Rough paths

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Calculus is the main mathematical tool used to describe systems with a local interaction. Examples of this abound throughout mathematics. Even the simplest case of a control problem without feedback has a huge importance. Differential equations of the form $dy^i = \sum_i f^{i,j} dx_i$ express the relationship between a controlling process x and a response y; x could be a path in a Lie algebra and y its development into a Lie group; x could represent the evolution of massive particles exerting a gravitational influence on the evolution of a satellite, whose state is represented by y.

However, an examination of many real world situations leads one to conclude that in many contexts, the controls that influence evolution are highly oscillatory and not at all adequately modelled on normal scales by classical tools of calculus. Itô showed us that it was possible to develop a theory of differential equations to allow the external influence to be modelled by a Brownian motion or a semi-martingale. This development was ground-breaking at the theoretical level and for its practical applications. The theory of rough paths considers the relationship between the control and the response, and identifies natural metrics making this functional uniformly continuous, and so well-defined on the completion. The completions of the smooth paths in these metrics are called rough paths; they are tractable and not particularly abstract objects (it is important to understand that they are a generalisation of the concept of a smooth path, rather than a restriction of the concept of a continuous path). They allow one to describe what is important about x on normal scales and to do numerical analysis on these scales, rather than tunnel deep into fine structure to use classical calculus. This deterministic extension of calculus is rich enough to capture the main Itô-Stratonovich examples. A key feature of the rough path approach is a natural and universal representation of the class of paths in $\bbR^n$ with concatenation into a subgroup of the free tensor algebra.