NUMERICAL MODELLING OF NONLINEAR WAVES IN SHALLOW WATER

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ABSTRACT

The project report is divided into two parts. In the first part, we reported the behaviour of solitons based on the paper of Zabusky & Kruskal(1965). The KdV equation is solved numerically using leap frog method. Numerical results presented include wave elevation profiles for $\delta = 0.000001$ *and* $\delta = 0.05$*. The results obtained showed that the number of solitons increases with increasing* t *and decreases with increasing value of* δ*. In the second part, we simulate the real case event Tsunami 2004 by using COMCOT(2007) model. Time series of the wave height in five cities in the coastal line of India Ocean have been drawn.*

Part 1: Soliton Modeling using Korteweg - De Vries Equation

Zabusky, N. J. & Kruskal, M. D. (1965). Interactions of “solitons” in a collisionless plasma and the recurrence of initial states. Physical Review Letters. Vol 15. no. 6 pg. 240 – 243.

1. INTRODUCTION

Nowadays, solitons are one of the exciting new technologies emerging in optical networking and they are poised to benefit the commercial ultra long haul all-optical multi-terabit networks spanning distances of up to many kilometers. It could well become one of the fundamental technologies in the current communication revolution. Solitons are localized nonlinear wave that have highly stable properties that allow them to propagate very long distance with a very little change [1].

Most expositions of soliton theory outline the history of the Korteweg - De Vries (KdV) equation, beginning with the physical observation of Scott Rusell of the boat wave in a canal in 1834. The equations were first written down by Boussinesq in 1871 and in 1895 by Korteweg De Vries. In addition to describing water waves, the KdV equation also arises as a universal limit of lattice vibrations as the spacing goes to zero. The surprising numerical experiment of Ferma, Pasta and Ulam in 1955 on an anharmonic lattice and ingenious explanation by Zabusky and Kruskal in 1965 in terms of solitons of the KdV equations were quickly followed by a ground-breaking paper of Gardner, Green, Kruskal and Miura in 1967, which introduced the method of solving KdV using the inverse-scattering transform for the Hill’s operator. This brings us to the modern era [2]. Now, in this project, we try to reproduce the work by Zabusky and Kruskal [3] in terms of different initial and boundary conditions as well as different values of δ.

2. PROBLEM FORMULATION

Following Zabusky & Kruskal (1965), the KdV equation is:

$$u_t + uu_x + \delta^2 u_{xxx} = 0 \qquad (1)$$

where u is the wave elevation(m), x is distance(m), t is time(s) and δ is an arbitrary constant. The terms uu_x and $\delta^2 u_{xxx}$ in Eq. (1) respectively is the nonlinear effects causing the steeping of the wave form and the dispersion effect causing waveform to spread.

In this project, to generate the initial and boundary condition, a general solitary wave equation

$$u(x,t) = H \,\mathrm{sech}^2\left[\sqrt{\frac{3H}{4h_0^3}}\left(x - ct\right)\right] \qquad (2)$$

is used, with the speed of wave, $c = \sqrt{g\left(H + h_0\right)}$ where, g is gravity acceleration(ms^{-2}), H is the wave amplitude(m) and h_0 is the water depth(m).

According to [3], the leapfrog scheme for equation (1) is:

$$u_i^{j+1} = u_i^{j-1} - \frac{1}{3}\left(\frac{k}{h}\right)\left(u_{i+1}^j + u_i^j + u_{i-1}^j\right)\left(u_{i+1}^j - u_{i-1}^j\right) - \left(\frac{\delta^2 k}{h^3}\right)\left(u_{i+2}^j - u_{i+1}^j + u_{i-1}^j - u_{i-2}^j\right)$$

for $i = 0, 1, ..., n-2$ and $j = 0, 1, ..., m$.

where $k = \Delta x$ and $h = \Delta t$ is the distance and time interval respectively.

By substituting $x = 0$ into Eq. (2) , we obtain the boundary condition as follows:

$$u(0, t) = H \operatorname{sech}^2\left[-\sqrt{\frac{3H}{4h_0^3}}ct\right] \tag{3}$$

and, by substituting $t = 0$ into Eq. (2) , we obtain the following initial condition:

$$u(x, 0) = H \operatorname{sech}^2\left[\sqrt{\frac{3H}{4h_0^3}}x\right] \tag{4}$$

3. RESULTS AND DISCUSSION

A Fortran program was written to solve the KdV equation (1) by means of the leap frog method subject to the boundary and initial condition (3) and (4) respectively. In our project, we have used the step sizes of $k = 0.1$ and $h = 0.01$, the wave amplitude, $H = 1$ and the water depth, $h_0 = 1$. In all cases we choose x_{max} = $20\,m$ and t_{max} = $31.92\,s$. Figures 1(a) - (d) give some graphs of the characteristics of elevation profiles as a function of x for $\delta = 0.000001$ at t = 1.02s, 4.05s, 11.05s and 31.02s respectively. From the figures we can see the breaking of single soliton to many solitons as the time increases. Figures 1(e) - (h) displays the elevation profiles for $\delta = 0.005$. From Figures 1(a) – (h), it is shown that the increasing value of δ will decrease the number of solitons.

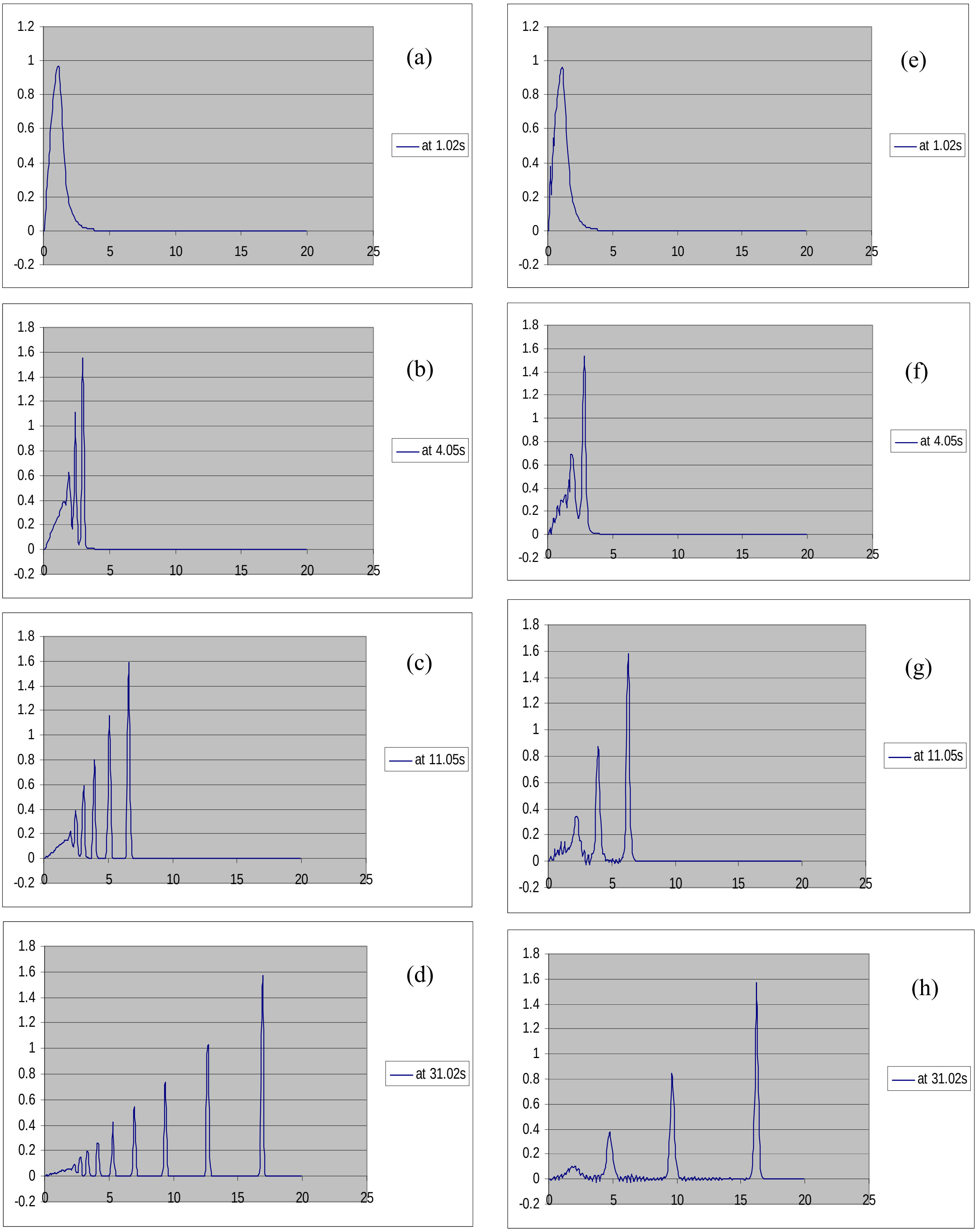

Figure 1: The development of the wave form.

4. CONCLUSIONS

In this study we can draw the following conclusions:
- The number of solitons increases with increasing time.
- The increasing value of δ will decrease the number of solitons.

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Part 2. Tsunami Simulation using Boussinesq Equations model (COMCOT, 2007)

1. INTRODUCTION

Tsunami induced by submarine earthquake and landslide became an object of great concern since the recent severe earthquakes. A severely one is the Indian Ocean tsunami induced by Sumatra earthquake on the 26th December 2004 caused disastrous damages to the countries along the coastal lines of the Indian ocean. To improve the anti-disaster abilities of the human beings, more knowledge on the onset and propagation of the earthquake induced tsunamis is needed.
For the tsunami hazard mitigation, it is very important to construct inundation maps along those coastlines vulnerable to tsunami flooding. These maps should be developed based on the historical tsunami events and hypothetical scenarios. To produce realistic and reliable inundation estimates, it is essential to use a numerical model that calculates accurately the tsunami propagation from a source region to the coastal areas of concern and the subsequent tsunami runup and inundation[1].

2. COMCOT

COMCOT (Cornell Multi-grid Coupled Tsunami model) is one of the models based on Shallow Water Equations. It is capable of simulating the entire lifespan of a tsunami, from its generation, propagation and runup/rundown in coastal regions. As the real tsunami waves propagate around, this two dimensional model has more advantages than the one dimensional model.
Wang (2008) extended COMCOT to weakly nonlinear weakly dispersive waves over a slowly varying bathymetry so that the numerical dispersion in the shallow water equation model can be used to recover the dispersion relationship of traditional Boussinesq equations[1].

3. FORMULATION [1]

For tsunamis in deep ocean, the amplitude is much smaller than the water depth and linear shallow water equations in Spherical Coordinates can be applied.

$$\frac{\partial \eta}{\partial t} + \frac{1}{R\cos\varphi}\left\{\frac{\partial P}{\partial \psi} + \frac{\partial}{\partial \varphi}(\cos\varphi Q)\right\} = -\frac{\partial h}{\partial t}$$

$$\frac{\partial P}{\partial t} + \frac{gh}{R\cos\varphi}\frac{\partial \eta}{\partial \psi} - fQ = 0$$

$$\frac{\partial Q}{\partial t} + \frac{gh}{R}\frac{\partial \eta}{\partial \varphi} + fP = 0$$

f represents the Coriolis force coefficient due to the rotation of the Earth and

$$f = \Omega \sin\varphi$$

When a simulation involving a relatively small region in which the Earth rotation effect is not prominent, shallow water equations in Cartesian Coordinates are preferred.

$$\frac{\partial \eta}{\partial t} + \left\{ \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right\} = -\frac{\partial h}{\partial t}$$
$$\frac{\partial P}{\partial t} + gh\frac{\partial \eta}{\partial x} - fQ = 0$$
$$\frac{\partial Q}{\partial t} + gh\frac{\partial \eta}{\partial y} + fP = 0$$

The nonlinear shallow water equations in Spherical Coordinates can be expressed

$$\frac{\partial \eta}{\partial t} + \frac{1}{R\cos\varphi} \left\{ \frac{\partial P}{\partial \psi} + \frac{\partial}{\partial \varphi}(\cos\varphi Q) \right\} = -\frac{\partial h}{\partial t}$$
$$\frac{\partial P}{\partial t} + \frac{1}{R\cos\varphi}\frac{\partial}{\partial \psi}\left\{ \frac{P^2}{H} \right\} + \frac{1}{R}\frac{\partial}{\partial \varphi}\left\{ \frac{PQ}{H} \right\} + \frac{gH}{R\cos\varphi}\frac{\partial \eta}{\partial \psi} - fQ + F_x = 0$$
$$\frac{\partial Q}{\partial t} + \frac{1}{R\cos\varphi}\frac{\partial}{\partial \psi}\left\{ \frac{PQ}{H} \right\} + \frac{1}{R}\frac{\partial}{\partial \varphi}\left\{ \frac{Q^2}{H} \right\} + \frac{gH}{R}\frac{\partial \eta}{\partial \varphi} + fP + F_y = 0$$

The nonlinear shallow water equations in Cartesian Coordinates can be expressed

$$\frac{\partial \eta}{\partial t} + \left\{ \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right\} = -\frac{\partial h}{\partial t}$$
$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left\{ \frac{P^2}{H} \right\} + \frac{\partial}{\partial y}\left\{ \frac{PQ}{H} \right\} + gH\frac{\partial \eta}{\partial x} + F_x = 0$$
$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left\{ \frac{PQ}{H} \right\} + \frac{\partial}{\partial y}\left\{ \frac{Q^2}{H} \right\} + gH\frac{\partial \eta}{\partial y} + F_y = 0$$

The mechanism of the earthquake is shown in figure 1. *h* is focal depth which means the distance between the epicenter and the sea floor. There are three important angles indicate the feature of an earthquake. They are strike angle θ, dip angle δ and slip angle λ.

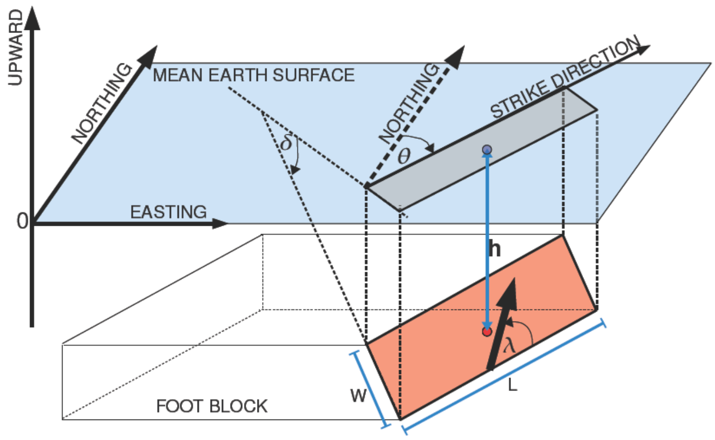
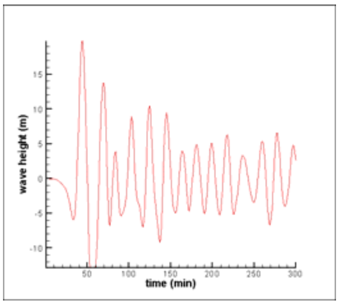


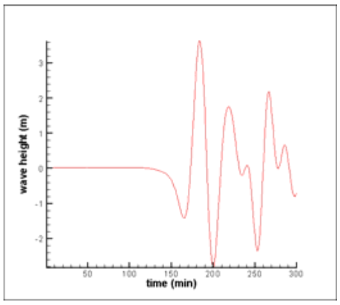
Figure 1. Sketch of a Fault Plane [1]

4. SIMULATION OF 2004 EVENT

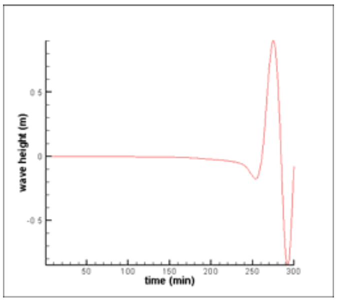
By using COMCOT, the 2004 event has been simulated. Time series of the wave height in five cities in the coastal line of India Ocean have been drawn. Figure 2 shows that the first tsunami wave reach Banda Aceh in 40 minutes which has the wave height of about 20m. In Phuket, the first wave is more than 3m. In Barguna, the first wave height is about 1m. In Batticaloa, the highest wave is more than 2m. In Colombo, the highest wave is more than 3m.



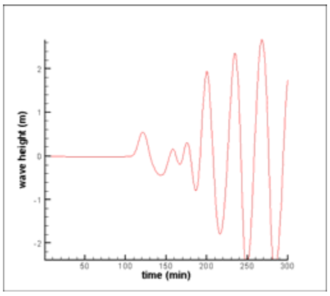
Banda Aceh, Indonesia



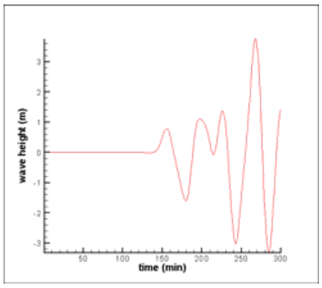
Phuket, Thailand



Barguna, Bangladesh



Batticaloa, Sri Lanka



Colombo, Sri Lanka

Figure 2. Time series of the cities in the coastal line of India Ocean

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