Malliavin Calculus and Computation Finance

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Computational Finance

Fimancial Model

Diffusion Process : Mainly used in Practice

Jump Diffusion Process

Stochastic Differential Equation

$(\Omega, \mathcal{F}, P)$ Probability Space

$B(t) = (B^1(t), \ldots, B^d(t))$, $t \geq 0$,

d-dimensinal Standard Wiener Process

State Space : N-dim Euclidean Space

SDE

$$dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt, \qquad t \in [0, T],$$

$$X(0) = x_0 \in \mathbf{R}^N$$

$\sigma : [0, T] \times \mathbf{R}^N \to \mathbf{R}^N \otimes \mathbf{R}^d$

$b : [0, T] \times \mathbf{R}^N \to \mathbf{R}^N$

Mainly continuous but necessary smooth

(e.g. Affine Model, Heston Model, SABAR Model)

One of Main Problems

Compute $E[f(X(T))]$ $\qquad f : \mathbf{R}^N \to \mathbf{R}$

Example of f

$$f(x) = (x^1 - K) \vee 0, \qquad f(x) = 1_{[a,\infty)}(x^1)$$

Usually not C^1

2nd Differential Operator L_t

$$L_t = \frac{1}{2}\sum_{i,j=1}^{N} a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{N} b^i(t,x)\frac{\partial}{\partial x^i}$$

$$a^{ij}(t,x) = \sum_{k=1}^{N} \sigma_k^i(t,x)\sigma_k^j(t,x), \qquad (t,x) \in [0,T] \times \mathbf{R}^N$$

PDE

$$\frac{\partial}{\partial t}u(t,x) + L_t u(t,x) = 0, \qquad (t,x) \in (0,T) \times \mathbf{R}^N$$

$$u(T,x) = f(x)$$

$$u(0,x_0) = E[f(X(T))]$$

One can also use Numerical Analysis for PDE

Problem: In the case that $N \geqq 5$,

Finite Difference Methods etc. are not efficient.

Bonds : Interest rate

Term structure (Short term, long Term) 2,3 Factors

Currency: Yen, Dollar, Euro

Exotic Derivatives:

The dimension of state space $\geqq$ number of Factors

We want to compute in the case N is big.

Main Tool: Euler-Maruyama Approximation

$$X_n(k/n) - X_n(k - 1/n)$$

$$= \sigma(X_n(k - 1/n))(B^k(k/n) - B^k(k - 1/n)) + \frac{1}{n}b(X_n(k - 1/n)),$$

$$X_n(0) = x_0 \in \mathbf{R}^N$$

Computation of $E[f(X_n([nT]/n))]$

Fineite dimensinal Integration: dimensions $= nd$

About integration

Mote Carlo Method : efficient for big dimensions,

slow : Error $\sim M^{-1/2}$

Quasi Monte Carlo

Method by using low discrepancy sequence:

Fast : Error $\sim M^{-1}$, dimension $\leqq 1000$

Error Estimate for Euler-Maruyama method

$|E[f(X(T))] - E[f(X_n(T))]|$ at least $O(n^{-1})$

Error $\leqq 10^{-5} \Rightarrow n \geqq 10^5$ (in priciple)

Needs to use Monte Carlo Method

Stratonovich type SDE

$$dX(t,x) = \sum_{i=0}^{d} V_i(X(t,x)) \circ dB^i(t) \tag{1}$$

$$X(0,x) = x \in \mathbf{R}^N$$

$B^0(t) = t$, $V_i \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$, $i = 0, 1, \ldots, d$,

Ito's formula

$$g(X(t,x)) = g(X(0,x)) + \sum_{i=0}^{d} \int_0^t (V_i g)(X(s,x)) \circ dB^i(s)$$

Stochastic Talylor Expansion

$$g(X(t,x))$$

$$= g(x) + \sum_{\ell=1}^{m} \sum_{i_1,\ldots,i_\ell=0}^{d} (V_{i_1} \cdots V_{i_\ell} g)(x)$$

$$\times \int_0^t \int_0^{s_1} \cdots \int_0^{s_{\ell-1}} \circ dB_{i_1}(s_1) \cdots \circ dB_{i_\ell}(s_\ell)) + R_m(t,x)$$

Extension of Euler-Maruyama methods

$$X_n^{(m)}(k/n,x)$$

$$= X_n^{(m)}(k-1/n,x)+\sum_{\ell=1}^{m}\sum_{i_1,\ldots,i_\ell=0}^{d}(V_{i_1}\cdots V_{i_\ell})(X_n^{(m)}(k-1/n,x))$$

$$\times\int_{k-1/n}^{k/n}\int_{k-1/n}^{s_1}\cdots\int_{k-1/n}^{s_{\ell-1}}\circ dB_{i_1}(s_1)\cdots\circ dB_{i_\ell}(s_\ell))$$

Problem.

Joint distribution of itterated stochastic integrals

is not known

K-Scheme

Lie Algebra + Malliavin Calculus

If one can construct a family of Markov linear operators

$Q_{(s)}$, $s > 0$, in $C_b(\mathbf{R}^N)$ of special type,

then $Q_{T/n}^n f(x) \to P_t f(x) = E[f(X(T,x))]$ very fast

ODE

$W \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$

$$\frac{d}{dt} y(t,x) = W(y(t,x)),$$

$$y(0,x) = x$$

$\exp(W)(x)$ is defined by $y(1,x)$

Examples of K-Scheme of order 5

(1) Ninomiya-Victoir

$$(Q_{(s)}f)(x)$$

$$= \frac{1}{2}E[f(\exp(\frac{s}{2}V_0)(\exp(B^1(s)V_1)(\cdots \exp(B^d(s)V_d)(\exp(\frac{s}{2}V_0)(x))\cdots))))]$$

$$+\frac{1}{2}E[f(\exp(\frac{s}{2}V_0)(\exp(B^d(s)V_d)(\cdots \exp(B^1(s)V_1)(\exp(\frac{s}{2}V_0)(x))\cdots))))]$$

(2) Ninomiya-Ninomiya

$$(Q_{(s)}f)(x)$$

$$= \mathrm{E}\ [f(\exp(\sqrt{s}\sum_{i=1}^{d} Z_{i,1}V_i)(\exp(sV_0 + \sqrt{s}\sum_{i=1}^{d} Z_{i,2}V_i)(x)))]$$

$Z_{i,j}$ $i = 1, \ldots, d$, $j = 1, 2$, Gaussian random variables

$$E[Z_{i,j}] = 0, \quad E[Z_{i,j}Z_{i',j'}] = \delta_{ii'}R_{j,j'} \qquad i, i' = 1, \ldots, d,\ j, j' = 1, 2$$

$$R_{1,1} = \frac{1}{2}, \qquad R_{2,2} = \frac{3}{2}, \qquad R_{1,2} = -\frac{1}{2}$$

$\exp(W)(x)$ can be replaced by Runge-Kutta formula of one step.

Theorem 1 Assume that (UFG) is satisfied. Then for any $T > 0$, there exists $C > 0$ such that for any Lipschitz continuous function $f : \mathbf{R}^N \to \mathbf{R}$ and $x_0 \in \mathbf{R}^N$, there exists $c(f, x_0) \in \mathbf{R}$ such that

$$|(Q^n_{(T/n)} f)(x_0) - E[f(X(T, x_0))] + \frac{c(f, x_0)}{n^2}| \leqq C n^{-3} ||\nabla f||_\infty$$

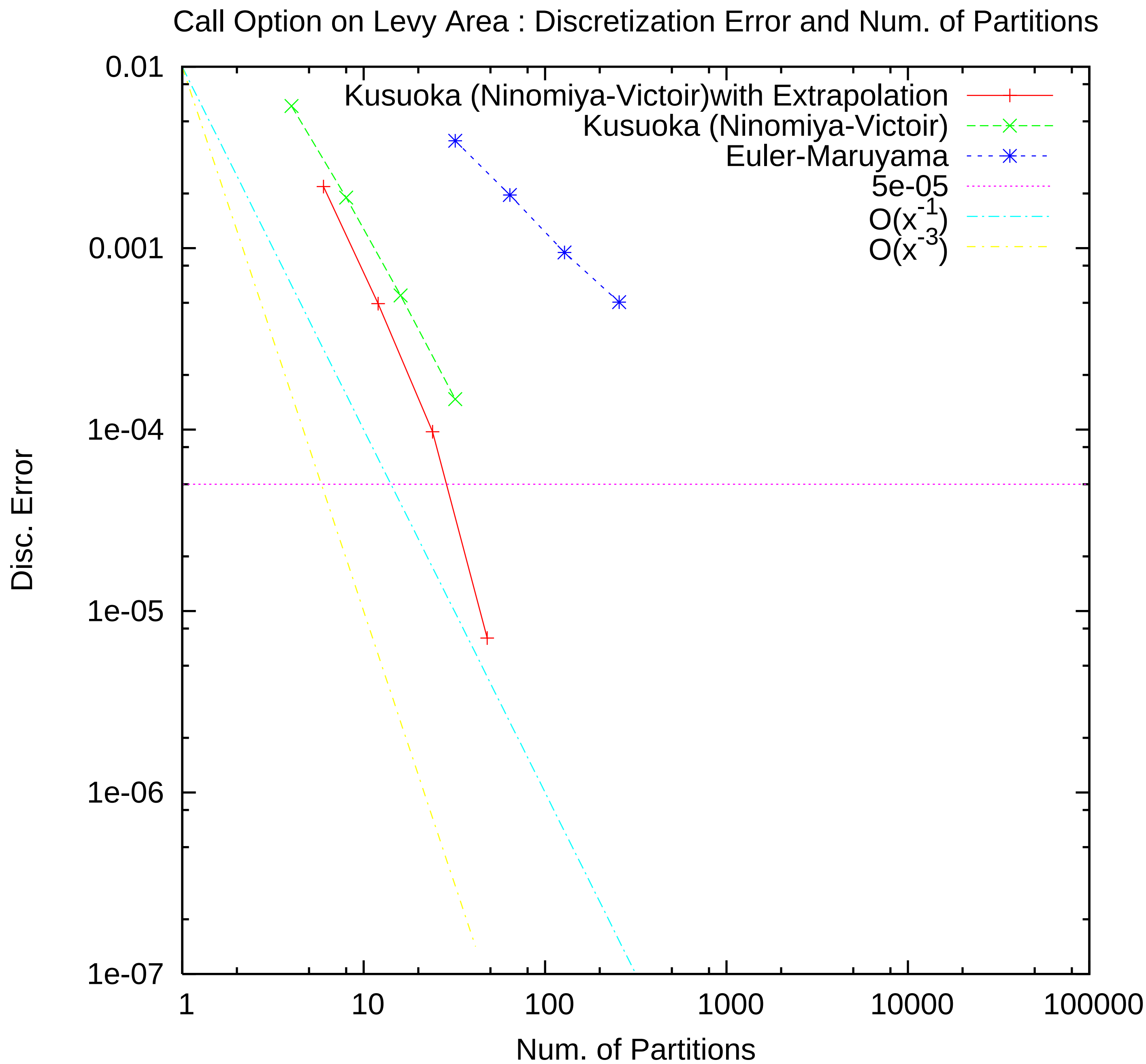
Romberg Extrapolation

Let $a(n) = (Q^n_{(T/n)} f)(x_0)$, $n \geqq 1$

$$\frac{4}{3} a(2n) - \frac{1}{3} a(n) = E[f(X(T, x_0))] + O(n^{-3})$$

Good Approximation

Another Example : Fujiwara



Call Option on Levy Area : Discretization Error and Num. of Partitions
Kusuoka (Ninomiya-Victoir)with Extrapolation
Kusuoka (Ninomiya-Victoir)
Euler-Maruyama
5e-05
O(x-1)
O(x-3)
0.01
0.001
1e-04
1e-05
1e-06
1e-07
1
10
100
1000
10000
100000
Disc. Error
Num. of Partitions

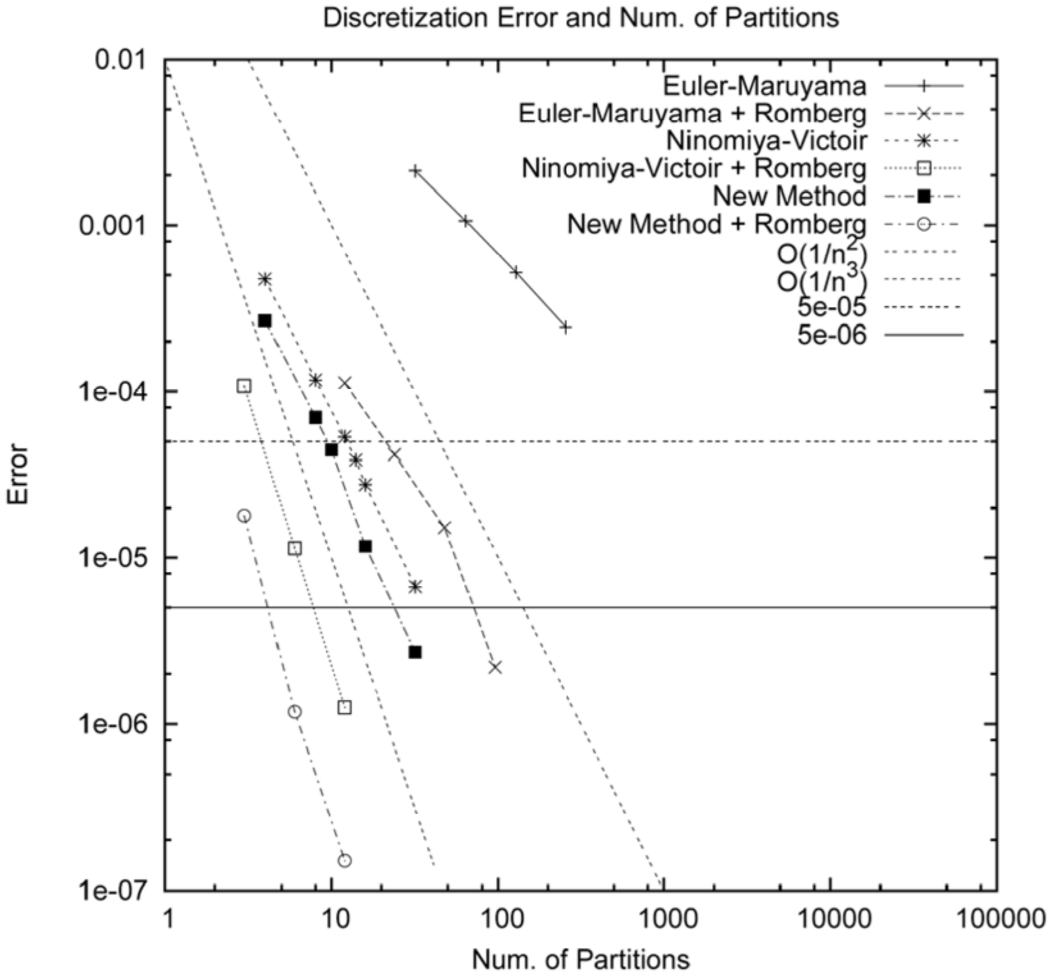


FIGURE 6.1. Error coming from the discretization

$\alpha = (\alpha_1, \dots, \alpha_n) \in \{0, 1, \dots, d\}^n$, $n \geqq 1$,

$$||\alpha|| = n + \#\{k = 1, \dots, n;\ \alpha_i = 0\}$$

$$V_\alpha = [V_{\alpha_1}, [V_{\alpha_2}, \cdots [V_{\alpha_{n-1}}, V_{\alpha_n}] \cdots]$$

$$A_{00} = \bigcup_{n=1}^{\infty} \{0, 1, \dots, d\}^n \setminus \{0\}$$

(UFG) $\exists$ a finite subset $L \subset A_{00}$

$\exists \phi_{\alpha,\beta} \in C_b^\infty$, $\alpha \in A_{00}$, $\beta \in L$

$$V_\alpha = \sum_{\beta \in L} \phi_{\alpha,\beta} V_\beta \qquad \alpha \in A_{00}$$

Theorem 2 Assume that (UFG) is satisfied. For any $m \geqq 1$, there is a $C > 0$ such that

$$||V_{\alpha^1} \cdots V_{\alpha^k} P_t V_{\alpha^{k+1}} \cdots V_{\alpha_{k+k'}} f||_\infty \leqq C t^{-(||\alpha^1||+\cdots||\alpha_{k+k'}||)/2} ||f||_\infty$$

for any $k, k' \geqq 0$ with $k + k' \leqq m$, $\alpha^1, \ldots, \alpha^{k+k'} \in A_{00}$ and $f \in C_b^\infty(\mathbf{R}^N)$.

Theorem 3 Assume that (UFG) is satisfied. Let $0 < T_0 < T_1$. For any $m \geqq 1$, there is a $C > 0$ satisiying the following.

If $n \geqq 1$, $s \in (0,1]$, with $T_0 < ns < T_1$, there are linear operators $Q_0^{(s,n)}$, $Q_1^{(s,n)}$ in $C_b^\infty(\mathbf{R}^N)$ such that

(1) $Q_{(s)}^n = Q_{n,s}^{(0)} + Q_{n,s}^{(1)}$.

(2)

$$||V_{\alpha^1} \cdots V_{\alpha^k} Q_{n,s}^{(0)} V_{\alpha^{k+1}} \cdots V_{\alpha_{k+k'}} f||_\infty \leqq C||f||_\infty$$

for any $k, k' \geqq 0$ with $k + k' \leqq m$, $\alpha^1, \ldots, \alpha^{k+k'} \in A_{00}$ and $f \in C_b^\infty(\mathbf{R}^N)$.

(3)

$$||Q_{n,s}^{(1)} f||_\infty \leqq Cs^m ||f||_\infty \qquad f \in C_b^\infty(\mathbf{R}^N).$$

In the proof, we use partial Malliavin calculus.

$$Q^n_{(T/n)}f - P_T f$$

$$= \sum_{k=0}^{n-1} Q^k_{(T/n)}(Q_{(T/n)} - P_{T/n})P_{(n-k-1)T/n}f$$

$$(Q_{(s)} - P_s)(f(\exp(-sV_0)(\cdot)))$$

$$= \sum_{\alpha^1,\ldots,\alpha^k \in A_{00}, m+1 \leqq ||\alpha^1|| + \cdots + ||\alpha^k|| \leqq 4m} s^{(||\alpha^1|| + \cdots + ||\alpha^k||)/2} V_{\alpha^1} \cdots V_{\alpha^k} H_{\alpha^1,\ldots,\alpha^k} f$$

$$= \sum_{\alpha^1,\ldots,\alpha^k \in A_{00}, m+1 \leqq ||\alpha^1|| + \cdots + ||\alpha^k|| \leqq 4m} s^{(||\alpha^1|| + \cdots + ||\alpha^k||)/2} \tilde{H}_{\alpha^1,\ldots,\alpha^k} V_{\alpha^1} \cdots V_{\alpha^k} f$$

Expectation with Dirichle boundary condition.

Stratonovich type SDE

$$dX^i(t,x) = \sum_{i=0}^{d} V_i(X(t,x)) \circ dB^i(t)$$

$$X(0,x) = x \in \mathbf{R}^N$$

$B^0(t) = t$, $V_i \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$, $i = 0, 1, \ldots, d$,

$$P_T f(x) = E[f(X(T,x)), \min_{t \in [0,T]} X^1(t,x) > 0]$$

(Assumption)

(1) $V_1^1(x) = 1$, $V_1^j(x) = 0$, $j = 2, \ldots, N$.

(2) $V_i^1(x) = 0$, $i \neq 1$.

$$dX^i(t,x) = \sum_{i=0}^{d} V_i(X(t,x)) \circ dB^i(t)$$

$$X(0,x) = x \in \mathbf{R}^N$$

$$X^1(t) = x^1 + B^1(t)$$

$$P_T f(x) = E[f(X(T,x)), \min_{t \in [0,T]} (x^1 + B^1(t)) > 0]$$

$$A_{000} = \bigcup_{n=1}^{\infty} \{0, 1, \ldots, d\}^n \setminus \{0, 1\}$$

Theorem 4 Assume that (UFG) is satisfied. For any $m \geqq 1$, there is a $C > 0$ such that

$$||V_{\alpha^1} \cdots V_{\alpha^k} P_t V_{\alpha^{k+1}} \cdots V_{\alpha_{k+k'}} f||_\infty \leq C t^{-(||\alpha^1|| + \cdots ||\alpha_{k+k'}||)/2} ||f||_\infty$$

for any $k, k' \geq 0$ with $k + k' \leqq m$, $\alpha^1, \ldots, \alpha^{k+k'} \in A_{000}$
and $f \in C_b^\infty(\mathbf{R}^N)$.

Malliavin's trick does not work directly

$$\min_{t\in[0,T]}(x^1+B^1(t)) \text{ does not belong to } W_p^1,\ 1<p<\infty$$

But we have

$$\min_{t\in[0,T]}(x^1+B^1(t)) \in W_{p(\varepsilon)}^{1-\varepsilon}$$

Real interpolation : fractional derivatives

In the next step, we need localization