A NONLOCAL VECTOR CALCULUS WITH APPLICATION TO NONLOCAL BOUNDARY-VALUE PROBLEMS

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- **MOTIVATION**
  - ◦ analysis and numerical analysis of peridynamic model for materials
    - characterization of “boundary” conditions
    - well posedeness
    - characterization of solution and data spaces
    - finite element, e.g., discontinuous Galerkin, discretizations

- **IN THIS TALK**
  - ◦ we treat scalar-valued problems
    - extension to vector case is (formally) straightforward

Related papers

- Gilboa and Osher

  G. Gilboa and S. Osher, Nonlocal operators with applications to image processing; *Multiscale Model. Simul.* **7** 2008, 1005–1028.

- Rossi and co-workers

  F. Andreu, J. Mazon, J. Rossi, and J. Toledo, A nonlocal p-Laplacian evolution equation with nonhomogeneous Dirichlet boundary conditions; *SIAM J. Math. Anal.* **40** 2009, 1815–1851.

- Emmrich and Weckner

E. Emmrich and O. Weckner, On the well-posedness of the linear peridynamic model and its convergence towards the Navier equation of Linear elasticity; *Commun. Math. Sci.* **5** 2007, 851–864.

- Alali and Lipton

B. Alali and R. Lipton, Multiscale analysis of heterogeneous media in the peridynamic formulation; *IMA preprint* 2009.

- Du and Zhou

Q. Du and K. Zhou, Multiscale analysis of heterogeneous media in the peridynamic formulation; in preparation.

A NONLOCAL GAUSS'S THEOREM

- $\Omega \subset \mathbb{R}^d$

- Let $p(\mathbf{x}, \mathbf{x}') : \Omega \times \Omega \to \mathbb{R}$ denote a skew-symmetric function

$$\Rightarrow p(\mathbf{x}', \mathbf{x}) = -p(\mathbf{x}, \mathbf{x}')$$

- One easily sees that

$$\int_{\widehat{\Omega}} \int_{\widehat{\Omega}} p(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}' d\mathbf{x} = 0 \qquad \forall\, \widehat{\Omega} \subseteq \Omega$$

- Let
  - $\alpha(\mathbf{x}, \mathbf{x}') : \Omega \times \Omega \to \mathbb{R}$ denote a symmetric function

    $$\Rightarrow \alpha(\mathbf{x}', \mathbf{x}) = \alpha(\mathbf{x}, \mathbf{x}')$$

  - $f(\mathbf{x}, \mathbf{x}') : \Omega \times \Omega \to \mathbb{R}$ denote a skew-symmetric function

    $$\Rightarrow f(\mathbf{x}', \mathbf{x}) = -f(\mathbf{x}, \mathbf{x}')$$

- Let $\widetilde{\Omega} \subset \Omega$ such that both $\widetilde{\Omega}$ and $\Omega \setminus \widetilde{\Omega}$ have finite measure
  - then

$$\int_{\widetilde{\Omega}} \int_{\Omega} f(\mathbf{x}, \mathbf{x}') \alpha(\mathbf{x}', \mathbf{x}) \, d\mathbf{x}' d\mathbf{x} = - \int_{\Omega \setminus \widetilde{\Omega}} \int_{\Omega} f(\mathbf{x}, \mathbf{x}') \alpha(\mathbf{x}', \mathbf{x}) \, d\mathbf{x}' d\mathbf{x}$$

  - the right-hand side corresponds to a nonlocal "flux"

- Let $\mathcal{D}$ denote the operator mapping functions $f(\cdot,\cdot)$ defined over $\widetilde{\Omega} \times \Omega$ into functions defined over $\widetilde{\Omega}$ given by

$$
\begin{aligned}
(\mathcal{D}f)(\mathbf{x}) &= 2\int_{\Omega} f(\mathbf{x},\mathbf{x}')\alpha(\mathbf{x}',\mathbf{x})\,d\mathbf{x}' \\
&= \int_{\Omega} \big(f(\mathbf{x},\mathbf{x}') - f(\mathbf{x}',\mathbf{x})\big)\alpha(\mathbf{x}',\mathbf{x})\,d\mathbf{x}' \qquad \text{for } \mathbf{x} \in \widetilde{\Omega}
\end{aligned}
$$

- Similarly, let $\mathcal{N}$ denote the operator mapping functions $f(\cdot,\cdot)$ defined over $(\Omega \setminus \widetilde{\Omega}) \times \Omega$ into functions defined over $\Omega \setminus \widetilde{\Omega}$ given by

$$
\begin{aligned}
(\mathcal{N}f)(\mathbf{x}) &= -2\int_{\Omega} f(\mathbf{x},\mathbf{x}')\alpha(\mathbf{x}',\mathbf{x})\,d\mathbf{x}' \\
&= -\int_{\Omega} \big(f(\mathbf{x},\mathbf{x}') - f(\mathbf{x}',\mathbf{x})\big)\alpha(\mathbf{x}',\mathbf{x})\,d\mathbf{x}' \qquad \text{for } \mathbf{x} \in \Omega \setminus \widetilde{\Omega}
\end{aligned}
$$

- Then, we have the nonlocal Gauss's theorem

$$\int_{\widetilde{\Omega}} (\mathcal{D}f)(\mathbf{x})\, d\mathbf{x} = \int_{\Omega\setminus\widetilde{\Omega}} (\mathcal{N}f)(\mathbf{x})\, d\mathbf{x}$$

- This result is analogous to the classical Gauss's theorem

$$\int_{\widetilde{\Omega}} (\mathcal{D}f)(\mathbf{x})\, d\mathbf{x} = \int_{\Omega\setminus\widetilde{\Omega}} (\mathcal{N}f)(\mathbf{x})\, d\mathbf{x} \qquad \int_{\Omega} \nabla \cdot \mathbf{q}\, d\mathbf{x} = \int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n}\, d\mathbf{x}$$

  - thus, we have a derived a "nonlocal Gauss's theorem" that is analogous to the classical Gauss's theorem
  - amazingly, the two theorems have a much more direct relation

Relation to the classical Gauss's theorem

- We apply two remarkable results due to Walter Noll

W. Noll, Die Herleitung der Grundgleichungen der Thermomechanik der Kontinua aus der statistischen Mechanik; *Indiana Univ. Math. J.* **4** 1955, 627–646. Originally published in *J. Rational Mech. Anal.*

W. Noll, Derivation of the fundamental equations of continuum thermodynamics from statistical mechanics; Translation with corrections by R. Lehoucq and O. A. von Lilienfeld, to appear in *J. Elasticity*, 2009

- Let the vector field $\mathbf{q}\colon \Omega \longrightarrow \mathbb{R}^d$ be defined by

$$\mathbf{q}(\mathbf{x}) = -\int_{\Omega} (\mathbf{x}' - \mathbf{x})\varphi(\mathbf{x}, \mathbf{x}' - \mathbf{x})\, d\mathbf{x}'$$

  – the function $\varphi(\cdot,\cdot)$ is given by, with $\mathbf{z} = \mathbf{x}' - \mathbf{x}$,

$$\varphi(\mathbf{x}, \mathbf{z}) = \int_0^1 f\big(\mathbf{x} + \lambda\mathbf{z}, \mathbf{x} - (1-\lambda)\mathbf{z}\big)\alpha\big(\mathbf{x} + \lambda\mathbf{z}, \mathbf{x} - (1-\lambda)\mathbf{z}\big)\, d\lambda$$

- Lemma I in the Noll paper implies

$$\nabla \cdot \mathbf{q} = \int_{\Omega} \big(f(\mathbf{x}, \mathbf{x}') - f(\mathbf{x}', \mathbf{x}) \big) \alpha(\mathbf{x}, \mathbf{x}') \, d\mathbf{x}' \qquad \text{for } \mathbf{x} \in \widetilde{\Omega}$$

  – using the definition of the operator $\mathcal{D}(\cdot)$, we then have

$$\nabla \cdot \mathbf{q} = \mathcal{D} f \qquad \text{for } \mathbf{x} \in \widetilde{\Omega}$$

- Lemma II in the Noll paper implies

$$\int_{\partial\widetilde{\Omega}} \mathbf{q}(\mathbf{x}) \cdot \mathbf{n}\, dA = 2 \int_{\widetilde{\Omega}} \int_{\Omega\setminus\widetilde{\Omega}} f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}' d\mathbf{x}$$

  – it is a simple matter to show that, due to the skew-symmetry of $f(\cdot, \cdot)$ and symmetry of $\alpha(\cdot, \cdot)$,

$$\int_{\widetilde{\Omega}} \int_{\Omega\setminus\widetilde{\Omega}} f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}' d\mathbf{x} = - \int_{\Omega\setminus\widetilde{\Omega}} \int_{\Omega} f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}' d\mathbf{x}$$

  so that

$$\int_{\partial\widetilde{\Omega}} \mathbf{q}(\mathbf{x}) \cdot \mathbf{n}\, dA = -2 \int_{\Omega\setminus\widetilde{\Omega}} \int_{\Omega} f(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}' d\mathbf{x}$$

  – using the definition of the operator $\mathcal{N}(\cdot)$, we then have

$$\int_{\partial\widetilde{\Omega}} \mathbf{q}(\mathbf{x}) \cdot \mathbf{n}\, dA = \int_{\Omega\setminus\widetilde{\Omega}} \mathcal{N}(f)\, d\mathbf{x}$$

- By substituting the results in the last two slides into the nonlocal Gauss's theorem, we have that the vector-valued function $\mathbf{q}$ satisfies

$$\int_{\widetilde{\Omega}} \nabla \cdot \mathbf{q}\, d\mathbf{x} = \int_{\partial\widetilde{\Omega}} \mathbf{n} \cdot \mathbf{q}\, dA$$

i.e., the classical, local Gauss's theorem for the vector function $\mathbf{q}(\cdot)$

- Thus, we have shown that

the nonlocal Gauss's theorem for the

nonlocal scalar-valued function $f(\cdot,\cdot)$

is exactly equivalent to the

classical Gauss's theorem for the

local vector-valued function $\mathbf{q}(\cdot)$ derived from $f(\cdot,\cdot)$

– i.e., we have that if

$$\mathbf{q}(\mathbf{x}) = -\int_{\Omega} \mathbf{z} \int_0^1 f\big(\mathbf{x} + \lambda\mathbf{z}, \mathbf{x} - (1-\lambda)\mathbf{z}\big)\alpha\big(\mathbf{x} + \lambda\mathbf{z}, \mathbf{x} - (1-\lambda)\mathbf{z}\big)\, d\lambda\, d\mathbf{z}$$

then

$$\int_{\widetilde{\Omega}} (\mathcal{D}f)(\mathbf{x})\, d\mathbf{x} = \int_{\Omega\setminus\widetilde{\Omega}} (\mathcal{N}f)(\mathbf{x})\, d\mathbf{x} \quad \Longleftrightarrow \quad \int_{\Omega} \nabla\cdot\mathbf{q}\, d\mathbf{x} = \int_{\partial\Omega} \mathbf{q}\cdot\mathbf{n}\, d\mathbf{x}$$

An application of the nonlocal Gauss's theorem

- In the sequel, we frequently let

$$u = u(\mathbf{x}) \qquad u' = u(\mathbf{x}') \qquad v = v(\mathbf{x}) \qquad v' = v(\mathbf{x}')$$

$$f = f(\mathbf{x}, \mathbf{x}') \qquad f' = f(\mathbf{x}', \mathbf{x}) = -f, \qquad \alpha = \alpha(\mathbf{x}, \mathbf{x}') \qquad \alpha' = \alpha(\mathbf{x}', \mathbf{x}) = \alpha$$

- Let $U(\Omega)$ and $V(\Omega)$ denote Banach spaces of scalar-valued functions defined over Ω

- Define a skew-symmetric, nonlinear operator $\mathcal{K}\big(u(\mathbf{x}), u(\mathbf{x}'); \mathbf{x}, \mathbf{x}'\big)$ on $U(\Omega)\times U(\Omega) \times \Omega \times \Omega$

$$\Rightarrow \quad \mathcal{K} = \mathcal{K}(u, u'; \mathbf{x}, \mathbf{x}') = -\mathcal{K}(u', u; \mathbf{x}', \mathbf{x}) = -\mathcal{K}'$$

- Then, for $u \in U(\Omega)$ and $v \in V(\Omega)$, set

$$f = (v + v')\mathcal{K}$$

so that, because $\mathcal{K}' = -\mathcal{K}$

$$f = (v + v')\mathcal{K} = (v' - v)\mathcal{K} + 2v\mathcal{K} = v(\mathcal{K} - \mathcal{K}') + (v' - v)\mathcal{K}$$

- Substituting into

$$\int_{\widetilde{\Omega}} \int_{\Omega} f\alpha\, d\mathbf{x}' d\mathbf{x} = -\int_{\Omega \setminus \widetilde{\Omega}} \int_{\Omega} f\alpha\, d\mathbf{x}' d\mathbf{x}$$

it can be shown that

$$\int_{\widetilde{\Omega}} \int_{\Omega} v(\mathcal{K} - \mathcal{K}')\alpha\, d\mathbf{x}' d\mathbf{x} + \int_{\Omega} \int_{\Omega} (v' - v)\mathcal{K}\alpha\, d\mathbf{x}' d\mathbf{x} = -2\int_{\Omega \setminus \widetilde{\Omega}} \int_{\Omega} v\mathcal{K}\alpha\, d\mathbf{x}' d\mathbf{x}$$

- The composite operators $\mathcal{D}(\mathcal{K})$ and $\mathcal{N}(\mathcal{K})$ acting on functions belonging to $U(\Omega)$ are given by

$$\mathcal{D}(\mathcal{K}) = \int_{\Omega} (\mathcal{K} - \mathcal{K}')\alpha \, d\mathbf{x}' = 2 \int_{\Omega} \mathcal{K}\alpha \, d\mathbf{x}' \qquad \text{for } \mathbf{x} \in \widetilde{\Omega}$$

and

$$\mathcal{N}(\mathcal{K}) = -\int_{\Omega} (\mathcal{K} - \mathcal{K}')\alpha \, d\mathbf{x}' = -2 \int_{\Omega} \mathcal{K}\alpha \, d\mathbf{x}' \qquad \text{for } \mathbf{x} \in \Omega \setminus \widetilde{\Omega}$$

respectively

- Define the operator $\mathcal{G}$ acting on functions belonging to $V(\Omega)$ by

$$\mathcal{G}(v) = (v' - v)\alpha \qquad \text{for } \mathbf{x}, \mathbf{x}' \in \Omega$$

- Combining the results of the previous two slides, we obtain

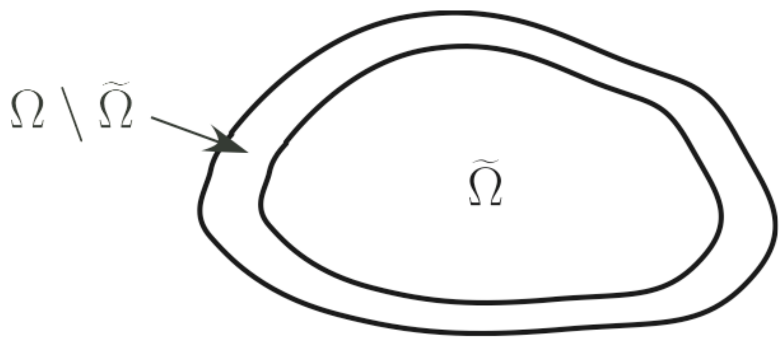
$$\int_{\widetilde{\Omega}} v\mathcal{D}(\mathcal{K})\,d\mathbf{x} + \int_{\Omega}\int_{\Omega} \mathcal{G}(v)\mathcal{K}\,d\mathbf{x}'d\mathbf{x} = \int_{\Omega\setminus\widetilde{\Omega}} v\mathcal{N}(\mathcal{K})\,d\mathbf{x}$$

  – this is the nonlocal analog of the classical result

$$\int_{\Omega} v\nabla\cdot\mathbf{p}\,d\mathbf{x} + \int_{\Omega}\mathbf{p}\cdot\nabla v\,d\mathbf{x} = \int_{\partial\Omega} v\mathbf{p}\cdot\mathbf{n}\,dA$$

NONLINEAR, NONLOCAL BOUNDARY VALUE PROBLEMS

- It is useful to think of a subdivision of Ω along the lines of



- Let

$$V_0(\Omega) = \{u \in V(\Omega) \quad : \quad v = 0 \;\text{ for } \mathbf{x} \in \Omega \setminus \widetilde{\Omega}\}$$

and let the mappings

$$\begin{aligned} b &: \; \widetilde{\Omega} \times \Omega \to \mathbb{R} \\ h_d &: \; \Omega \setminus \widetilde{\Omega} \to \mathbb{R} \\ h_n &: \; (\Omega \setminus \widetilde{\Omega}) \times \Omega \to \mathbb{R} \end{aligned}$$

be given

- Consider the variational problem

seek $u \in U(\Omega)$ *such that*

$$u = h_d \quad \text{for } \mathbf{x} \in \Omega \setminus \widetilde{\Omega}$$

and

$$\int_\Omega \int_\Omega \mathcal{G}(v)\mathcal{K}\, d\mathbf{x}' d\mathbf{x} = \int_{\widetilde{\Omega}} v \int_\Omega b\, d\mathbf{x}' d\mathbf{x} \qquad \forall\, v \in V_0(\Omega)$$

- – using the nonlocal Gauss's theorem, this can be viewed as a weak formulation of the "boundary" value problem

$$\begin{aligned} -\mathcal{D}(\mathcal{K}) &= \int_\Omega b\, d\mathbf{x}' && \text{for } \mathbf{x} \in \widetilde{\Omega}, \\ u &= h_d && \text{for } \mathbf{x} \in \Omega \setminus \widetilde{\Omega} \end{aligned}$$

- – the second equation is a "Dirichlet boundary" condition that is essential for the variational formulation

- Next, assume that the compatibility condition

$$\int_{\widetilde{\Omega}}\int_{\Omega} b\,d\mathbf{x}'d\mathbf{x} + \int_{\Omega\setminus\widetilde{\Omega}}\int_{\Omega} h_n\,d\mathbf{x}'d\mathbf{x} = 0$$

holds and consider the variational problem

seek $u \in U(\Omega)$ *such that*

$$\int_{\Omega}\int_{\Omega}\mathcal{G}(v)\mathcal{K}\,d\mathbf{x}'d\mathbf{x} = \int_{\widetilde{\Omega}} v\int_{\Omega} b\,d\mathbf{x}'d\mathbf{x} + \int_{\Omega\setminus\widetilde{\Omega}} v\int_{\Omega} h_n\,d\mathbf{x}'d\mathbf{x} \qquad \forall\, v \in V(\Omega)\setminus\mathbb{R}$$

  - using the nonlocal Gauss's theorem, this can be viewed as a weak formulation of the "boundary" value problem

$$-\mathcal{D}(\mathcal{K}) = \int_{\Omega} b\,d\mathbf{x}' \quad \text{for } \mathbf{x}\in\widetilde{\Omega}$$

$$\mathcal{N}(\mathcal{K}) = \int_{\Omega} h_n\,d\mathbf{x}' \quad \text{for } \mathbf{x}\in\Omega\setminus\widetilde{\Omega}$$

  - the second equation is a "Neumann boundary" condition that is natural for the variational formulation

LINEAR NONLOCAL GREEN'S IDENTITIES

- We specialize the nonlocal Gauss's theorem to the case of $U(\Omega) = V(\Omega)$ and to linear operators

- Let

$$\mathcal{K}(u, u'; \mathbf{x}, \mathbf{x}') = \beta \mathcal{G}(u) = (u' - u)\alpha\beta$$

where $\beta(\mathbf{x}, \mathbf{x}') \colon \Omega \times \Omega \to \mathbb{R}$ is a symmetric function and $u \in U(\Omega)$

- Then, the nonlocal Gauss's theorem results in the nonlocal Green's first identity

$$\int_{\widetilde{\Omega}} v \mathcal{D}\big(\beta \mathcal{G}(u)\big)\, d\mathbf{x} + \int_{\Omega}\int_{\Omega} \beta \mathcal{G}(v) \mathcal{G}(u)\, d\mathbf{x}' d\mathbf{x} = \int_{\Omega \setminus \widetilde{\Omega}} v \mathcal{N}\big(\beta \mathcal{G}(u)\big)\, d\mathbf{x}$$

- One then easily obtains the nonlocal Green's second identity

$$\int_{\widetilde{\Omega}} v\mathcal{D}\big(\beta\mathcal{G}(u)\big)\,d\mathbf{x} - \int_{\widetilde{\Omega}} u\mathcal{D}\big(\beta\mathcal{G}(v)\big)\,d\mathbf{x} = \int_{\Omega\setminus\widetilde{\Omega}} \Big(v\mathcal{N}\big(\beta\mathcal{G}(u)\big) - u\mathcal{N}\big(\beta\mathcal{G}(v)\big)\Big)\,d\mathbf{x}$$

- These are analogous to (generalizations) of the classical Green's identities

$$\int_{\Omega} v\Delta u\,d\mathbf{x} + \int_{\Omega} \nabla v\cdot\nabla u\,d\mathbf{x} = \int_{\partial\Omega} v\mathbf{n}\cdot\nabla u\,dA$$

and

$$\int_{\Omega} v\Delta u\,d\mathbf{x} - \int_{\Omega} u\Delta v\,d\mathbf{x} = \int_{\partial\Omega} v\mathbf{n}\cdot\nabla u\,dA - \int_{\partial\Omega} u\mathbf{n}\cdot\nabla v\,dA$$

Linear, nonlocal Dirichlet and Neumann problems

- In the linear case, the first nonlocal variational problem reduces to

seek $u \in V(\Omega)$ *such that*

$$u = h_d \quad \text{for } \mathbf{x} \in \Omega \setminus \widetilde{\Omega}$$

and

$$\int_\Omega \int_\Omega \beta \mathcal{G}(v) \mathcal{G}(u)\, d\mathbf{x}' d\mathbf{x} = \int_{\widetilde{\Omega}} v \int_\Omega b\, d\mathbf{x}' d\mathbf{x} \qquad \forall\, v \in V_0(\Omega)$$

and the corresponding "Dirichlet boundary" value problem reduces to the linear problem

$$\begin{aligned} -\mathcal{D}(\beta \mathcal{G}(u)) &= \int_\Omega b\, d\mathbf{x}' && \text{for } \mathbf{x} \in \widetilde{\Omega} \\ u &= h_d && \text{for } \mathbf{x} \in \Omega \setminus \widetilde{\Omega} \end{aligned}$$

where again the second equation is a "Dirichlet boundary" condition that is essential for the variational formulation

- Similarly, assuming b and h_n satisfy the compatibility condition, the second nonlocal variational problem reduces to

seek $u \in V(\Omega)$ *such that*

$$\int_{\Omega}\int_{\Omega} \beta\mathcal{G}(v)\mathcal{G}(u)\, d\mathbf{x}' d\mathbf{x} = \int_{\widetilde{\Omega}} v \int_{\Omega} b\, d\mathbf{x}' d\mathbf{x} + \int_{\Omega\setminus\widetilde{\Omega}} v \int_{\Omega} h_n\, d\mathbf{x}' d\mathbf{x} \qquad \forall\, v \in V(\Omega) \setminus \mathbb{R}$$

and the corresponding "Neumann boundary" value problem reduces to the linear problem

$$-\mathcal{D}(\beta\mathcal{G}(u)) = \int_{\Omega} b\, d\mathbf{x}' \qquad \text{for } \mathbf{x} \in \widetilde{\Omega}$$

$$\mathcal{N}(\beta\mathcal{G}(u))\alpha = \int_{\Omega} h_n\, d\mathbf{x}' \qquad \text{for } \mathbf{x} \in \Omega \setminus \widetilde{\Omega}$$

where again the second equation is a "Neumann boundary" condition that is natural for the variational formulation

- Substituting the definitions for $\mathcal{D}$ and $\mathcal{G}$ we have

$$\int_{\Omega}\int_{\Omega}\beta\mathcal{G}(v)\mathcal{G}(u)\,d\mathbf{x}'d\mathbf{x} = \int_{\Omega}\int_{\Omega}(v'-v)(u'-u)\alpha^2\beta\,d\mathbf{x}'d\mathbf{x}$$

$$-\mathcal{D}(\beta\mathcal{G}(u)) = 2\int_{\widetilde{\Omega}}(u'-u)\alpha^2\beta\,d\mathbf{x}' \qquad \mathbf{x}\in\widetilde{\Omega}$$

$$\mathcal{N}(\beta\mathcal{G}(u)) = -2\int_{\widetilde{\Omega}}(u'-u)\alpha^2\beta\,d\mathbf{x}' \qquad \mathbf{x}\in\Omega\setminus\widetilde{\Omega}$$

- The relation

$$\mathcal{K}(u, u'; \mathbf{x}, \mathbf{x}') = \beta \mathcal{G}(u)$$

  is a "constitutive" relation

- To define a general form of the constitutive function β, we let

  $\gamma(\mathbf{x}, \mathbf{x}') : \Omega \times \Omega \to \mathbb{R}$ denote a symmetric function

  $\mathbf{K}(\mathbf{x}, \mathbf{x}') : \Omega \times \Omega \to \mathbb{R}^{d \times d}$ a symmetric positive definite tensor such that $\mathbf{K}_{ij}(\mathbf{x}, \mathbf{x}') = \mathbf{K}_{ij}(\mathbf{x}', \mathbf{x})$ for all $i, j = 1, \ldots, d$

- Then, a general constitutive function β is given by

$$\beta = \gamma \left(\mathbf{x}' - \mathbf{x}\right) \cdot \mathbf{K} \cdot \left(\mathbf{x}' - \mathbf{x}\right)$$

Nonlocal Green's functions and a nonlocal Green's third identity

- For each $\mathbf{y}$ in Ω, define the nonlocal Green's function $g(\cdot;\mathbf{y}) : \Omega \longrightarrow \mathbb{R}$ as the solution of

$$\mathcal{D}(\beta\mathcal{G}(g(\mathbf{x};\mathbf{y}))) = \delta(\mathbf{x}-\mathbf{y}) \qquad \text{for } \mathbf{x}\in\Omega$$

  where $\delta(\cdot)$ denotes the Dirac delta function

- For each $y \in \Omega$, set $v(\cdot) = g(\cdot;\mathbf{y})$ in the second Green's identity so that we obtain the nonlocal Green's third identity

$$\int_{\widetilde{\Omega}} g\mathcal{D}(\beta\mathcal{G}(u))\,d\mathbf{x} - u(\mathbf{y}) = \int_{\Omega\setminus\widetilde{\Omega}} \Big(g\mathcal{N}(\beta\mathcal{G}(u)) - u\mathcal{N}(\beta\mathcal{G}(g))\Big)\,d\mathbf{x}$$

– this is analogous to a (generalization) of the classical Green's third identity

$$\int_\Omega g\Delta u\, d\mathbf{x} - u(\mathbf{y}) = \int_{\partial\Omega} (g\mathbf{n}\cdot\nabla u - u\mathbf{n}\cdot\nabla g)\, dA$$

- Using Fourier transforms, we have also identified fundamental solutions for the nonlocal operator $\mathcal{D}(\beta\mathcal{G}(\cdot))$

- Now, assume that, for each $\mathbf{y}\in\Omega$, the Green's function $g(\cdot,\mathbf{y})$ satisfies the homogeneous "Dirichlet boundary" condition $g(\mathbf{x},\mathbf{y}) = 0$ for all $\mathbf{x}\in\Omega\setminus\widetilde{\Omega}$

  – then we have

$$u(\mathbf{y}) = -\int_{\widetilde{\Omega}} g\int_\Omega b\, d\mathbf{x}' d\mathbf{x} + \int_{\Omega\setminus\widetilde{\Omega}} h_d\mathcal{N}(\beta\mathcal{G}(g))\, d\mathbf{x} \qquad \text{for} \quad \mathbf{y}\in\widetilde{\Omega}$$

  is the solution of the "Dirichlet boundary" value problem

- If, instead, assume that, for each $\mathbf{y} \in \Omega$, the Green's function $g(\cdot, \mathbf{y})$ satisfies the homogeneous "Neumann boundary" condition $\mathcal{N}(\beta \mathcal{G}(g)) = 0$

  - then we have

$$u(\mathbf{y}) = -\int_{\widetilde{\Omega}} g \int_{\Omega} b \, d\mathbf{x}' d\mathbf{x} - \int_{\Omega \setminus \widetilde{\Omega}} g \int_{\Omega} h_n \, d\mathbf{x}' d\mathbf{x} \qquad \text{for} \quad \mathbf{y} \in \Omega$$

    is the solution of the "Neumann boundary" value problem

- These are analogous to the classical formulas for solutions of boundary-value problems in terms of Green's functions

$$u(\mathbf{y}) = -\int_{\Omega} g b \, d\mathbf{x} + \int_{\partial\Omega} h_d \mathbf{n} \cdot \nabla g \, dA$$

$$u(\mathbf{y}) = -\int_{\Omega} g b \, d\mathbf{x} - \int_{\partial\Omega} g h_n \, dA$$

LOCAL SMOOTH LIMITS

- We connect the linear nonlocal "boundary" value problems to the classical Dirichlet and Neumann problems for second-order elliptic partial differential equations

- To do so, we make two assumptions
  - solutions of the nonlocal "boundary" value problems are smooth
  - operators are asymptotically local

- These assumptions are made only to make the connection to classical problems for PDEs
  - they are not required for the well posedness of the nonlocal "boundary" value problems

- In addition, the nonlocal “boundary” value problems admit solutions that are not solutions, even in the usual sense of weak solutions, of the PDEs
  - thus, one can view solutions of the nonlocal “boundary” value problems as further generalizations of solutions of the PDEs
    - they are nonlocal
    - they lack the smoothness needed to be standard weak solutions of the PDEs

- Assume that $\mathbf{K}$, α, and γ are radial functions, e.g.,

$$\mathbf{K}_{ij}(\mathbf{x}, \mathbf{x}') = \mathbf{K}_{ij}(\mathbf{x}' - \mathbf{x}), \qquad i, j = 1, \ldots, d$$

- Assume that the radial function $\gamma(\mathbf{x}' - \mathbf{x})$ satisfies, for a specified $\varepsilon > 0$

$$\gamma_\varepsilon(\mathbf{x}' - \mathbf{x}) = 0 \qquad |\mathbf{x}' - \mathbf{x}| \geq \varepsilon$$

- Then, the "constitutive" function β_ε is the radial function given by

$$\beta_\varepsilon = \gamma_\varepsilon\left(\mathbf{x}' - \mathbf{x}\right) \cdot \mathbf{K} \cdot \left(\mathbf{x}' - \mathbf{x}\right)$$

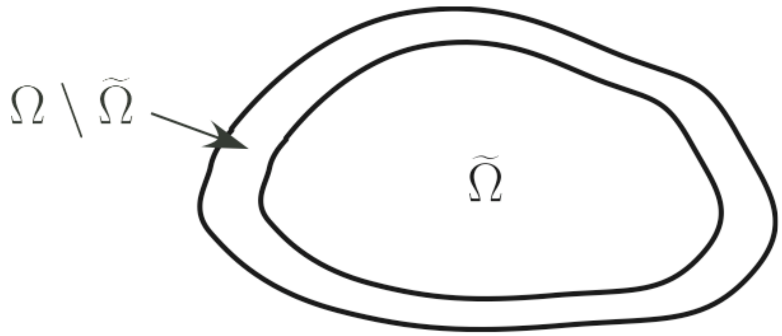
- Similarly, we assume that the data b and h_n are localized, i.e.,

$$b = b_\varepsilon \qquad \text{and} \qquad h_n = h_{n,\varepsilon}$$

where $b_\varepsilon(\cdot, \mathbf{x}')$ and $h_{n,\varepsilon}(\cdot, \mathbf{x}')$ vanish outside of $B_\varepsilon(\cdot)$ with

$B_\varepsilon(\mathbf{x})$ denoting the ball of radius ε centered at $\mathbf{x}$

- Referring to the sketch



we assume that the thickness of the "boundary" domain $\Omega \setminus \widetilde{\Omega}$ if of $O(\varepsilon)$ so that

$$|\Omega| - |\widetilde{\Omega}| = O(\varepsilon)$$

where $|\cdot|$ denotes the volume

- For $\mathbf{x} \in \Omega$, let

$$\Omega_\varepsilon(\mathbf{x}) = B_\varepsilon(\mathbf{x}) \cap \Omega$$

- We further assume that

test and trial functions $u \in U(\Omega)$ and $v \in U(\Omega)$ are smooth

- Note that no assumptions are made about the smoothness of the functions α, γ_ε, and the elements of the matrix function $\mathbf{K}$

- We do assume that all integrals encountered are well defined

- We then have that

$$\mathcal{G}(v) = \big(v' - v\big)\alpha = \Big((\mathbf{x}' - \mathbf{x}) \cdot \nabla v(\mathbf{x}) + O(\varepsilon^2)\Big)\alpha \qquad \forall\, \mathbf{x}' \in \Omega_\varepsilon(\mathbf{x}),$$

- It can then be shown that

$$\int_\Omega \int_\Omega \beta_\varepsilon \mathcal{G}(v)\mathcal{G}(u)\, d\mathbf{x}' d\mathbf{x} = \int_\Omega \nabla v \cdot \big(\mathbf{D}_\varepsilon \nabla u\big)\, d\mathbf{x} + \text{h.o.t.}$$

where the second-order tensor $\mathbf{D}_\varepsilon$ is given by

$$\mathbf{D}_\varepsilon(\mathbf{x}) = \int_{\Omega_\varepsilon(x)} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})\, \mathbf{K}\, (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) \gamma_\varepsilon \alpha^2\, d\mathbf{x}'$$

- Using Noll's results, it can be shown that

$$\mathcal{D}\big(\beta_\varepsilon \mathcal{G}(u)\big) = \nabla \cdot \big(\mathbf{D}_\varepsilon \cdot \nabla u\big) + \text{h.o.t.}$$

and

$$\int_{\partial\Omega} v\big(\mathbf{D}_\varepsilon \cdot \nabla u\big) \cdot \mathbf{n}\, dA = \int_{\Omega\setminus\widetilde{\Omega}} v\mathcal{N}\big(\beta_\varepsilon \mathcal{G}(u)\big)\, d\mathbf{x} + \text{h.o.t.}$$

- Let

$$\mathbf{D} = \lim_{\varepsilon\to 0} \mathbf{D}_\varepsilon$$

  – note that for $\mathbf{D}_\varepsilon$ to not vanish as $\varepsilon \to 0$, we must have that the integrand in its definition is singular at $\mathbf{x}' = \mathbf{x}$

- Then

$$\lim_{\varepsilon\to 0}\int_\Omega\int_\Omega \beta_\varepsilon \mathcal{G}(v)\mathcal{G}(u)\,d\mathbf{x}'d\mathbf{x} = \int_\Omega \nabla v\cdot\mathbf{D}\cdot\nabla u\,d\mathbf{x}$$

$$\lim_{\varepsilon\to 0}\mathcal{D}\big(\beta_\varepsilon\mathcal{G}(u)\big) = \nabla\cdot\big(\mathbf{D}\cdot\nabla u\big)$$

$$\lim_{\varepsilon\to 0}\int_{\Omega\setminus\widetilde{\Omega}} v\mathcal{N}\big(\beta_\varepsilon\mathcal{G}(u)\big)\,d\mathbf{x} = \int_{\partial\Omega} v\big(\mathbf{D}\cdot\nabla u\big)\cdot\mathbf{n}\,dA$$

- These results easily imply that the nonlocal Green's first and second identities reduce to the corresponding classical Green's identities

- Moreover, the nonlocal "Dirichlet" variational problem reduces to

$$\begin{cases} \displaystyle\int_\Omega \nabla v \cdot \mathbf{D} \cdot \nabla u = \int_\Omega v\widehat{b}\, d\mathbf{x} & \text{in } \Omega \\ u = h_d & \text{on } \partial\Omega \end{cases}$$

where

$$\widehat{b} = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon(x)} b_\varepsilon \, d\mathbf{x}'$$

which, of course, corresponds to the classical Dirichlet problem

$$\begin{cases} -\nabla \cdot \big(\mathbf{D} \cdot \nabla u\big) = \widehat{b} & \text{in } \Omega \\ u = h_d & \text{on } \partial\Omega. \end{cases}$$

Similarly, the nonlocal "Neumann" variational principle reduces to

$$\int_\Omega \nabla v \cdot \mathbf{D} \cdot \nabla u = \int_\Omega v\widehat{b}\, d\mathbf{x} + \int_{\partial\Omega} v\widehat{h}_n\, dA \qquad \text{in } \Omega$$

where

$$\widehat{h}_n = \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon(x)} h_{n,\varepsilon}\, d\mathbf{x}'$$

which, of course, corresponds to the classical Neumann problem

$$\begin{cases} -\nabla \cdot \big(\mathbf{D} \cdot \nabla u\big) = \widehat{b} & \text{in } \Omega \\ \big(\mathbf{D} \cdot \nabla u\big) \cdot \mathbf{n} = \widehat{h}_n & \text{on } \partial\Omega \end{cases}$$

WELL POSEDENESS OF NONLOCAL LINEAR BOUNDARY-VALUE PROBLEMS

- We restrict attention to the case $U(\Omega) = V(\Omega)$ and consider the linear nonlocal variational problems

  – the "Dirichlet" problem has the form

  $$\int_{\Omega}\int_{\Omega}(v'-v)(u'-u)\alpha^2\beta\,d\mathbf{x}'d\mathbf{x} = \int_{\widetilde{\Omega}} v \int_{\Omega} b\,d\mathbf{x}'d\mathbf{x} \qquad \forall\, v \in V_0(\Omega)$$

  where we assume a homogeneous "boundary" condition so that $u \in V_0(\Omega)$

  – the "Neumann" problem has the form

  $$\int_{\Omega}\int_{\Omega}(v'-v)(u'-u)\alpha^2\beta\,d\mathbf{x}'d\mathbf{x}$$
  $$= \int_{\widetilde{\Omega}} v \int_{\Omega} b\,d\mathbf{x}'d\mathbf{x} + \int_{\Omega\setminus\widetilde{\Omega}} v \int_{\Omega} h_n\,d\mathbf{x}'d\mathbf{x} \qquad \forall\, v \in V(\Omega)\setminus\mathbb{R}$$

  where $u \in V(\Omega)\setminus\mathbb{R}$

Bilinear forms, norms, and inner products

- Define the symmetric bilinear form

$$B(u,v) = \int_\Omega \int_\Omega (v' - v)(u' - u)\beta\alpha^2 \, d\mathbf{x}' d\mathbf{x} \qquad \forall\, u, v \in V(\Omega)$$

  - ssume that $\beta(\mathbf{x}, \mathbf{x}') > 0$ for all $\mathbf{x}, \mathbf{x}' \in \Omega$
  - then $B(u,u) \geq 0$

- Let

$$((u,v)) = B(u,v) \qquad\qquad |||u||| = \big(B(u,u)\big)^{1/2}$$

$$V(\Omega) = \{u \; : \; |||u||| < \infty\}$$

- We show that $|||\cdot|||$ and $((\cdot,\cdot))$ define a norm and an inner product, respectively, on both $V_0(\Omega)$ and $V(\Omega)\setminus\mathbb{R}$
  - note that $|||\cdot|||$ only defines a semi-norm on $V(\Omega)$

- Let $\Omega\setminus\widetilde{\Omega}\subset\Omega$ have finite measure and let $u\in V_0(\Omega)$ so that

$$u(\mathbf{x})=0 \quad \forall\,\mathbf{x}\in\Omega\setminus\widetilde{\Omega}$$

  - then, it is easily shown that

$$B(u,u)\geq \int_{\widetilde{\Omega}} u^2\left(\int_{\Omega\setminus\widetilde{\Omega}}\beta\alpha^2\,d\mathbf{x}'\right)d\mathbf{x}$$

  - note that

$$0<\int_{\Omega\setminus\widetilde{\Omega}}\beta\alpha^2\,d\mathbf{x}' \qquad \forall\,\mathbf{x}\in\widetilde{\Omega}$$

– then

$$B(u,u) = 0 \qquad \Longrightarrow \qquad u = 0 \quad \forall\, \mathbf{x} \in \widetilde{\Omega}$$

– but, $u = 0$ in $\Omega \setminus \widetilde{\Omega}$ as well so that

$$B(u,u) = 0 \qquad \Longrightarrow \qquad u = 0 \quad \forall\, \mathbf{x} \in \Omega$$

– thus, we have that $|||\cdot|||$ defines a norm and $((\cdot,\cdot))$ defines and inner product on $V_0(\Omega) \subset V(\Omega)$

- Note that we have assumed that

$$\int_{\Omega\setminus\widetilde{\Omega}} \beta\alpha^2\, d\mathbf{x}' < \infty \qquad \forall\, \mathbf{x} \in \widetilde{\Omega}$$

- Also, note that

$$B(u,u) = 0 \quad \text{only if} \quad (u' - u)^2 \beta \alpha^2 = 0 \quad \forall\, \mathbf{x}, \mathbf{x}' \in \Omega$$

i.e.,

$$\text{only if} \quad u = \text{constant} \quad \forall\, \mathbf{x} \in \Omega$$

  - thus, we again conclude that $|||\cdot|||$ defines a norm and $((\cdot,\cdot))$ defines and inner product on $V(\Omega) \setminus \mathbb{R} \subset V(\Omega)$

Decomposition of the solution space

- Let $S(\Omega)$ denote the functions $u \in V(\Omega)$ that satisfy

$$-\mathcal{D}\big(\mathcal{G}(u)\big) = \int_{\Omega} (u' - u)\beta\alpha^2 \, d\mathbf{x}' = 0 \qquad \forall\, \mathbf{x} \in \widetilde{\Omega}.$$

  – then, from the nonlocal Green's first identity, we have that

$$\int_{\Omega}\int_{\Omega} \mathcal{G}(v)\mathcal{G}(u) \, d\mathbf{x}' d\mathbf{x} = 0 \qquad \forall\, u \in S, \;\; v \in V_0(\Omega)$$

  so that

$$\begin{aligned} B(u,v) = ((u,v)) &= \int_{\Omega}\int_{\Omega} (v' - v)(u' - u)\beta\alpha^2 \, d\mathbf{x}' d\mathbf{x} \\ &= \int_{\Omega}\int_{\Omega} \mathcal{G}(v)\mathcal{G}(u) \, d\mathbf{x}' d\mathbf{x} = 0 \qquad \forall\, u \in S, \;\; v \in V_0(\Omega) \end{aligned}$$

- Thus, we conclude that

$$V(\Omega) = V_0(\Omega) \oplus S(\Omega)$$

  - any function in $V(\Omega)$ can be written as a sum of two functions that are orthogonal with respect to the inner product $((\cdot,\cdot))$
    - the first a function that vanishes on $\Omega \setminus \widetilde{\Omega}$
    - the second a "harmonic" function, i.e., a function $u \in S(\Omega)$

- Of course, this is entirely analogous to the decomposition of the Sobolev space $H^1(\Omega)$ into functions belonging to $H_0^1(\Omega)$ and harmonic functions

Nonlocal dual and nonlocal trace spaces

- Let

$$|||b|||_* = \sup_{v \in V_0(\Omega)} \frac{\displaystyle\int_{\widetilde{\Omega}} v(\mathbf{x}) \int_{\Omega} b(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}' d\mathbf{x}}{|||v|||}$$

- Define the "dual" space

$$V_0^*(\Omega) = \{ b \;\; : \;\; |||b|||_* < \infty \}$$

- Define the "trace" space

$$V_d = \{ \chi_{\Omega \setminus \widetilde{\Omega}} u \;\; : \;\; u \in V(\Omega) \}$$

where $\chi_{(\cdot)}$ denotes the characteristic function, along with the norm

$$|||u|||_d = |||\chi_{\Omega \setminus \widetilde{\Omega}} u|||$$

- Finally, define the norm

$$|||h|||_n = \sup_{v \in V_d} \frac{\displaystyle\int_{\Omega\setminus\widetilde{\Omega}} v(\mathbf{x}) \int_{\Omega} h(\mathbf{x},\mathbf{x}')\, d\mathbf{x}' d\mathbf{x}}{|||v|||_d}$$

and the second "trace" space

$$V_n = \{h \;\;:\;\; |||h|||_n < \infty\}$$

Well-posedness of variational problems

- The linear nonlocal variational problems take the form of the homogeneous "Dirichlet" problem

$$\begin{cases} \textit{given } b \in V_0^* \textit{ and } h_d \in V_d, \textit{ seek } u \in V_0(\Omega) \textit{ such that} \\ \qquad B(u,v) = F_d(v) \qquad \forall\, v \in V_0(\Omega) \end{cases}$$

and the "Neumann" problem

$$\begin{cases} \textit{given } b \in V^* \textit{ and } h_n \in V_n, \textit{ seek } u \in V(\Omega) \setminus \mathbb{R} \textit{ such that} \\ \qquad B(u,v) = F_n(v) \qquad \forall\, v \in V(\Omega) \setminus \mathbb{R} \end{cases}$$

- The linear functionals $F_d(\cdot)$ and $F_n(\cdot)$ are defined by

$$F_d(v) = \int_{\widetilde{\Omega}} v \int_{\Omega} b \, d\mathbf{x}' d\mathbf{x} \qquad \forall\, v \in V_0(\Omega)$$

and

$$F_n(v) = \int_{\widetilde{\Omega}} v \int_{\Omega} b \, d\mathbf{x}' d\mathbf{x} + \int_{\Omega\setminus\widetilde{\Omega}} v \int_{\Omega} h_n \, d\mathbf{x}' d\mathbf{x} \qquad \forall\, v \in V(\Omega) \setminus \mathbb{R}$$

- Because $B(\cdot,\cdot)$ defines an inner product on $V_0(\Omega)$ and $V(\Omega) \setminus \mathbb{R}$, it is continuous and coercive on those spaces

- Then, if we assume that the data is such that the functionals $F_d(\cdot)$ and $F_n(\cdot)$ are continuous, then, the Lax-Milgram theorem can be applied to show that both the nonlocal Dirichlet and Neumann problems have unique solutions and, moreover, those solutions satisfy

$$|||u||| \leq |||b|||_* \qquad \text{and} \qquad |||u||| \leq |||b|||_* + |||h_n|||_n$$

GENERAL "SECOND-ORDER ELLIPTIC" PROBLEMS

- The nonlocal variational problems and the corresponding nonlocal "boundary" value problems mimic the classical setting described by Poisson type equations

- Nonlocal analogs of more general second-order elliptic boundary value problems can also be defined

- For example, consider the nonlocal variational principle

$$\begin{cases} \textit{seek } u \in V(\Omega) \textit{ such that} \\[2ex] \qquad u = h_d \quad \text{for } \mathbf{x} \in \Omega \setminus \widetilde{\Omega} \\[2ex] \textit{and} \\[2ex] \qquad \displaystyle\int_\Omega \int_\Omega \beta \mathcal{G}(v)\mathcal{G}(u)\, d\mathbf{x}' d\mathbf{x} + \int_\Omega v \int_\Omega \sigma \mathcal{G}(u)\, d\mathbf{x}' d\mathbf{x} \\[2ex] \qquad\qquad \displaystyle + \int_\Omega v \int_\Omega \omega(u' + u)\, d\mathbf{x}' d\mathbf{x} = \int_{\widetilde{\Omega}} v \int_\Omega b\, d\mathbf{x}' d\mathbf{x} \qquad \forall\, v \in V_0(\Omega) \end{cases}$$

where

$\sigma(\mathbf{x}, \mathbf{x}')$ is a skew-symmetric function

$\omega(\mathbf{x}, \mathbf{x}')$ is a symmetric function

- The corresponding nonlocal "Dirichlet" boundary-value problem is given by

$$\begin{cases} -\mathcal{D}(\beta\mathcal{G}(u)) + \sigma\mathcal{G}(u) + \omega(u' + u) = \int_{\Omega} b\, d\mathbf{x}' & \text{for } \mathbf{x} \in \widetilde{\Omega} \\ u = h_d & \text{for } \mathbf{x} \in \Omega \setminus \widetilde{\Omega} \end{cases}$$

- General problems may be defined by letting

  $\mathbf{a}(\mathbf{x}, \mathbf{x}')$ be a symmetric vector-valued function

  $\xi(\mathbf{x}, \mathbf{x}')$, $\eta(\mathbf{x}, \mathbf{x}')$, and $r(\mathbf{x}, \mathbf{x}')$ be symmetric functions

  and then setting

$$\beta = ((\mathbf{x}' - \mathbf{x}) \cdot \mathbf{K} \cdot (\mathbf{x}' - \mathbf{x})$$

$$\sigma = \xi\mathbf{a} \cdot (\mathbf{x}' - \mathbf{x})$$

$$\omega = \eta r$$

- For smooth solutions and asymptotically local operators,
  - let $\mathbf{D}$ and $\widehat{b}$ be as before
  - analogously, let

$$\mathbf{w}(\mathbf{x}) = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon(x)} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) \cdot \mathbf{a}\, \xi_\varepsilon \alpha \, d\mathbf{x}'$$

and

$$c(\mathbf{x}) = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon(x)} r \eta_\varepsilon \, d\mathbf{x}'$$

- Then, the nonlocal "boundary-value" problem reduces to the classical linear convection–diffusion–reaction problem

$$-\nabla \cdot (\mathbf{D} \cdot \nabla u) + \mathbf{w} \cdot \nabla u + cu = \widehat{b}$$

along with a Dirichlet boundary condition

CURRENT WORK

- Develop functional analytic characterizations of the solution, trace, and data spaces used
- Develop the equivalent multidomain formulations for the nonlocal boundary-value problems and applying them to “interface” problems
- Develop and analyze finite element discretization methods, including discontinuous Galerkin methods, for the nonlocal variational problems
- Extend the nonlocal vector calculus to vector-valued functions and develop nonlocal variational problems and the corresponding nonlocal “boundary” value problems for vector-valued functions
  - of particular interest is the application of the nonlocal vector calculus to the peridynamic model for materials