p}) = \{s : \tau_s > \tau_{s-}\}$$

$$\mathbf{p}_s(t) = \begin{cases} X_{\tau_{s-}+t} & t \in [0, \tau_s - \tau_{s-}) \\ a & t \geq \tau_s - \tau_{s-}, \end{cases} \qquad s \in \mathcal{D}(\mathbf{p}).$$

If $\ell_\infty < \infty$, then $\mathbf{p}_{\ell_\infty}$ is a non-returning excursion.

Assume additionally that a is a recurrent point;

$$\mathbf{P}_x(\sigma_a < \infty) = 1, \quad \forall x \in E,$$

then $\{\tau_t\}$ is a subordinator and
$\mathbf{p}$ is a W^+-valued Poisson point process under $\mathbf{P}_0$

Denote by $\mathbf{n}$ the characteristic measure of $\mathbf{p}$, $\mathbf{n}$ is a σ-finite measure on W. Let

$$\mu_t(B) = \mathbf{n}(w_t \in B,\, t < \sigma_a(w)) \quad t > 0, \quad B \in \mathcal{B}(E\backslash\{a\}).$$

Let X^0 be the process obtained from X by stopping at time σ_a and $\{p_t^0; t \geq 0\}$ be its transition function.

1. X is determined by X^0 and $\mathbf{n}$.
2. $\int_U (1 - e^{-\sigma_a})\, \mathbf{n}(dw) \leq 1$.
3. $\{\mu_t; t > 0\}$ is a $\{p_t^0\}$-entrance law: $\mu_t p_s^0 = \mu_{t+s}$
4. $\mathbf{n}$ is Markovian with semigroup p_t^0 and entrance law $\{\mu_t\}$:

$$\begin{aligned} &\int_W f_1(w(t_1)) f_2(w(t_2)) \cdots f_n(w(t_n))\mathbf{n}(dw) \\ = \;& \mu_{t_1} f_1 P^0_{t_2-t_1} f_2 \cdots P^0_{t_{n-1}-t_{n-2}} f_{n-1} P^0_{t_n-t_{n-1}} f_n. \end{aligned}$$

5. When $\mathbf{n}(w_0 = a) = 0$ (discontinuous entry), then $\mu_t = k p_t^0$ for some σ-finite measure k on $E \setminus \{a\}$ with

$$\int_{E\backslash\{a\}} \mathbf{E}_x \left[1 - e^{-\sigma_a}\right] k(dx) < \infty,$$

   and conversely any such measure gives rise to a jump-in extension of X^0.

Itô stated 5 explicitly in a unpublished lecture note [I-1] and, as an application, he determined all possible extensions of the absorbed diffusion process on a half line with exit but non-entrance boundary yelding a generalization of a part of his joint paper [IM] with McKean.

P.A. Meyer, Processus de Poisson ponctuels, d'apré K. Itô,
Séminaire de Probab. V, in: Lecture Notes in Math. Vol 191, Springer, Berlin, 1971, 177-190

removed the recurrence condition for a from [I-2],
and proved that Itô's point process $\mathbf{p}$ is equivalent to an *absorbed(stopped) Poisson point process* in the following sense:
Decompose the excursion space as $W = W^+ + W^- + \{\partial\}$.
There exists a σ-finite measure $\widetilde{\mathbf{n}}$ on W such that

- $\widetilde{\mathbf{n}}$ is Markovian with transition function $\{p_t^0\}$
- $\widetilde{\mathbf{n}}(W^- \cup \{\partial\}) < \infty$
- Let $\widetilde{\mathbf{p}}$ the W-valued Poissson point process with characteristic measure $\widetilde{\mathbf{n}}$ and $\widetilde{T}$ be the first occurance time of $W^- \cup \{\partial\}$. $\{\mathbf{p}_\cdot\}$ is then equivalent to the stopped point process $\{\widetilde{\mathbf{p}}_{\cdot \wedge \widetilde{T}}\}$.

X is said to be *of continuous entry* from a if $\widetilde{\mathbf{n}}(w(0) \neq a) = 0$.

Problem I (Uniqueness): when and how is the measure $\widetilde{\mathbf{n}}$ on W uniquely determined by the minimal process X^0 ?

Problem II (Construction): when and how does X^0 admit extentions X with continuous entry from a ?

L. C. G. Rogers, Itô excursion theory via resolvents, *Z. Wahrsch. Verw. Gebiete* **63** (1983), 237-255
T. S. Salisbury, On the Itô excursion process, *PTRF* **73** (1986), 319-350.
T. S. Salisbury, Construction of right processes from excursions, *PTRF* **73** (1986), 351-367
R. M. Blumenthal, *Excursions of Markov Processes*, Birkhäuser, 1992

These important articles have dealt with generalzations of [I-2] but the dependence and independece on X^0 of the involved quantities were not clearly separated.

[FT] M. Fukushima and H. Tanaka, Poisson point processes attached to symmetric diffusions, *Ann. Inst. Henri Poincaré Probab.Statist.* **41** (2005), 419-459
[CFY] Z.-Q. Chen, M. Fukushima and J. Ying, Extending Markov processes in weak duality by Poisson point processes of excursions.
in: *Stochastic Analysis and Applications* The Abel Symposium 2005 (Eds) F.E. Benth; G. Di Nunno; T. Lindstrom; B. Oksendal; T. Zhang, Springer, 2007, pp153-196.

give affirmative answers to **Problem I, II** under a symmetry or weak duality setting for X.

- If a pair of standard processes $X,\ \widehat{X}$ is in weak duality with respect to an excessive measure m, then $\widetilde{\mathbf{n}}$ is uniquely determined by $X^0,\ \widehat{X}^0$ and $m_0 = m\big|_{E\setminus\{a\}}$ up to non-negaitve parameters $\delta_0,\ \widehat{\delta}_0$ of the killing rate at a.
  In particular, if X^0 is m_0-symmetric, then its symmetric extension with no sojourn nor killing at a is unique.
- If $X^0,\ \widehat{X}^0$ are in weak duality with respect to m_0 and with no killing inside $E \setminus \{a\}$ and approachable to a, then they admit a pair of duality preserving extensions $X,\ \widehat{X}$ with continuous entry from a (by a time reversion argument).

§2. Uniqueness statements from [CFY]

§2.1. Description of $\widetilde{\mathrm{n}}$ by exit system

$\widetilde{\mathbf{n}}$ can be described in terms of the exit system due to

[Mai] B.Maisonneuve, Exit systems. *Ann. Probab.* **3** (1975), 399-411.

$X = (\Omega, X_t, \mathbf{P}_x)$: a right process on a Lusin space E.
A point $a \in E$ is assumed to be regular for itself.
ℓ_t : a local time of X at a
$\{p_t^0; t \geq 0\}$: the transition function of X^0 obtained from X by killing at σ_a

Ω: the space of all paths ω on $E_\Delta = E \cup \Delta$ which are cadlag up to the life time $\zeta(\omega)$ and stay at the cemetery Δ after ζ.
$X_t(\omega)$: t-th coordinate of ω.

The *shift operator* θ_t on Ω is defined by
$X_s(\theta_t \omega) = X_{s+t}(\omega), \ s \geq 0.$
The *killing operator* $k_t, \ t \geq 0$, on Ω defined by

$$X_s(k_t \omega) = \begin{cases} X_s(\omega) & \text{if} \quad s < t \\ \Delta & \text{if} \quad s \geq t. \end{cases}$$

The *excursion space* W is specified by

$$W = \{k_{\sigma_a} \omega : \omega \in \Omega, \sigma_a(\omega) > 0\},$$

which can be decomposed as

$$W = W^+ \cup W^- \cup \{\partial\}$$

with

$W^+ = \{w \in W : \sigma_a < \infty\}$, $W^- = \{w \in W : \sigma_a = \infty \text{ and } \zeta > 0\}$.

∂: the path identically equal to Δ.

k_{σ_a} is a measurable map from Ω to W.

Define the random time set $M(\omega)$ by

$$M(\omega) := \overline{\{t \in [0,\infty) : X_t(\omega) = a\}}.$$

The connected components of the open set $[0,\infty)\setminus M(\omega)$ are called the *excursion intervals.*

$G(\omega)$: the collection of positive left end points of excursion intervals

By [Mai], there exists a unique σ-finite measure $\mathbf{P}^*$ on Ω carried by $\{\sigma_a > 0\}$ and satisfying

$$\mathbf{E}^* \left[1 - e^{-\sigma_a}\right] < \infty$$

such that

$$\mathbf{E}_x \left[\sum_{s\in G} Z_s \cdot \Gamma \circ \theta_s\right] = \mathbf{E}^*(\Gamma)\cdot\mathbf{E}_x \left[\int_0^\infty Z_s d\ell_s\right] \qquad \text{for } x \in E,$$

for any non-negative predictable process Z and any non-negative random variable Γ on Ω.

($\mathbf{E}^*$: the expectation with respect to $\mathbf{P}^*$).

Let $Q^* = \mathbf{P}^* \circ k_{\sigma_a}^{-1}$. Q^* is a σ-finite measure on W and Markovian with semigroup $\{p_t^0; t \geq 0\}$.

Proposition 1 $Q^* = \widetilde{\mathbf{n}}$.

§2.2 Unique determination of $\widetilde{\mathbf{n}}$ by $\mathbf{X^0}$, $\widetilde{\mathbf{X}}^{\mathbf{0}}$

m : a σ-finite Borel measure on E with $m(\{a\}) = 0$.
$(u, v) = \int_E u(x)v(x)m(dx)$
$X = (X_t, \zeta, \mathbf{P}_x)$ and $\widehat{X} = (\widehat{X}_t, \widehat{\zeta}, \widehat{\mathbf{P}}_x)$:
a pair of Borel right processes on E that are in weak duality with respect to m; their resolvent G_α, $\widehat{G}_\alpha$ satisfy

$$(\widehat{G}_\alpha f, g) = (f, G_\alpha g), \quad \forall f, g \in \mathcal{B}^+(E),\ \forall \alpha > 0,$$

A point $a \in E$ is assumed to be regular for itself and non-m-polar with respect to X and $\widehat{X}$.

$$\varphi(x) = \mathbf{P}_x(\sigma_a < \infty), \quad u_\alpha(x) = \mathbf{E}_x\left[e^{-\alpha\sigma_a}\right], \quad x \in E.$$

The corresponding functions for $\widehat{X}$ is denoted by $\widehat{\varphi}$, $\widehat{u}_\alpha$
X^0, $\widehat{X}^0$: the killed processes of X, $\widehat{X}$ upon σ_a.
They are in weak duality with respect to m_0.
$\{p_t^0; t \geq 0\}$: the transition function of X^0

For an excessive measure η and an excessive function v of X^0, the *energy functional* is defined by

$$L^{(0)}(\eta, v) = \lim_{t \downarrow 0} \frac{1}{t}\langle \eta, v - p_t^0 v\rangle.$$

Let $\{\mu_t; t > 0\}$ be the $\{p_t^0\}$-entrance law associated with $\widetilde{\mathbf{n}}$:

$$\mu_t(B) = \widetilde{\mathbf{n}}(w_t \in B; t < \zeta(w)), \quad B \in \mathcal{B}(E \setminus \{a\})$$

Then

$$\begin{aligned}
&\int_W f_1(w(t_1)) f_2(w(t_2)) \cdots f_n(w(t_n))\mathbf{n}(dw) \\
= \ &\mu_{t_1} f_1 P^0_{t_2-t_1} f_2 \cdots P^0_{t_{n-1}-t_{n-2}} f_{n-1} P^0_{t_n-t_{n-1}} f_n.
\end{aligned}$$

We let $\delta_0 = \widetilde{\mathbf{n}}(\{\partial\})$

Theorem 2 (i) $\{\mu_t\}$ *satisfies*

$$\widehat{\varphi} \cdot m = \int_0^\infty \mu_t dt.$$

(ii) $\widetilde{\mathbf{n}}(W^-) = L^{(0)}(\widehat{\varphi} \cdot m, 1 - \varphi)$
(iii) *It holds that*

$$L^{(0)}(\widehat{\varphi} \cdot m, 1 - \varphi) + \delta_0 = L^{(0)}(\varphi \cdot m, 1 - \widehat{\varphi}) + \widehat{\delta}_0.$$

A general theorem due to Fitzsimmons (1987):
For a transient right process with transition function $\{q_t; t \geq 0\}$, any excessive measure η which is pure in the sense that $\eta q_t \to 0,\ t \to \infty$, can be represented by a unique $\{q_t\}$-entrance law $\{\nu_t; t > 0\}$ as

$$\eta = \int_0^\infty \nu_t dt.$$

Theorem 2 (i) means that the entrance law determining $\widetilde{\mathbf{n}}$ is uniquely decided by $\widehat{X}^0$ and m.

Theorem 2 means that the Itô point process $\mathbf{p}$ is uniquely determined by $X^0, \widehat{X}^0, m$ up to a pair of non-negative constants $\delta_0,\ \widehat{\delta}_0$ satisfying the above indentity.

Theorem 2 is a consequence of recent works by

P. J. Fitzsimmons and R. G. Getoor, Excursion theory revisited. *Illinois J. Math.* **50**(2006), 413-437

Z.-Q. Chen, M. Fukushima and J. Ying, Entrance law, exit system and Lévy system of time changed processes. *Illinois J. Math.* **50**(2006), 269-312

§3. One point extensions of Brownian motions on $\mathbb{R}^d$

Example 1

$D \subset \mathbb{R}^d$: bounded domain
$X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$: absorbing Brownian motion on D
The Dirichlet form of X^0 on $L^2(D)$ is the Sobolev space $(\frac{1}{2}\mathbf{D}, W_0^{1,2}(D))$, where $\mathbf{D}(u,u) = \int_D |\nabla u|^2(x)dx$. Let

$$\mathcal{F} = \{w = u + c : u \in W_0^{1,2}(D),\ c \text{ is constant}\}$$

$$\mathcal{E}(w,w) = \frac{1}{2}\mathbf{D}(u,u),$$

which is readily seen to be a regular Dirichlet form on $L^2(D^*; m)$
where $D^* = D \cup a$ is the one point compactification of D and
$m(dx) = 1_D(x)dx$.
The associated diffusion process X on D^* extends X^0.
a is regular for itself and recurrent with respect to X.
By Theorem 2 (i), the associated entrance law $\{\mu_t; t > 0\}$ equals

$$\mu_t(B)dt = \int_B \mathbf{P}_x^0(\zeta^0 \in dt)dx, \quad b \in \mathcal{B}(D).$$

Example 2(a current work with Zhen -Qing Chen)

$D \subset \mathbb{R}^d$, $d \geq 3$,: unbounded uniform domain. For instance, D can be an infinite cone or $\mathbb{R}^d$ itself.

Let $X = (X_t, \mathbf{P}_x)$ be the reflecting Brownian motion on $\overline{D}$. Then X is transient; it is conservative but, if a denotes the point at ifinity of $\overline{D}$, then

$$\lim_{t\to\infty} X_t = a \quad \mathbf{P}_x\text{−a.s.}$$

Let $m(dx) = m(x)dx$ be a finite measure with positive density $m \in L^1(\mathbb{R}^d)$.
Let $Y = (Y_t, \zeta, \mathbf{P}_x)$ be the time change of X by its PCAF $A_t = \int_0^t m(X_s)ds$.
Then $\mathbf{P}_x(\zeta < \infty) > 0$ and Y_t approaches to a as $t \to \zeta$.

Question How many symmetric conservative extensions does Y admit ?

Answer Only one, that can be realized as a one point extension of Y to $\overline{D} \cup a$ by Itô's ppp.

Define

$$\mathrm{BL}(D) = \{u \in L^2_{loc}(D) : \frac{\partial u}{\partial x_i} \in L^2(D),\ 1 \leq i \leq d\}$$

$$W_e^{1,2}(D) = \overline{\mathrm{BL}(D) \cap L^2(D)}^{\mathbf{D}}.$$

Then, $W_e^{1,2}(D)$ does not contain non-zero constants and

$$\mathrm{BL}(D) = \{u + c : u \in W_e^{1,2}(D),\ c \text{ is constant}\}$$

$(\frac{1}{2}\mathbf{D}, W_e^{1,2}(D) \cap L^2(D; m))$: Dirichlet form of Y on $L^2(D; m)$

$(\frac{1}{2}\mathbf{D}, \mathrm{BL}(D) \cap L^2(D; m))$: its maximal Dirichlet extension on $L^2(D; m)$