Fluid-Particles Interaction Models Asymptotic Models and Simulation II

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Outline

Final PDE Model

Vlasov-Euler-Fokker-Planck system:

We arrive at the system:

$$\partial_t f + \xi \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_\xi f = \frac{9\mu}{2a^2\rho_{\mathrm{P}}} \operatorname{div}_\xi \left((\xi - u) f + \frac{k\theta_0}{m_{\mathrm{P}}} \nabla_\xi f \right), \tag{1}$$

$$\partial_t n + \operatorname{div}_x (nu) = 0, \tag{2}$$

$$\rho_{\mathrm{F}} \left(\partial_t (nu) + \operatorname{Div}_x (nu \otimes u) + \alpha n \nabla_x \Phi \right) + \nabla_x p(n) = 6\pi\mu a \int_{\mathbb{R}^3} (\xi - u) f \, \mathrm{d}\xi. \tag{3}$$

where k stands for the Boltzmann constant, and $\theta_0 > 0$ controls the noise strength and $p(n)$ is a general pressure law, for instance $p(n) = C_\gamma \, n^\gamma$, $\gamma \geq 1$, $C_\gamma > 0$.

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DimensionLess Parameters

- The Stokes number $\varepsilon = \frac{\mathcal{T}_S}{T}$ is the ratio of the Stokes settling time $\mathcal{T}_S$ over the time scale of observation T. It measures the strength of the friction force.
- ρ_P/ρ_F is the ratio of the the density of particles over the typical density of the surrounding gas;
- $\beta = \frac{\mathcal{V}_{th}}{U}$ is the ratio of the thermal velocity $\mathcal{V}_{th} = \sqrt{\frac{k\theta_0}{m_P}}$ of the particles, which measures the fluctuation of particles velocities, over the typical velocity of the gas $U = L/T$.
- η' is, up to its sign, ε^{-1} times the ratio of the Stokes velocity, that enters into the scaling of the external forces, over the thermal velocity.

$$\eta' = \frac{g(1-\rho_F/\rho_P)T}{\sqrt{k\Theta/m_P}} \qquad |\eta'| = \frac{1}{\varepsilon}\,\frac{V_S}{\sqrt{k\Theta/m_P}}.$$

- η/η' with $\eta = \frac{gT}{U}$ measures the difference of the influence of the external forces on the different phases; it is a dimensionless coefficient with a sign.

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DimensionLess Change of Variables

Adopting the convention that primed quantities are dimensionless, we set

$$t = \mathrm{T}\, t', \qquad x = \mathrm{L}\, x', \qquad \xi = \mathcal{V}_{th}\xi'$$
$$n(\mathrm{T}t', \mathrm{L}x') = n'(t', x') \qquad u(\mathrm{T}t', \mathrm{L}x') = \mathrm{U}\, u'(t', x') \qquad p(\mathrm{T}t', \mathrm{L}x') = \mathcal{P}\, p'(t', x')$$

where $\mathcal{P} = \rho_{\mathrm{F}} U^2$ is a pressure unit, and

$$f'(t', x', \xi') = \frac{4}{3}\pi a^3\, \mathcal{V}_{th}^3\, f(\mathrm{T}t', \mathrm{L}x', \mathcal{V}_{th}\xi') \qquad \Phi(\mathrm{T}t', \mathrm{L}x') = \frac{\mathcal{V}_S \mathrm{L}}{\mathcal{T}_S}\, \Phi'(t', x').$$

DimensionLess Change of Variables

Therefore, the kinetic equation can be recast as

$$\frac{1}{\mathrm{T}}\partial_{t'}f' + \frac{\mathcal{V}_{th}}{\mathrm{L}}\,\xi'\cdot\nabla_{x'}f' - \frac{\mathcal{V}_S}{\mathcal{T}_S\mathcal{V}_{th}}\nabla_{x'}\Phi'\cdot\nabla_{\xi'}f' = \frac{1}{\mathcal{T}_S\mathcal{V}_{th}}\,\mathrm{div}_{\xi'}\left(\left(\mathcal{V}_{th}\xi' - \mathrm{U}u'\right)f' + \mathcal{V}_{th}\nabla_{\xi'}f'\right),$$

while the fluid equations become

$$\begin{aligned}
&\frac{1}{\mathrm{T}}\partial_{t'}n' + \frac{\mathrm{U}}{\mathrm{L}}\;\mathrm{div}_{x'}(n'u') = 0,\\
&\frac{\mathrm{U}}{\mathrm{T}}\,\partial_{t'}(n'u') + \frac{\mathrm{U}^2}{\mathrm{L}}\,\mathrm{Div}_{x'}(n'u'\otimes u') + \frac{\mathcal{P}}{\rho_{\mathrm{F}}\mathrm{L}}\,\nabla_{x'}p'(n') + \alpha\frac{\mathcal{V}_S}{\mathcal{T}_S}n'\nabla_{x'}\Phi'\\
&\qquad\qquad = \frac{1}{\mathcal{T}_S}\,\frac{\rho_{\mathrm{P}}}{\rho_{\mathrm{F}}}\,\int_{\mathbb{R}^3}\left(\mathcal{V}_{th}\xi' - \mathrm{U}u'\right)f'\,\mathrm{d}\xi'.
\end{aligned}$$

DimensionLess PDE Model

DimensionLess Vlasov-Euler-Fokker-Planck system:

$$\partial_t f + \beta\xi \cdot \nabla_x f - \eta' \nabla_x \Phi \cdot \nabla_\xi f = \frac{1}{\epsilon} \nabla_\xi \cdot \Big((\xi - \frac{1}{\beta} u) f + \nabla_\xi f \Big), \tag{4}$$

$$\partial_t n + \mathrm{div}_x (nu) = 0, \tag{5}$$

$$\partial_t (nu) + \mathrm{Div}_x (nu \otimes u) + \nabla_x p(n) + \eta\, n \nabla_x \Phi = \frac{1}{\epsilon} \frac{\rho_{\mathrm{P}}}{\rho_{\mathrm{F}}} (J - \rho u). \tag{6}$$

where

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,\xi)\, \mathrm{d}\xi, \qquad J(t,x) = \beta \int_{\mathbb{R}^3} \xi\, f(t,x,\xi)\, \mathrm{d}\xi.$$

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DimensionLess PDE Model

Comments on the Model:

- Density-Dependent Viscosity: Typically the viscosity is temperature dependent and the equation of state reads $n = (RT)^{1/(\gamma-1)}$. Thus

$$\partial_t f + \beta\xi \cdot \nabla_x f - \eta' \nabla_x \Phi \cdot \nabla_\xi f = \frac{n^\alpha}{\epsilon} \nabla_\xi \cdot \left((\xi - \frac{1}{\beta} u) f + \nabla_\xi f \right)$$

- Non-viscous Fluid: We neglect viscosity in the fluid equation since

$$\frac{1}{\mathrm{Re}} = \frac{2}{9} \left(\rho_P / \rho_F\right) (a/L)^2 \frac{1}{\varepsilon}$$

and typically $a \ll L$.

Notation Fokker-Planck operator:

$$L_u(f) = \nabla_\xi \cdot \left((\xi - u) f + \nabla_\xi f \right) = \nabla_\xi \cdot \left(M_u \nabla_\xi (f/M_u) \right), \qquad M_u(\xi) = \frac{e^{-|\xi - u|^2/2}}{(2\pi)^{3/2}}$$

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Boundary conditions

Let $x \in \Omega \subset \mathbb{R}^3$, with either $\Omega = \mathbb{R}^3$ itself or a bounded domain with smooth boundary. In such a case we denote by $\nu(x)$ the outward unit vector at $x \in \partial\Omega$.

- Fluid Equations: it is natural to require $u \cdot \nu(x) = 0, \qquad$ for $x \in \partial\Omega$.
- Kinetic Equation: Let us denote by $f_\pm(t,x,\xi)$ the trace of f on the set

$$\Sigma_\pm = \{(t,x,\xi) \in \mathbb{R}^+ \times \partial\Omega \times \mathbb{R}^3,\ \pm\xi \cdot \nu(x) \geq 0\}.$$

The boundary condition relates the incoming trace to the outgoing one as follows

$$|\xi \cdot \nu(x)| f_-(t,x,\xi) \quad = \quad \int_{\xi' \cdot \nu(x) > 0} K(x,\xi,\xi')\, f_+(t,x,\xi')\, \xi' \cdot \nu(x)\, \mathrm{d}\xi'$$

for $(t,x,\xi) \in \Sigma_-$ with K satisfying the following properties: nonnegative, normalization and preservation of maxwellians. They imply the mass conservation and

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \int_{\mathbb{R}^3} f \, \mathrm{d}\xi \, \mathrm{d}x = 0.$$

Main example: Reflection boundary conditions.

Dissipation Property

Entropy Decay:

Assumme the scaling:

$$\frac{\rho_{\mathrm{P}}}{\rho_{\mathrm{F}}}\,\beta^2 = 1, \qquad \eta' = \varsigma\beta, \qquad \text{with } \varsigma = \pm 1.$$

Defining the free energies associated respectively to the particles and the fluid as:

$$\mathcal{F}_{\mathrm{P}}(t) = \int_\Omega \int_{\mathbb{R}^3} \left(f\ln(f) + \frac{\xi^2}{2} f + \varsigma\Phi f \right) \mathrm{d}\xi\,\mathrm{d}x,$$

$$\mathcal{F}_{\mathrm{F}}(t) = \int_{\mathbb{R}^3} \left(n\frac{|u|^2}{2} + \Pi(n) + \eta\Phi n \right) \mathrm{d}x,$$

where $\Pi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is defined by $s\Pi''(s) = p'(s)$. Then, we have the crucial dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}_{\mathrm{P}} + \mathcal{F}_{\mathrm{F}}\right) + \frac{1}{\varepsilon}\int_\Omega \int_{\mathbb{R}^3} \left| (\xi - \beta^{-1}u)\sqrt{f} + 2\nabla_\xi \sqrt{f} \right|^2 \mathrm{d}\xi\,\mathrm{d}x \leq 0.$$

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Dissipation Property: Proof

- Entropy and kinetic energy of the particles

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(f\ln(f)+\frac{\xi^2}{2}f\right)\mathrm{d}\xi\,\mathrm{d}x$$
$$=-\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left((\xi-\beta^{-1}u)f+\nabla_\xi f\right)\cdot\left(\frac{\nabla_\xi f}{f}+\xi\right)\mathrm{d}\xi\,\mathrm{d}x$$
$$-\eta'\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\nabla_x\Phi\cdot\xi f\,\mathrm{d}\xi\,\mathrm{d}x.$$

- Potential energy of the particles

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Phi f\,\mathrm{d}\xi\,\mathrm{d}x=\beta\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\nabla_x\Phi\cdot\xi f\,\mathrm{d}\xi\,\mathrm{d}x.$$

Dissipation Property: Proof

- Kinetic energy of the fluid

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3} n\frac{|u|^2}{2}\,\mathrm{d}x = -\int_{\mathbb{R}^3} up'(n)\cdot\nabla_x n\,\mathrm{d}x - \eta\int_{\mathbb{R}^3} nu\cdot\nabla_x\Phi\,\mathrm{d}x + \frac{\beta}{\varepsilon}\frac{\rho_{\mathrm{P}}}{\rho_{\mathrm{F}}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(\xi-\beta^{-1}u)f\cdot u\,\mathrm{d}\xi\,\mathrm{d}x.$$

- Entropy of the fluid

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3}\Pi(n)\,\mathrm{d}x = -\int_{\mathbb{R}^3}\Pi'(n)\mathrm{div}_x(nu)\,\mathrm{d}x = \int_{\mathbb{R}^3}\Pi''(n)\nabla_x n\cdot nu\,\mathrm{d}x.$$

- Potential energy of the fluid

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3} n\Phi\,\mathrm{d}x = \int_{\mathbb{R}^3} nu\cdot\nabla_x\Phi\,\mathrm{d}x.$$

Dissipation Property: Proof

Now, we sum these relations. Taking into account

$$\frac{\rho_P}{\rho_F}\,\beta^2 = 1, \qquad \eta' = \varsigma\beta, \qquad \text{with } \varsigma = \pm 1$$

and using the fact that

$$\int_{\mathbb{R}^3} u \cdot \nabla_\xi f \, \mathrm{d}\xi = 0,$$

we arrive at

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathcal{F}(f(t), n(t), u(t))\Big) = -\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big((\xi - \beta^{-1}u)^2 f + \frac{|\nabla_\xi f|^2}{f} + (\xi - \beta^{-1}u) f \cdot \frac{\nabla_\xi f}{f} + \xi \cdot \nabla_\xi f - \beta^{-1} u \cdot \nabla_\xi f\Big)\,\mathrm{d}\xi\,\mathrm{d}x,$$

which ends the proof in the whole space.

Dissipation Property: Proof

When considering boundary conditions, integration by parts yields an additional boundary term, which reads

$$\int_{\partial\Omega}\int_{\mathbb{R}^3}\left(f\ln(f)+\frac{\xi^2}{2}+\Phi f\right)\xi\cdot\nu(x)\,\mathrm{d}\xi\,\mathrm{d}\sigma(x)$$

with $\mathrm{d}\sigma(x)$ the Lebesgue measure on $\partial\Omega$. All boundary terms from the fluid equation vanish, by using $u\cdot\nu=0$. The mass conservation property satisfied by the kernel K implies that

$$\int_{\partial\Omega}\Phi(x)\left(\int_{\mathbb{R}^3}f\xi\cdot\nu(x)\,\mathrm{d}\xi\right)\mathrm{d}\sigma(x)=0.$$

Then, using the properties of K, we can check that the remainder term

$$\int_{\mathbb{R}^3}\left(f\ln(f)+\frac{\xi^2}{2}\right)\xi\cdot\nu(x)\,\mathrm{d}\xi=\int_{\mathbb{R}^3}f\ln\left(\frac{f}{e^{-\xi^2/2}}\right)\xi\cdot\nu(x)\,\mathrm{d}\xi$$

is non positive, as a consequence of the Jensen inequality, a property known as the Darrozès-Guiraud inequality.

Dissipation Properties 2

Comments:

- Typical Pressure laws: If $p(n) = n^\gamma$, we have $\Pi(n) = n^\gamma/(\gamma - 1)$ for $\gamma > 1$ and $\Pi(n) = n\ln(n)$ for $\gamma = 1$.
- Entropy Dissipation: This claim helps in understanding the asymptotic regime $\varepsilon \ll 1$: we infer that f has essentially a hydrodynamic behavior

$$f(t,x,\xi) \simeq \rho(t,x)\,(2\pi)^{-3/2}\,\exp\left(-|\xi - \beta^{-1}u(t,x)|^2/2\right) = \rho(t,x)M_{u(t,x)/\beta}(\xi).$$

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Stability - Settling Equilibria

Equilibrium - Kinetic:

Let us fix $\varsigma = 1$ and set

$$f_S(x,\xi) = Z_{\mathcal{M}_P}\, e^{-\Phi(x)}\, M(\xi)$$

with

$$M(\xi) = \frac{e^{-\xi^2/2}}{(2\pi)^{3/2}},$$

and the normalization condition

$$Z_{\mathcal{M}_P} = \frac{\mathcal{M}_P}{\displaystyle\int_\Omega e^{-\Phi(x)}\,\mathrm{d}x}.$$

Such a definition makes sense provided Φ fulfils the confinement condition:

(HC1) $x \mapsto e^{-\Phi(x)} \in \mathrm{L}^1(\Omega)$.

Then, f_S is a (non homogeneous) stationary solution of the kinetic equation with $u = 0$, since $(\xi\cdot\nabla_x - \nabla_x\Phi\cdot\nabla_\xi)f_S = 0$ as well as $\mathrm{div}_\xi(\xi f_S + \nabla_\xi f_S) = 0$.

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Let us fix $\varsigma = 1$ and set

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with

$$M(\xi) = \frac{e^{-\xi^2/2}}{(2\pi)^{3/2}},$$

and the normalization condition

$$Z_{\mathcal{M}_P} = \frac{\mathcal{M}_P}{\displaystyle\int_\Omega e^{-\Phi(x)}\,\mathrm{d}x}.$$

Such a definition makes sense provided Φ fulfils the confinement condition:

(HC1) $x \mapsto e^{-\Phi(x)} \in \mathrm{L}^1(\Omega)$.

Then, f_S is a (non homogeneous) stationary solution of the kinetic equation with $u = 0$, since $(\xi\cdot\nabla_x - \nabla_x\Phi\cdot\nabla_\xi)f_S = 0$ as well as $\mathrm{div}_\xi(\xi f_S + \nabla_\xi f_S) = 0$.

Stability

Equilibrium - Kinetic:

We select the equilibrium state f_S by

$$\mathcal{M}_{\mathrm{P}} = \int_\Omega \int_{\mathbb{R}^3} f \, \mathrm{d}\xi \, \mathrm{d}x = \int_\Omega \int_{\mathbb{R}^3} f_0 \, \mathrm{d}\xi \, \mathrm{d}x = \int_\Omega \int_{\mathbb{R}^3} f_S \, \mathrm{d}\xi \, \mathrm{d}x.$$

To have finite free energy, we further assume that $\mathcal{F}_{\mathrm{P}}(f_S) < \infty$,

(HC2) $x \mapsto \Phi(x) e^{-\Phi(x)} \in \mathrm{L}^1(\Omega)$.

Finally, we remark that

$$\begin{aligned}\mathcal{F}_{\mathrm{P}}(f) &= \int_\Omega \int_{\mathbb{R}^3} \left(f \ln\left(\frac{f}{f_S}\right) - f + f_S \right) \mathrm{d}\xi \, \mathrm{d}x + \ln\left(\frac{Z_{\mathcal{M}_{\mathrm{P}}}}{(2\pi)^{3/2}}\right) \mathcal{M}_{\mathrm{P}} \\ &:= RE_{\mathrm{P}}(f|f_S) + \ln\left(\frac{Z_{\mathcal{M}_{\mathrm{P}}}}{(2\pi)^{3/2}}\right) \mathcal{M}_{\mathrm{P}}.\end{aligned}$$

and the first term is nonnegative and vanishes if and only if $f = f_S$.

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We select the equilibrium state f_S by

$$\mathcal{M}_P = \int_\Omega \int_{\mathbb{R}^3} f \, d\xi \, dx = \int_\Omega \int_{\mathbb{R}^3} f_0 \, d\xi \, dx = \int_\Omega \int_{\mathbb{R}^3} f_S \, d\xi \, dx.$$

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Stability

Relative Entropy - Fluid:

We define the functional $E_F : L^1_+(\Omega) \to \mathbb{R} \cup \{\infty\}$

$$E_F(n) = \begin{cases} \int_\Omega \left(\eta n \Phi + \Pi^+(n)\right) dx - \int_\Omega \Pi^-(n)\, dx & \text{if } \Pi^-(n) \in L^1(\Omega) \\ \infty & \text{else,} \end{cases}$$

where $L^1_+(\Omega) = \{n \in L^1(\Omega) : n \geq 0\}$.

Restricting this functional to the set of $L^1_+(\Omega)$ functions with total fluid mass

$$\mathcal{M}_F = \int_\Omega n \, dx = \int_\Omega n_0 \, dx,$$

and including this restriction as a Lagrange multiplier for (10), we obtain the formal Euler-Lagrange condition, whenever $n_S > 0$:

$$\Pi'(n_S(x)) + \eta\Phi(x) = Z_{\mathcal{M}_F} \in \mathbb{R}$$

to be satisfied for a minimizer n_S in this set, where $Z_{\mathcal{M}_F}$ is a normalization constant.

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In fact, $h = \Pi'$ is a diffeomorphism from $(0, \infty)$ onto its range $(h(0^+), h(\infty))$. The generalized inverse σ of h is defined as

$$\sigma : \mathbb{R} \to [0, \infty], \quad \sigma(s) = \begin{cases} 0 & \text{for} \quad s \leq h(0+), \\ h^{-1}(s) & \text{for} \quad h(0+) < s < h(\infty), \\ \infty & \text{for} \quad h(\infty) \leq s. \end{cases}$$

Candidates to be minimizers of the functional $E_{\mathrm{F}}(n)$ on the set of $\mathrm{L}^1_+(\Omega)$ functions with total fluid mass $\mathcal{M}_{\mathrm{F}}$ are

$$n_S(x) = \sigma\left(Z_{\mathcal{M}_{\mathrm{F}}} - \eta\Phi(x)\right),$$

where $Z_{\mathcal{M}_{\mathrm{F}}}$ is fixed by imposing the conservation of fluid mass by

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Stability

Theorem

Under suitable conditions on the pressure function and the potential, then the functional $E_{\mathrm{F}}(n)$ *has a unique minimizer given by*

$$n_S(x) = \sigma\left(Z_{\mathcal{M}_{\mathrm{F}}} - \eta\Phi(x)\right),$$

in the set of $\mathrm{L}^1_+(\Omega)$ *functions with total fluid mass* $\mathcal{M}_{\mathrm{F}}$*. Moreover:*

$$E_{\mathrm{F}}(n) - E_{\mathrm{F}}(n_S) \geq \int_\Omega \left[\Pi(n) - \Pi(n_S) - \Pi'(n_S)\,(n - n_S)\right](x)\,dx$$

with equality if and only if

$$\eta\Phi(x) + h(n_S(x)) = Z_{\mathcal{M}_{\mathrm{F}}}, \quad \text{for almost all } x \in \Omega.$$

Stability

Relative Entropy - Fluid:

Thus, previous theorem allows us to rewrite the fluid free energy functional as

$$\mathcal{F}_{\mathrm{F}}(n(t),u(t)) = \int_{\Omega} n\frac{|u|^2}{2}\,\mathrm{d}x + (E_{\mathrm{F}}(n) - E_{\mathrm{F}}(n_S)) + E_{\mathrm{F}}(n_S)$$

and we observe due to (1) that again

$$RE_{\mathrm{F}}((n,u)|(n_S,u_S)) = \int_{\Omega} n\frac{|u|^2}{2}\,\mathrm{d}x + (E_{\mathrm{F}}(n) - E_{\mathrm{F}}(n_S))$$

is a functional that controls the distance from the pair $(n(t),u(t))$ to the equilibrium solution $(n_S,u_S=0)$.

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Therefore, the whole free energy functional $\mathcal{F}(f,n,u)$ can be considered, up to a constant, a relative entropy functional towards the equilibrium solution $(f_S, n_S, u_S = 0)$, i.e., defining the relative entropy functional from (f,n,u) to $(f_S, n_S, u_S = 0)$ as

$$RE((f,n,u)|(f_S,n_S,u_S)) = RE_{\mathrm{P}}(f|f_S) + RE_{\mathrm{F}}((n,u)|(n_S,u_S)).$$

Thus, we have

$$RE((f,n,u)|(f_S,n_S,u_S)) = \mathcal{F}(f,n,u) - \ln\left(\frac{Z_{\mathcal{M}_{\mathrm{P}}}}{(2\pi)^{3/2}}\right)\mathcal{M}_{\mathrm{P}} - \chi E_{\mathrm{F}}(n_S) \geq 0,$$

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Stability

Relative Entropy - Full Model:

- In the gravity settling case and as we discussed before, the physical boundary conditions of interest are periodic boundary conditions in the x_1,x_2 directions and no-flux boundary conditions for the x_3-direction. All boundary terms in the periodic boundary conditions disappear while boundary terms in the x_3-direction are treated as above. All conclusions of the last subsections apply equally well to this case.
- For the gravity settling case, in the particular case of $\alpha > 0$, which means that gravity dominates over buoyancy force, f_S and n_S represent the typical sedimentation profiles of particles and fluid respectively. It is interesting to remark that the steady density of dispersed particles will be always positive according to (8) while the steady fluid density given by (1) might be compactly supported for pressure functions of the form $p(n) = n^\gamma$, $\gamma > 1$.
- Let us remark that $(f_S, n_S, u_S = 0)$ is a stationary classical solution wherever n_S is regular.

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Stability

Theorem

Given $(f(t), n(t), u(t))$ *a solution to the Cauchy problem for the VEFP system with the scaling assumpions above and* $(f_S, n_S, u_S = 0)$ *with* f_S *and* n_S *given as above, such that*

$$\int_\Omega \int_{\mathbb{R}^3} f_0 \, \mathrm{d}\xi \, \mathrm{d}x = \int_\Omega \int_{\mathbb{R}^3} f_S \, \mathrm{d}\xi \, \mathrm{d}x \qquad \text{and} \qquad \int_\Omega n_0 \, \mathrm{d}x = \int_\Omega n_S \, \mathrm{d}x,$$

then for any $\epsilon > 0$, *there exists* $\delta > 0$ *such that if*

$$RE((f_0, n_0, u_0)|(f_S, n_S, u_S)) \leq \delta,$$

we conclude the solution satisfies

$$\|f(t) - f_S\|_{\mathrm{L}^1(\Omega \times \mathbb{R}^3)} \leq \epsilon, \ \|n(t) - n_S\|_{\mathrm{L}^1(\Omega)} \leq \epsilon \text{ and } \int_\Omega n(t) \frac{|u(t)|^2}{2} \, \mathrm{d}x \leq \epsilon$$

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Dissipation Property

Entropy Decay:

Assumme the scaling:

$$\frac{\rho_{\mathrm{P}}}{\rho_{\mathrm{F}}}\,\beta^2 = 1, \qquad \eta' = \varsigma\beta, \qquad \text{with } \varsigma = \pm 1.$$

Defining the free energies associated respectively to the particles and the fluid as:

$$\mathcal{F}_{\mathrm{P}}(t) = \int_\Omega \int_{\mathbb{R}^3} \left(f\ln(f) + \frac{\xi^2}{2} f + \varsigma\Phi f \right) \mathrm{d}\xi\,\mathrm{d}x,$$

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where $\Pi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is defined by $s\Pi''(s) = p'(s)$. Then, we have the crucial dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}_{\mathrm{P}} + \mathcal{F}_{\mathrm{F}}\right) + \frac{1}{\varepsilon}\int_\Omega \int_{\mathbb{R}^3} \left| (\xi - \beta^{-1}u)\sqrt{f} + 2\nabla_\xi \sqrt{f} \right|^2 \mathrm{d}\xi\,\mathrm{d}x \leq 0.$$

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Dissipation Property 2

Entropy Dissipation:

(i) $f(1+\xi^2+|\Phi|+|\ln(f)|)$ is bounded in $L^\infty(\mathbb{R}^+;L^1(\Omega\times\mathbb{R}^3))$.

(ii) n, $|\Pi(n)|$ and Φn are bounded in $L^\infty(\mathbb{R}^+;L^1(\Omega))$.

(iii) $\sqrt{n}\,u$ is bounded in $L^\infty(\mathbb{R}^+;L^2(\Omega))$.

(iv) $\frac{1}{\sqrt{\varepsilon}}\left((\xi-\beta^{-1}u)\sqrt{f}+2\nabla_\xi\sqrt{f}\right)=\frac{D}{\sqrt{\varepsilon}}$ is bounded in $L^2(\mathbb{R}^+\times\Omega\times\mathbb{R}^3)$.

Moments:

We have the following expansions

$$J=\rho u+\beta\sqrt{\varepsilon}K,\qquad \mathbb{P}=\rho\boldsymbol{I}+\frac{1}{\beta^2}J\otimes u+\sqrt{\varepsilon}\mathbb{K}$$

where the components of the vector K and of the matrix $\mathbb{K}$ are bounded in $L^2(\mathbb{R}^+;L^1(\Omega))$. Here,

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Bubbling Regime

We set

$$\beta = \frac{1}{\sqrt{\varepsilon}}, \qquad |\eta'| = \frac{1}{\sqrt{\varepsilon}}, \qquad \frac{\rho_{\mathrm{P}}}{\rho_{\mathrm{F}}} = \varepsilon.$$

meaning that:

Stokes velocity $\simeq$ Typical velocity of the fluid $\ll$ Thermal velocity.

Gravity:

The scaling assumption is

$$\mathrm{Ri}_{\mathrm{F}} = \eta = \frac{\beta^2}{|1-\beta^2|} \qquad \text{and} \qquad \mathrm{Ri}_{\mathrm{P}} = |\eta'| = \beta.$$

As $\varepsilon \to 0$ we have $\rho_P/\rho_F \ll 1$, $\mathrm{Ri}_{\mathrm{F}} \sim 1$ and $\mathrm{Ri}_{\mathrm{P}} >> 1$ and thus, the dispersed phase is buoyancy driven while the flow is gravity driven. Here, $\eta' < 0$ and the external forces act in opposite directions on the particles and on the fluid.

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$$\text{Stokes velocity} \simeq \text{Typical velocity of the fluid} \ll \text{Thermal velocity}.$$

Gravity:

The scaling assumption is

$$\mathrm{Ri}_{\mathrm{F}} = \eta = \frac{\beta^2}{|1-\beta^2|} \qquad \text{and} \qquad \mathrm{Ri}_{\mathrm{P}} = |\eta'| = \beta.$$

As $\varepsilon \to 0$ we have $\rho_P/\rho_F \ll 1$, $\mathrm{Ri}_{\mathrm{F}} \sim 1$ and $\mathrm{Ri}_{\mathrm{P}} >> 1$ and thus, the dispersed phase is buoyancy driven while the flow is gravity driven. Here, $\eta' < 0$ and the external forces act in opposite directions on the particles and on the fluid.

Bubbling Regime: Moments

We are concerned with the behavior as $\varepsilon \to 0$ of

$$\begin{cases} \partial_t f_\varepsilon + \dfrac{1}{\sqrt{\varepsilon}}\Big(\xi \cdot \nabla_x f_\varepsilon + \nabla_x \Phi \cdot \nabla_\xi f_\varepsilon\Big) = \dfrac{1}{\varepsilon}\,\mathrm{div}_\xi\Big((\xi - \sqrt{\varepsilon}u_\varepsilon)f + \nabla_\xi f_\varepsilon\Big), \\ \partial_t n_\varepsilon + \mathrm{div}_x(n_\varepsilon u_\varepsilon) = 0, \\ \partial_t(n_\varepsilon u_\varepsilon) + \mathrm{Div}_x(n_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla_x p(n_\varepsilon) + \eta n_\varepsilon \nabla_x \Phi = (J_\varepsilon - \rho_\varepsilon u_\varepsilon), \end{cases}$$

with

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}^3} f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi, \qquad J_\varepsilon(t,x) = \int_{\mathbb{R}^3} \frac{1}{\sqrt{\varepsilon}}\,\xi\, f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi$$

and

$$\mathbb{P}_\varepsilon(t,x) = \int_{\mathbb{R}^3} \xi \otimes \xi\, f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi.$$

Then, we obtain the following moment equations

$$\begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}_x J_\varepsilon = 0, \\ \varepsilon \partial_t J_\varepsilon + \mathrm{Div}_x \mathbb{P}_\varepsilon - \rho_\varepsilon \nabla_x \Phi = -(J_\varepsilon - \rho_\varepsilon u_\varepsilon). \end{cases}$$

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Using the formal ansatz, the distribution function can be approximated as $f_\varepsilon \simeq \frac{\rho_\varepsilon}{(2\pi)^{3/2}} e^{-\xi^2/2}$, and $\mathbb{P}_\varepsilon \simeq \rho_\varepsilon \boldsymbol{I}$, but it remains to describe the behavior of J_ε. Letting ε go to 0 in the first order moment equation yields

$$\nabla_x \rho - \rho \nabla_x \Phi = -J + \rho u.$$

Inserting this result in the continuity equation, and passing to the limit in the fluid equation, we are led to the following claim.

Assuming formally the limit $\epsilon \to 0$, we get that (ρ, n, u) satisfy the following system

$$\begin{cases} \partial_t \rho + \operatorname{div}_x\big(\rho(u + \nabla_x \Phi) - \nabla_x \rho\big) = 0, \\ \partial_t n + \operatorname{div}_x(nu) = 0, \\ \partial_t (nu) + \operatorname{Div}_x(nu \otimes u) + \nabla_x(p(n) + \rho) + (\eta n - \rho)\nabla_x \Phi = 0. \end{cases}$$

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Flowing Regime

We set

$$\beta^2 \frac{\rho_P}{\rho_F} = 1, \qquad \beta = |\eta'| \text{ a fixed positive constant}$$

meaning that

Stokes velocity $\ll$ Typical velocity of the fluid $\simeq$ Thermal velocity.

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The scaling assumption is

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As $\varepsilon \to 0$ and β is smaller than 1, the two phases are driven by gravity, but with much more influence on the fluid.

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$$\begin{cases} \partial_t f_\varepsilon + \beta\Big(\xi \cdot \nabla_x f_\varepsilon - \varsigma \nabla_x \Phi \cdot \nabla_\xi f_\varepsilon\Big) = \dfrac{1}{\varepsilon}\,\mathrm{div}_\xi\Big(\big(\xi - \dfrac{1}{\beta}\,u_\varepsilon\big) f + \nabla_\xi f_\varepsilon\Big), \\ \partial_t(n_\varepsilon) + \mathrm{div}_x(n_\varepsilon u_\varepsilon) = 0, \\ \partial_t(n_\varepsilon u_\varepsilon) + \mathrm{Div}_x(n_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla_x p(n_\varepsilon) + \eta\, n_\varepsilon \nabla_x \Phi = \dfrac{1}{\varepsilon\beta^2}(J_\varepsilon - \rho_\varepsilon u_\varepsilon), \end{cases}$$

with

$$\begin{pmatrix} \rho_\varepsilon \\ J_\varepsilon \\ \mathbb{P}_\varepsilon \end{pmatrix}(t,x) = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \beta\xi \\ \xi \otimes \xi \end{pmatrix} f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi.$$

The macroscopic quantities satisfy the following moment system

$$\begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}_x J_\varepsilon = 0, \\ \dfrac{1}{\beta^2}\partial_t J_\varepsilon + \mathrm{Div}_x \mathbb{P}_\varepsilon + \varsigma \rho_\varepsilon \nabla_x \Phi = -\dfrac{1}{\varepsilon\beta^2}\,(J_\varepsilon - \rho_\varepsilon u_\varepsilon). \end{cases}$$

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Using the fluid momentum equation, we are led to

$$\partial_t(n_\varepsilon u_\varepsilon + \beta^{-2} J_\varepsilon) + \mathrm{Div}_x(n_\varepsilon u_\varepsilon \otimes u_\varepsilon + \mathbb{P}_\varepsilon) + \nabla_x p(n_\varepsilon) + (\eta n_\varepsilon + \varsigma \rho_\varepsilon)\nabla_x \Phi = 0.$$

Now, using the dissipation consequences it is tempting to infer

$$J_\varepsilon \simeq \rho_\varepsilon u_\varepsilon, \qquad \mathbb{P}_\varepsilon \simeq \rho_\varepsilon \boldsymbol{I} + \beta^{-2}\, \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon.$$

Assuming formally the limit $\epsilon \to 0$, we get that (ρ, n, u) satisfy the following system

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \\ \\ \partial_t n + \mathrm{div}_x(n u) = 0, \\ \\ \partial_t\big((n + \beta^{-2}\rho)u\big) + \mathrm{Div}_x\big((n + \beta^{-2}\rho)u \otimes u\big) \\ \qquad\qquad + \nabla_x\big(\rho + p(n)\big) + (\eta n + \varsigma \rho)\nabla_x \Phi = 0. \end{cases}$$

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Asymptotics: Hyperbolicity

Consider the equivalent flowing regime system to (8) given by

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\ \partial_t r + \operatorname{div}_x(ru) = 0, \\ \partial_t (ru) + \operatorname{Div}_x(ru \otimes u) + \nabla_x(\rho + p(n)) + \eta r \nabla_x \Phi = 0, \end{cases} \tag{7}$$

with $r = n + \beta^{-2}\rho$. Therefore, the first order term has a flux function $F_{fr} : \mathbb{R}^5 \longrightarrow M_{5\times 3}(\mathbb{R})$ given by:

$$F_{fr}(U) = \left(\frac{j\rho}{r}, j, \frac{j \otimes j}{r} + (p(r - \beta^{-2}\rho) + \rho)\, \boldsymbol{I} \right)$$

with $U = (\rho, r, j)$ and $j = ru$. Taking the component in the x_1-direction given by

$$F^1_{fr}(U) = \left(\frac{j_1\rho}{r}, j_1, \frac{j_1^2}{r} + p(r - \beta^{-2}\rho) + \rho, \frac{j_1 j_2}{r}, \frac{j_1 j_3}{r} \right),$$

it is easy to check (exercise) that its jacobian matrix has real eigenvalues given by $\frac{j_1}{r}$ (triple) and two simple eigenvalues $\frac{j_1}{r} \pm \sqrt{\frac{\rho}{r} + p'(n)\frac{n}{r}}$. Therefore, the system is hyperbolic. For the bubbling limit, the first order part is hyperbolic.

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Asymptotics: Stability

DimensionLess Bubbling Regime:

- Bubbling: one finds the following free energy functional

$$\mathcal{F}_{br}(\rho, n, u) = \int_{\Omega} \left[n\frac{|u|^2}{2} + \Pi(n) + \rho \ln \rho + (\eta n - \rho)\Phi \right] \mathrm{d}x,$$

which in this case is dissipated along the flow, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{br}(\rho(t), n(t), u(t)) = -\int_{\Omega} \rho \left| -\nabla_x \Phi + \nabla_x \ln \rho \right|^2 \mathrm{d}x \leq 0.$$

- Flowing: one has the following free energy functional:

$$\mathcal{F}_{fr}(\rho, n, u) = \int_{\Omega} \left[(n + \beta^{-2}\rho)\frac{|u|^2}{2} + \Pi(n) + \rho \ln \rho + (\eta n + \rho)\Phi \right] \mathrm{d}x,$$

for the flowing regime system. It can be checked easily that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{fr}(\rho(t), n(t), u(t)) = 0.$$

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