EIGENVALUE PROBLEM OF MAGNETIC SCHRÖDINGER OPERATORS

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§1. Lowest Eigenvalue $\mu(\mathbf{A})$

Given a vector field $\mathbf{A}$, let $\mu(\mathbf{A})$ denote the lowest eigenvalue of the following problem :

$$-\nabla_{\mathbf{A}}^2\psi = \mu\psi \quad \text{in } \Omega, \qquad (\nabla_{\mathbf{A}}\psi)\cdot\nu = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where ν is the unit outer normal to $\partial\Omega$.

$$\nabla_{\mathbf{A}}\psi = \nabla\psi - i\mathbf{A}\psi,$$

$$\nabla_{\mathbf{A}}^2\psi = \Delta\psi - i[2\mathbf{A}\cdot\nabla\psi + \psi\,\mathrm{div}\,\mathbf{A}] - |\mathbf{A}|^2\psi.$$

Variational characterization

$$\mu(\mathbf{A}) = \inf_{\phi \in H^1(\Omega,\mathbb{C})} \frac{\int_\Omega |\nabla_{\mathbf{A}}\phi|^2 dx}{\|\phi\|^2_{L^2(\Omega)}}.$$

Gauge invariance

$$\nabla_{\mathbf{A}+\nabla\chi}(e^{i\chi}\psi) = e^{i\chi}\nabla_{\mathbf{A}}\psi,$$

$$\mu(\mathbf{A}+\nabla\chi) = \mu(\mathbf{A}).$$

For a bounded domain Ω, $\mathbf{A}$ has decomposition

$$\mathbf{A} = \mathcal{A} + \nabla\chi,$$

where

$$\operatorname{div}\mathcal{A} = 0 \text{ in } \Omega, \quad \mathcal{A}\cdot\nu = 0 \text{ on } \partial\Omega.$$

In fact we may choose χ so that

$$\Delta\chi = \operatorname{div}\mathbf{A} \text{ in } \Omega, \quad \frac{\partial\chi}{\partial\nu} = \mathbf{A}\cdot\nu \text{ on } \partial\Omega.$$

Then $\mathcal{A} = \mathbf{A} - \nabla\chi$ satisfies the requirement.

Physical Motivation

1. Ginzburg-Landau energy for superconductors

$$\int_\Omega \{|\nabla_{\kappa\mathcal{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\}dx + \kappa^2\int_{\mathbb{R}^3}|\operatorname{curl}\mathcal{A} - \mathcal{H}|^2dx. \quad (2)$$

$|\psi|^2 \sim$ density of superconducting electron pairs,

Superconducting state: $\psi \not\equiv 0$.

$\mathcal{H}$: applied magnetic field, κ: GL parameter,

$\kappa = \frac{\lambda}{\xi}$, λ: penetration depth, ξ: coherence length.

Consider a superconductor occupying a bounded and simply-connected domain Ω in $\mathbb{R}^3$ and subjected to an applied magnetic field $\mathcal{H}$. Assume $\mathcal{H} = \sigma\mathbf{h}$, where $\sigma > 0$ is a constant, and $\mathbf{h}$ is a unit vector. Set $\mathcal{A} = \sigma\mathbf{A}$. (2) can be written as

$$\begin{aligned}&\mathcal{G}[\psi, \mathbf{A}]\\ =& \int_\Omega \{|\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\}dx + \kappa^2\sigma^2\int_{\mathbb{R}^3}|\mathrm{curl}\,\mathbf{A} - \mathbf{h}|^2 dx.\end{aligned}$$

The GL functional $\mathcal{G}$ has gauge invariance.

Normal state

$\mathcal{G}$ has a trivial critical point $(0, \mathbf{F_h})$, which describes the normal state, where $\mathbf{F_h}$ is a vector field satisfying the conditions

$$\operatorname{curl} \mathbf{F_h} = \mathbf{h}, \quad \operatorname{div} \mathbf{F_h} = 0.$$

Upper critical field H_{c_3} (Physica D 1999, JDE 2000).

$$H_{c_3}(\kappa, \mathbf{h}) = \inf\{\sigma > 0 : \ (0, \mathbf{F_h}) \text{ is a global minimizer of } \mathcal{G}\}.$$

2-dimensional superconductors

If a superconductor occupies a cylinder of infinite height with cross section $\Omega \subset \mathbb{R}^2$, and if the applied magnetic field is parallel to the axis of the cylinder, one may consider ψ and $\mathbf{A} = (A_1, A_2)$ to be defined on Ω, and the GL energy is reduced to:

$$\mathcal{G}[\psi, \mathbf{A}] = \int_\Omega \{|\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + \kappa^2\sigma^2|\operatorname{curl}\mathbf{A} - 1|^2\}dx.$$

We look for minimizers of $\mathcal{G}$ in $H^1(\Omega, \mathbb{C}) \times H_n(\Omega, \operatorname{div} 0)$, where

$$H_n(\Omega, \operatorname{div} 0) = \{\mathbf{A} \in H^1(\Omega, \mathbb{R}^2) : \ \operatorname{div}\mathbf{A}\Big|_\Omega = 0, \ \mathbf{A}\cdot\nu\Big|_{\partial\Omega} = 0\}.$$

For $\mathbf{A} = (A_1, A_2)$,

$$\operatorname{curl} \mathbf{A} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2},$$

$$\operatorname{curl}^2 \mathbf{A} = (\partial_2 \operatorname{curl} \mathbf{A}, -\partial_1 \operatorname{curl} \mathbf{A}).$$

The trivial critical point is $(0, \mathbf{F})$, where $\mathbf{F}$ satisfies

$$\operatorname{curl} \mathbf{F} = 1 \quad \text{and} \quad \operatorname{div} \mathbf{F} = 0 \quad \text{in } \Omega, \quad \nu \cdot \mathbf{F} = 0 \quad \text{on } \partial\Omega.$$

In this case the critical value $H_{c_3}(\kappa)$ can be defined similarly.

Eigenvalue problem and H_{c3}

Lemma. *If* $\mu(\sigma\kappa\mathbf{F_h}) < \kappa^2$ *(resp.* $\mu(\sigma\kappa\mathbf{F}) < \kappa^2$*) then the functional* $\mathcal{G}$ *has a non-trivial global minimizer.*

On the other hand, if $\mathcal{G}$ *has a non-trivial global minimizer* $(\psi, \mathbf{A})$ *then* $\mu(\sigma\kappa\mathbf{A}) < \kappa^2$*.*

The value of σ such that $\mu(\sigma\kappa\mathbf{F_h}) = \kappa^2$ provides information of H_{c3}.

2. Landau-de Gennes energy of liquid crystals

$$\mathcal{E}[\psi, \mathbf{n}] =$$

$$\int_\Omega \Big\{ |\nabla_{q\mathbf{n}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + K_1|\mathrm{div}\ \mathbf{n}|^2 + K_2|\mathbf{n}\cdot \mathrm{curl}\ \mathbf{n} + \tau|^2$$

$$+ K_3|\mathbf{n}\times \mathrm{curl}\,\mathbf{n}|^2 + (K_2+K_4)[\mathrm{tr}(\nabla\mathbf{n})^2 - (\mathrm{div}\,\mathbf{n})^2]\Big\} dx.$$

$|\psi|^2$: $\sim$ intensity of the smectic layering,

$\mathbf{n} : \bar{\Omega} \to \mathbb{S}^2$, director field,

q: wave number, intensity of layering of smectics,

τ: chiral constant,

$q\tau$: joint chirality constant,

K_j: elastic coefficients,

K_1 (splay), K_2 (twist), K_3 (bend) are positive,

$K_2 + K_4$ (saddle-splay),

κ: $= \sqrt{\frac{|r|}{c}}$,

r is a constant in the smectic energy density and c: coupling constant.

Nematic state: Molecules are aligned parallel to a particular direction indicated by *director* $\mathbf{n}$.

Smectic state: layered structure. Inside these layers, molecules tend to arrange themselves in the same direction; but they cannot move freely between the layers.

Smectic state: $\psi \not\equiv 0$.

Critical wave number (CMP 2003)

$$Q_{c_3}(K_1, K_2, \kappa, \tau) = \inf\{q > 0 : \ \mathcal{E} \text{ has only trivial minimizers}\},$$

$$Q_{c_3}(\kappa, \tau) = \inf_{K_1, K_2 > 0} Q_{c_3}(K_1, K_2, \kappa, \tau).$$

Observation: Q_{c_3} is an analogy of H_{C_3} and H_c.

Conjecture (CMP 2003). *Surface smectic state exists, which is an analogy of surface superconducting state.*

Related works

S. J. Chapman, P. Bauman, D. Phillips, Q. Tang, A. Bernoff, P. Sternberg, T. Giorgi, H. Jadallah, C. Bolley, B. Helffer, H. Morame, S. Fournais, V, Bonnaillie,

and many more.

§2. 2-dimensional Problem

Ω is a bounded, simply-connected domain in $\mathbb{R}^2$ with smooth boundary. Let us begin with the leading term estimate.

$$\omega = \frac{1}{2}(-x_2, x_1), \quad \operatorname{curl}\omega = 1,$$

$$\beta_0 = \text{the lowest eigenvalue of } -\nabla_\omega^2 \text{ on } \mathbb{R}^2_+, \quad 0 < \beta_0 < 1,$$

$$\alpha_0(\operatorname{curl}\mathbf{A}) = \min\Big\{ \inf_{x\in\Omega} |\operatorname{curl}\mathbf{A}(x)|, \quad \beta_0 \inf_{x\in\partial\Omega} |\operatorname{curl}\mathbf{A}(x)| \Big\}.$$

Theorem 1 (Lu-P). *Assume* $\operatorname{curl}\mathbf{A} \in C^{\alpha}(\bar{\Omega})$, $0 < \alpha < 1$. *Then*

$$\lim_{b \to +\infty} \frac{\mu(b\mathbf{A})}{b} = \alpha_0(\operatorname{curl}\mathbf{A}).$$

As $b \to +\infty$, *the eigenfunctions concentrate at* $\Omega_m \cup (\partial\Omega)_m$.

$$\Omega_m = \{x \in \Omega : \ |\operatorname{curl}\mathbf{A}(x)| = \inf_{y \in \Omega} |\operatorname{curl}\mathbf{A}(y)|\},$$

$$(\partial\Omega)_m = \{x \in \partial\Omega : \ |\operatorname{curl}\mathbf{A}(x)| = \inf_{y \in \partial\Omega} |\operatorname{curl}\mathbf{A}(y)|\}.$$

Case 1. $\operatorname{curl}\mathbf{A}$ **is constant**

Theorem 2 (Lu-P). *Assume* $\operatorname{curl}\mathbf{A} = 1$. *Then*

$$\lim_{b\to+\infty} \frac{\mu(b\mathbf{A})}{b} = \beta_0,$$

and the eigenfunctions concentrate at the boundary $\partial\Omega$ *as* $b \to +\infty$.

In order to prove the above results, one may use blow-up technique and then classify the solutions of limiting equations. Let us consider the case where

$$\inf_{x\in\bar{\Omega}} |\mathrm{curl}\,(x)| > 0.$$

Decomposition of vector fields

Let $\mathbf{A} \in C^{2+\alpha}(\bar{B}_R, \mathbb{R}^2)$. Then there exists $\chi \in C^{\infty}(\bar{B}_R)$ s.t.

$$\mathbf{A}(x) = \mathbf{A}(0) + \nabla\chi(x) + \mathrm{curl}\,\mathbf{A}(0)\omega(x) - \frac{1}{2}|x|^2\mathrm{curl}\,^2\mathbf{A}(0) + \mathbf{D}(x),$$

where $|\mathbf{D}(x)| = O(|x|^{2+\alpha})$.

Interior blowing up

Let $b = 1/\varepsilon^2$, $x = \varepsilon y$. Assume $\mathbf{A} \in C^2$. Then

$$\mathbf{A}(\varepsilon y) = \mathbf{A}(0) + (\nabla\chi)(\varepsilon y) + \varepsilon\, H\omega(y) + O(\varepsilon^2|y|^2),$$

where $H = \operatorname{curl}\mathbf{A}(0)$. Given a function ψ with compact support in Ω, set

$$\psi_\varepsilon(y) = \psi(x),$$
$$\phi_\varepsilon(y) = \exp(-\frac{i}{\varepsilon}[\mathbf{A}(0)\cdot y + \frac{1}{\varepsilon}\chi(\varepsilon y)])\psi_\varepsilon(y).$$

$$\nabla_x \psi - \frac{i}{\varepsilon^2}\mathbf{A}(x)\psi(x) = \frac{1}{\varepsilon}[\nabla_y \psi_\varepsilon(y) - \frac{i}{\varepsilon}\mathbf{A}(\varepsilon y)\psi_\varepsilon(y)]$$
$$= \frac{1}{\varepsilon}\exp(\frac{i}{\varepsilon}[\mathbf{A}(0)\cdot y + \frac{1}{\varepsilon}\chi(\varepsilon y)])$$
$$\cdot \Big\{\nabla_y \phi_\varepsilon(y) - i[H\omega(y) + O(\varepsilon^2|y|^2)]\phi_\varepsilon(y)\Big\},$$

$$\int_\Omega |\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}}\psi|^2 dx = \int_{\frac{\Omega}{\varepsilon}} |\nabla_{H\omega(y)+O(\varepsilon^2|y|^2)}\phi_\varepsilon(y)|^2 dy$$
$$\sim \int_{\frac{\Omega}{\varepsilon}} |\nabla_{H\omega(y)}\phi_\varepsilon(y)|^2 dy,$$

$$\frac{\int_\Omega |\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}}\psi|^2 dx}{\int_\Omega |\psi|^2 dx} \sim \frac{1}{\varepsilon^2}\frac{\int_{\frac{\Omega}{\varepsilon}} |\nabla_{H\omega(y)}\phi_\varepsilon(y)|^2 dy}{\int_{\frac{\Omega}{\varepsilon}} |\phi|^2 dy}.$$

Limiting problem

$$\alpha(H\omega) = \inf_{\phi} \frac{\int_{\mathbb{R}^2} |\nabla_{H\omega}\phi|^2 dx}{\int_{\mathbb{R}^2} |\phi|^2 dy}.$$

We have $\alpha(H\omega) = |H|\alpha(\omega) = |H|\alpha(\mathbf{E})$, where $\mathbf{E} = (-x_2, 0)$ and $\omega = \mathbf{E} + \nabla(\frac{1}{2}x_1x_2)$. Let

$$\alpha_0 = \alpha(\mathbf{E}) = \inf_{\phi} \frac{\int_{\mathbb{R}^2} |\nabla_{\mathbf{E}}\phi|^2 dx}{\int_{\mathbb{R}^2} |\phi|^2 dy}.$$

α_0 is achieved and the minimizers satisfy

$$-\nabla^2_{\mathbf{E}}\psi \equiv -\Delta\phi - 2ix_2\partial_1\phi + |x_2|^2\phi = \alpha\psi \quad \text{in } \mathbb{R}^2. \tag{3}$$

Boundary blowing up

Limiting problem

$$\beta(H\omega) = \inf_{\phi} \frac{\int_{\mathbb{R}^2_+} |\nabla_{H\omega}\phi|^2 dx}{\int_{\mathbb{R}^2_+} |\phi|^2 dy}.$$

We have $\beta(H\omega) = |H|\beta(\omega) = |H|\beta(\mathbf{E})$. Let $\beta_0 = \beta(\mathbf{E})$. β_0 is not achieved and the equation for bounded eigenfunctions is

$$\begin{cases} -\Delta\phi - 2ix_2\partial_1\phi + |x_2|^2\phi = \beta\phi \quad \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial\phi}{\partial x_2} = 0 \quad \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{4}$$

Eigenvalue problem in $\mathbb{R}^2$

Lemma 1. *(i) The eigenvalues of (3) are* $\alpha_n = 2n+1,\ n = 0, 1, 2, \cdots$.

(ii) The L^2 *eigenspace associated with* $\alpha_0 = 1$ *is given by*

$$\{f(x)e^{-r^2/2} \in L^2(\mathbb{R}^2) : \ f \textit{ is an entire function}\}.$$

Proof of (i). Formally make Fourier transform in x_1 and let

$$f(z, x_2) = \mathcal{F}_{x_1}[\phi(\cdot, x_2)].$$

Then we get

$$-\frac{d^2 f}{dx_2^2} + (z + x_2)^2 f = \alpha f, \quad f(\pm\infty) = 0,$$

Let $f(z,t) = y(z, z+t)$. We have

$$-y'' + t^2 y = \alpha y, \quad y(\pm\infty) = 0.$$

It has eigenvalues $\alpha = 2n+1$, $n = 0, 1, 2, \cdots$, with associated eigenfunctions $e^{-t^2/2} H_n(t)$, where H_n is the n-th Hermite polynomial

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}.$$

Let

$$f_n(z, x_2) = H_n(x_2 + z) \exp(-\frac{1}{2}(x_2 + z)).$$

Then the eigenfunctions of (3) associated with $\alpha = 2n + 1$ are given by

$$\phi(x) = \mathcal{F}_{x_1}^{-1}[a(z) f_n(z, x_2)].$$

In particular if $a(z) = \delta(z - z_0)$

$$\phi(x) = H_n(x_2 + z_0) \exp(iz_0 x_1 - \frac{1}{2}(x_2 + z)).$$

$\square$

Eigenvalue problem in $\mathbb{R}^2_+$

Lemma 2. *The lowest eigenvalue of (4) is*

$$\beta_0 = \min_{z \in \mathbb{R}} \beta(z) = \beta(z_0),$$

where $\beta(z)$ is the lowest eigenvalue of the following ODE

$$-u'' + (t+z)^2 u = \beta(z) u \quad \text{for } t > 0, \quad u'(0) = 0, \tag{5}$$

and z_0 is the unique minimum point of $\beta(z)$. The eigenfunctions are given by $\phi = ce^{iz_0 x_1} u(x_2)$, where u is an eigenfunction of (5) for $z = z_0$.

Proof of Theorem 1.

Upper bound estimate: Using the eigenfunctions of (3) and (4) to construct test functions.

Lower bound estimate: Blowing up. The rescaled functions, after gauge transform, converge to a bounded solution of (3) or (4). □

Case 2. $\operatorname{curl}\mathbf{A}$ **has non-degenerate zeros**

We say that $\operatorname{curl}\mathbf{A}$ vanishes non-degenerately if

$$\mathcal{Z}(\operatorname{curl}\mathbf{A}) = \{x \in \bar{\Omega} : \ \operatorname{curl}\mathbf{A}(x) = 0\}$$

is the union of a finite number of smooth curves and $\nabla(\operatorname{curl}\mathbf{A}) \neq 0$ on $\mathcal{Z}(\operatorname{curl}\mathbf{A})$. The limiting equation of a blow-up sequence is an eigenvalue problem for the operator $-\nabla^2_{\mathbf{A}_\vartheta}$, where

$$\mathbf{A}_\vartheta = -\frac{|x|^2}{2}(\cos\vartheta, \sin\vartheta), \quad \vartheta \in (-\pi, \pi).$$

Interior blowing-up: limiting equation is

$$-\nabla^2_{\mathbf{A}_\vartheta}\phi = \lambda\phi \quad \text{in } \mathbb{R}^2. \tag{6}$$

(6) can be reduced to an eigenvalue variation problem for an ordinary differential operator

$$-\frac{d^2}{dt^2} + \frac{1}{4}(t^2 + 2\tau)^2$$

for $t \in \mathbb{R}$. Let $\lambda(\tau)$ denote the lowest eigenvalue of this operator and let

$$\lambda_0 = \inf_{\tau\in\mathbb{R}} \lambda(\tau).$$

Boundary blowing-up: limiting equation is

$$-\nabla^2_{\mathbf{A}_\vartheta}\phi = \lambda\phi \quad \text{in } \mathbb{R}^2_+, \qquad (\nabla_{\mathbf{A}_\vartheta}\phi)\cdot\nu = 0 \quad \text{on } \partial\mathbb{R}^2_+. \tag{7}$$

Let $\lambda(\mathbb{R}^2_+, \vartheta)$ denote the lowest eigenvalue of (7).

Let ν be the unit outer normal of $\partial\Omega$ and τ be the unit tangential vector such that the orientation of $\{\nu, \tau\}$ is the same as that of x_1x_2 coordinates.

Theorem 3 (Kwek-P). *Assume* $\operatorname{curl} A \in C^{1+\alpha}(\bar{\Omega})$ *with* $0 < \alpha < 1$ *and* $\operatorname{curl} \mathbf{A}$ *vanishes non-degenerately on* $\bar{\Omega}$*. For any* $x \in \partial\Omega$*, let* $\vartheta(x)$ *denote the angle between* $\operatorname{curl}^2 \mathbf{A}(x)$ *and* τ*. Then we have*

$$\lim_{b \to +\infty} \frac{\mu(b\mathbf{A})}{b^{2/3}} = [\alpha_1(\operatorname{curl} \mathbf{A})]^{2/3},$$

where

$$\alpha_1(\operatorname{curl} \mathbf{A}) = \min\Big\{ \lambda_0^{3/2} \inf_{x \in \Omega \cap \mathcal{Z}(\operatorname{curl} \mathbf{A})} |\nabla \operatorname{curl} \mathbf{A}(x)|, \\ \inf_{x \in \partial\Omega \cap \mathcal{Z}(\operatorname{curl} \mathbf{A})} \lambda(\mathbb{R}^2_+, \vartheta(x))^{3/2} |\nabla \operatorname{curl} \mathbf{A}(x)| \Big\}.$$

Eigenvalue problems in $\mathbb{R}^2_+$, a general approach

Consider the eigenvalue problem

$$-\nabla^2_{\mathbf{A}}\phi = \beta\phi \quad \text{in } \mathbb{R}^2_+, \quad \nabla_{\mathbf{A}}\phi\cdot\nu = 0 \quad \text{on } \partial\mathbb{R}^2_+. \tag{8}$$

The lowest eigenvalue is given by

$$\beta(\mathbf{A}) = \inf_{\phi} \frac{\int_{\mathbb{R}^2_+} |\nabla_{\mathbf{A}}\phi|^2 dx}{\int_{\mathbb{R}^2_+} |\phi|^2 dy}.$$

Formally, for a bounded eigenfunction ϕ of (8), we make a Fourier transform in x_1 in the sense of distribution. Let

$$f(z,t) = \mathcal{F}_{x_1}[\phi(\cdot,t)].$$

Fix z and let $u(t) = f(z,t)$. Assume (8) is changed to

$$-u'' + q(z,t)u = \beta u \quad \text{for } t > 0, \quad u'(0) = 0. \tag{9}$$

Let $\beta(z)$ be the lowest eigenvalue of (9) and let

$$\beta_* = \inf_z \beta(z).$$

Step 1. If we can show that $\beta(z)$ has a unique minimum point z_0, then $\beta_* = \beta(z_0)$.

Step 2. Show that $\beta(\mathbf{A}) \le \beta_*$.

Method: Using the eigenfunction of (9) associated with β_* to construct text functions.

Step 3. Let ϕ be an eigenfunction of (8) associated with $\beta(\mathbf{A})$. Then there exists $C > 0$ such that

$$\int_a^b dx_1 \int_0^\infty |\phi|^2 dx_2 \leq C(b - a + 1)\|\phi\|_{L^\infty}^2.$$

Step 4. Then we can show that, as a distribution with parameter x_2, $\tilde{\phi}(\cdot, x_2) = \mathcal{F}_{x_1}[\phi]$ must be supported at a single point z_0

(the unique minimum point of $\beta(z)$). Hence for each x_2,

$$\tilde{\phi}(z, x_2) = \sum_{k=0}^{N(x_2)} c_k(x_2) \frac{d^k}{dz^k} \delta(z - z_0),$$

$$\phi(z, x_2) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{N(x_2)} c_k(x_2)(-ix_1)^k \exp(iz_0 x_1).$$

Since ϕ is bounded, $c_k(x_2) = 0$ for all $k > 0$, and hence

$$\phi(x_1, x_2) = v(x_2) \exp(iz_0 x_1),$$

where $v(x_2) = \frac{c_0(x_2)}{\sqrt{2\pi}}$. Then we show that v must satisfy (9) for $z = z_0$ and $\beta = \beta(\mathbf{A})$. Hence $\beta(\mathbf{A}) \geq \beta_*$. $\square$

§3. 3-dimensional Problem

Ω is a bounded, smooth and simply-connected domain in $\mathbb{R}^3$.

Theorem 4 (Lu-P). *For any* $\mathbf{A} \in C^{1+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ *we have*

$$\lim_{b\to+\infty} \frac{\mu(b\mathbf{A})}{b} = \min\left\{ \inf_{x\in\Omega} \left|\mathrm{curl}\ \mathbf{A}(x)\right|, \inf_{x\in\partial\Omega} B(\theta(x))\left|\mathrm{curl}\ \mathbf{A}(x)\right| \right\}, \tag{10}$$

where $\theta(x)$ *is the angle between* curl $\mathbf{A}(x)$ *and the outer-normal vector* ν *on* $\partial\Omega$, $B(\theta)$ *is a positive function, decreasing on* $(0, \frac{\pi}{2})$, $B(0) = 1$, $B(\frac{\pi}{2}) = \beta_0 < 1$, *and* $B(\pi - \theta) = B(\theta)$.

In the case where $\mathbf{A} = \mathbf{F_h}$,

$$\lim_{b \to +\infty} \frac{\mu(b\mathbf{F_h})}{b} = \beta_0,$$

The eigenfunctions concentrate at the tangential set

$$(\partial\Omega)_{\mathbf{h}} = \{x \in \partial\Omega : \ \mathbf{h} \cdot \nu(x) = 0\},$$

which is a subset of the surface where $\mathbf{h}$ *is tangential to* $\partial\Omega$.

Problem 1. *Find location of concentration of eigenfunctions, and multiplicity of eigenvalues.*

In Theorem 1, if

$$\inf_{\Omega} |\mathrm{curl}\, \mathbf{A}| < \beta_0 \inf_{\partial\Omega} |\mathrm{curl}\, \mathbf{A}|,$$

then the eigenfunctions concentrate on Ω_m. If Ω_m consists of more than one point, should the eigenfunctions concentrate at only one point, or should they concentrate over all Ω_m? One may ask a similar question for $(\partial\Omega)_m$.

In Theorem 2, the eigenfunctions concentrate in $\mathcal{N}(\partial\Omega)$, the set of maximum points of the boundary curvature. If there are more than one maximum points, should the eigenfunctions concentrate at only one point, or should they concentrate over all $\mathcal{N}(\partial\Omega)$?

Problem 2. *Find an asymptotic estimate for* $\mu(b\mathbf{A})$ *when* $\operatorname{curl}\mathbf{A}$ *has higher order zeros.*

For recent progress on this problem see J. Aramaki.

Problem 3. *Examine the second eigenvalue $\lambda_2(b)$, or all other eigenvalues $\lambda_j(b)$ which satisfy asymptotically $\lambda_j(b) \leq (1 + o(1))b$ as $b \to \infty$.*

See recent results of Morame-Truc.

Problem 4. *Find an asymptotic estimate for the eigenvalue* $\mu(b\mathbf{A})$ *for large* b *where* $\operatorname{curl}\mathbf{A}$ *is in* $W^{1,2}$ *or in* L^2 *but is not smooth.*

This problem rises in the study of nucleation of smectics and is needed for the estimate of critical wave number Q_{c_3} for liquid crystals.

§4. Surface Superconductivity in 2-Dimensional Superconductors

Throughout this section we assume that Ω is a bounded and simply-connected domain in $\mathbb{R}^2$ with smooth boundary.

Theorem 5 (Helffer-P). *For large κ we have*

$$H_{c_3}(\kappa) = \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}} \kappa_{max} + O(\kappa^{-1/3}),$$

where C_1 is a positive constant, and κ_{max} is the maximum value of the curvature of $\partial\Omega$.

Behavior of minimizers

(i) As the applied field decreases from H_{c_3}, superconductivity nucleates first at the maximum points of the boundary curvature.

(ii) As the applied field is reduced again but is still close to H_{c_3}, the superconducting region expands gradually, and then a *thin* superconducting sheath forms on the entire boundary of the sample.

(iii) As the applied field is further reduced but is still kept away above H_{c_2}, the superconducting sheath becomes strong and a boundary layer gradually raises, while the interior of the sample remains in a normal state.

(iv) The sample will remain in a surface superconducting state until the applied field reaches H_{c_2}.

Remark. Conclusion (ii) has been improved by S. Fournails and B. Helffer.

Conclusion (iii) suggests that the equality $H_{c_2}(\kappa) \sim \kappa$ is asymptotically correct.

The behavior of minimizers with the applied field lying in between H_{c_2} and H_{c_3} was further investigated by E. Sandier and S. Serfaty.

Conjecture. *Assume* $\kappa_n \to \infty$, $H_n = (b + o(1))\kappa_n$, *where* $1 \leq b < \frac{1}{\beta_0}$; *When* $b = 1$ *we further require that* $1 \ll H_n - \kappa_n = o(\kappa_n)$. *Let* ψ_n *be order parameter corresponding to* $\kappa = \kappa_n$ *and* $H = H_n$. *Then there exists a positive constant* c_b *such that*

$$\lim_{n\to\infty} |\psi_n(x)| = \begin{cases} 0 & \text{if } x \in \Omega, \\ c_b & \text{if } x \in \partial\Omega. \end{cases}$$

Recently Y. Almog and B. Helffer proved that if $1 < b < 1/\beta_0$ then the order parameters indeed converge to a constant but in a rather weak sense.

Y. Almog and B. Helffer,

The distribution of surface superconductivity along the boundary: on a conjecture of X. B. Pan,

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[2]. K. Lu and X. B. Pan, *Eigenvalue problems of Ginzburg-Landau operator in bounded domains*, J. Math. Phys. **40** (1999), 2647–2670.

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