

Illiquidity, Portfolio Constraints and Diversification

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Introduction: Motivation

- Mutual funds are often restricted to allocate certain percentages of fund assets to certain securities (**Almazan, Brown, Carlson, and Chapman 2004, Clarke, de Silva, and Thorley 2002**).
- Mutual funds can also face significant illiquidity in trading securities (**Chalmers, Edelen, and Kadlec 1999, Delib and Varma 2002**)
- The coexistence of these restrictions and asset illiquidity and the interactions among them are important for the optimal trading strategy of a mutual fund.

Motivation (continued)

- The existing literature ignores the coexistence of portfolio restrictions and asset illiquidity and the interactions among them.
 - Portfolio selection with portfolio constraints: **Fleming and Zariphopoulou (1991), Cuoco (1997), Cuoco and Liu (2000).**
 - Portfolio selection with transaction costs: **Constantinides (1986), Davis and Norman (1990), Shreve and Soner (1994), Liu and Loewenstein (2002), Liu (2004), Dai and Yi (2009), Dai, Jiang, Li, and Yi (2009).**

The Model

- An investor with a finite horizon $T \in (0, \infty)$ maximizes his CRRA utility from terminal non-liquidated wealth:

$$u(W) = \frac{W^{1-\gamma} - 1}{1 - \gamma}.$$

- Two assets: 1 liquid stock, and 1 illiquid stock.
- The liquid stock price S_{Lt} evolves as

$$\frac{dS_{Lt}}{S_{Lt}} = \mu_L dt + \sigma_L dB_{Lt},$$

where μ_L and $\sigma_L > 0$ are both constants and B_{Lt} is a one-dimensional Brownian motion.

The Illiquid Stock

- The investor can buy the illiquid stock at the ask price $S_{It}^A = (1 + \theta)S_{It}$ and sell the stock at the bid price $S_{It}^B = (1 - \alpha)S_{It}$, where $\theta \geq 0$ and $0 \leq \alpha < 1$ represent the proportional transaction cost rates and S_{It} follows the process

$$\frac{dS_{It}}{S_{It}} = \mu_I dt + \sigma_I dB_{It},$$

where μ_I and $\sigma_I > 0$ are both constants and B_{It} is another one-dimensional Brownian motion that has a correlation of ρ with B_{Lt} with $|\rho| < 1$.

Portfolio Constraints

- Let x_t and y_t be the dollar amount invested in the liquid stock and the illiquid stock respectively.
- The investor is subject to the following exogenously given constraints on its trading strategy:

$$(1) \quad \underline{b} \leq \frac{y_t}{W_t} \leq \bar{b}, \quad \forall t \geq 0,$$

where W_t is the non-liquidated wealth process.

The Dynamic Budget Constraint

When $\alpha + \theta > 0$, we have

$$(2) \quad dx_t = \mu_L x_t dt + \sigma_L x_t dB_{Lt} - (1 + \theta)dI_t + (1 - \alpha)dD_t,$$

$$(3) \quad dy_t = \mu_I y_t dt + \sigma_I y_t dB_{It} + dI_t - dD_t,$$

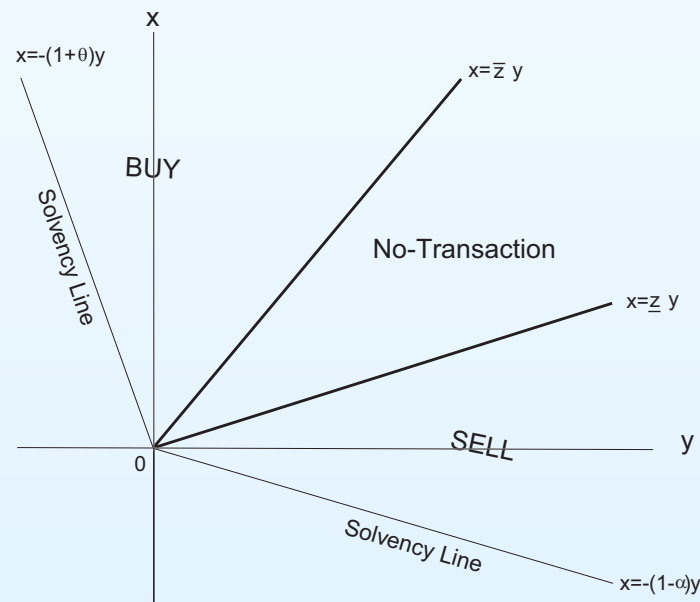
where the processes D and I represent the cumulative dollar amount of sales and purchases of the illiquid stock, respectively. D and I are nondecreasing and right continuous adapted processes with $D(0) = I(0) = 0$.

The Solvency Region

- Let $\Theta(x_0, y_0)$ denote the set of admissible trading strategies (D, I) such that (1), (2), (3), and

$$\hat{W}_t \geq 0, \quad \forall t \geq 0,$$

where $\hat{W}_t = x_t + (1 - \alpha)y_t^+ - (1 + \theta)y_t^-$ is the time t wealth after liquidation.



The Investor's Problem and HJB Equation

- The investor's problem is then

$$\sup_{(D,I) \in \Theta(x_0, y_0)} E [u(W_T)].$$

- HJB Equation

$$\max \{V_t + LV, (1 - \alpha)V_x - V_y, -(1 + \theta)V_x + V_y\} = 0,$$

with the boundary conditions

$$(1 - \alpha)V_x - V_y = 0 \text{ on } \frac{y}{x + y} = \bar{b}, \quad (1 + \theta)V_x - V_y = 0 \text{ on } \frac{y}{x + y} = \underline{b},$$

and the terminal condition

$$V(x, y, T) = \frac{(x + y)^{1-\gamma} - 1}{1 - \gamma},$$

where $LV = \frac{1}{2}\sigma_I^2 y^2 V_{yy} + \frac{1}{2}\sigma_L^2 x^2 V_{xx} + \rho\sigma_I\sigma_L xy V_{xy} + \mu_I y V_y + \mu_L x V_x$

Analytical Results: The No Transaction Cost Case

Theorem 1. Suppose that $\alpha = \theta = 0$. Then the optimal trading policy is given by

$$\pi_I^* = \begin{cases} \bar{b} & \text{if } \pi_I^M \geq \bar{b} \\ \pi_I^M & \text{if } \underline{b} < \pi_I^M < \bar{b} \\ \underline{b} & \text{if } \pi_I^M \leq \underline{b} \end{cases}, \quad \pi_L^* = 1 - \pi_I^*$$

where

$$\pi_I^M = \frac{\mu_I - \mu_L + \gamma\sigma_L(\sigma_L - \rho\sigma_I)}{\gamma(\sigma_L^2 + \sigma_I^2 - 2\rho\sigma_L\sigma_I)}$$

is the optimal fraction of wealth invested in illiquid stock in the unconstrained case.

The Transaction Cost Case Without Constraints

Proposition 2. Assume $-\frac{1}{\alpha} + 1 < \pi_I^M < \frac{1}{\theta} + 1$. We have $\forall t \in [0, T]$,

1. for the sell boundary, there exists $\bar{t} < T$ such that

$$\frac{1}{\alpha} = \bar{\pi}_I(s) \geq \bar{\pi}_I(t) \geq \frac{\pi_I^M}{1 - \alpha(1 - \pi_I^M)}, \text{ for any } t \text{ and all } s > \bar{t};$$

2. for the buy boundary, there exists $\underline{t} < T$ such that

$$-\frac{1}{\theta} = \underline{\pi}_I(s) \leq \underline{\pi}_I(t) \leq \frac{\pi_I^M}{1 + \theta(1 - \pi_I^M)}, \text{ for any } t \text{ and all } s > \underline{t}.$$

The Case with Transaction Costs and Constraints

The following verification theorem shows the existence and the uniqueness of the optimal trading strategy. It also ensures the smoothness of the value function except for a set of measure zero.

Theorem 3.

- (i) The HJB equation admits a unique viscosity solution, and the value function is the viscosity solution.
- (ii) The value function is $C^{2,2,1}$ in $\{(x, y, t) : x + (1 - \alpha)y^+ - (1 + \theta)y^- > 0, \underline{b} < y/(x + y) < \bar{b}, 0 \leq t < T\} \setminus (\{y = 0\})$.

The Case with Transaction Costs and Constraints

Proposition 4. We have

1. for the sell boundary, there exists $\bar{t}_b < T$ such that

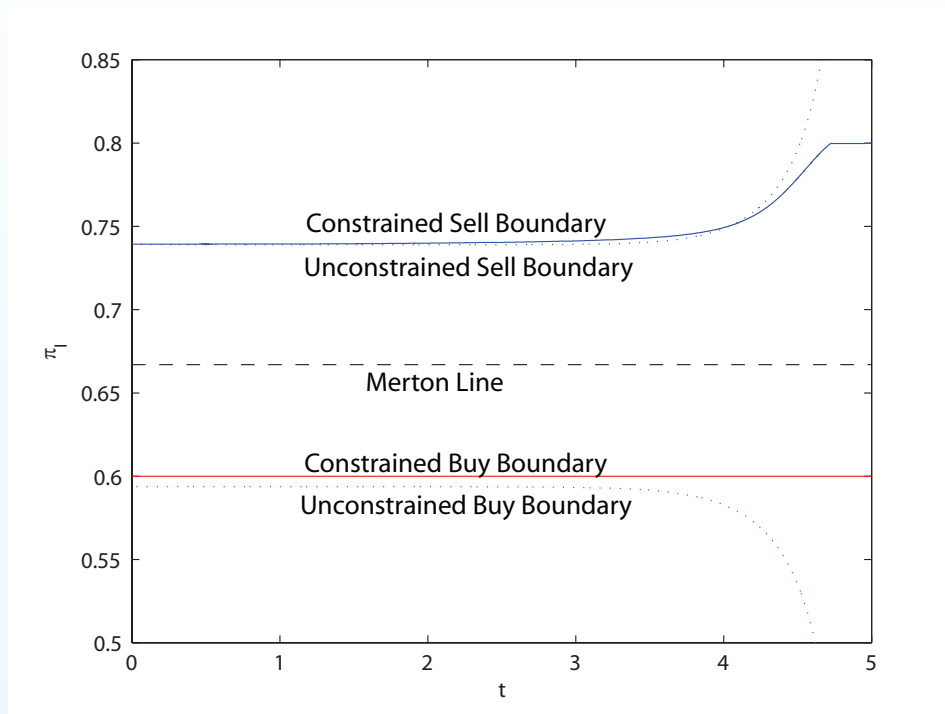
$$\bar{b} = \bar{\pi}_I^c(s; \underline{b}, \bar{b}) \geq \bar{\pi}_I^c(t; \underline{b}, \bar{b}) \geq \max \left(\min \left(\frac{\pi_I^M}{1 - \alpha(1 - \pi_I^M)}, \bar{b} \right), \underline{b} \right), \text{ for any } t \text{ and } s > \bar{t}_b;$$

2. for the buy boundary, there exists $\underline{t}_b < T$ such that

$$\min \left(\max \left(\frac{\pi_I^M}{1 + \theta(1 - \pi_I^M)}, \underline{b} \right), \bar{b} \right) \geq \underline{\pi}_I^c(t; \underline{b}, \bar{b}) \geq \underline{\pi}_I^c(s; \underline{b}, \bar{b}) = \underline{b}, \text{ for any } t \text{ and } s > \underline{t}_b.$$

3. both $\bar{\pi}_I^c(t; \underline{b}, \bar{b})$ and $\underline{\pi}_I^c(t; \underline{b}, \bar{b})$ are increasing in \underline{b} and \bar{b} for all $t \in [0, T]$;

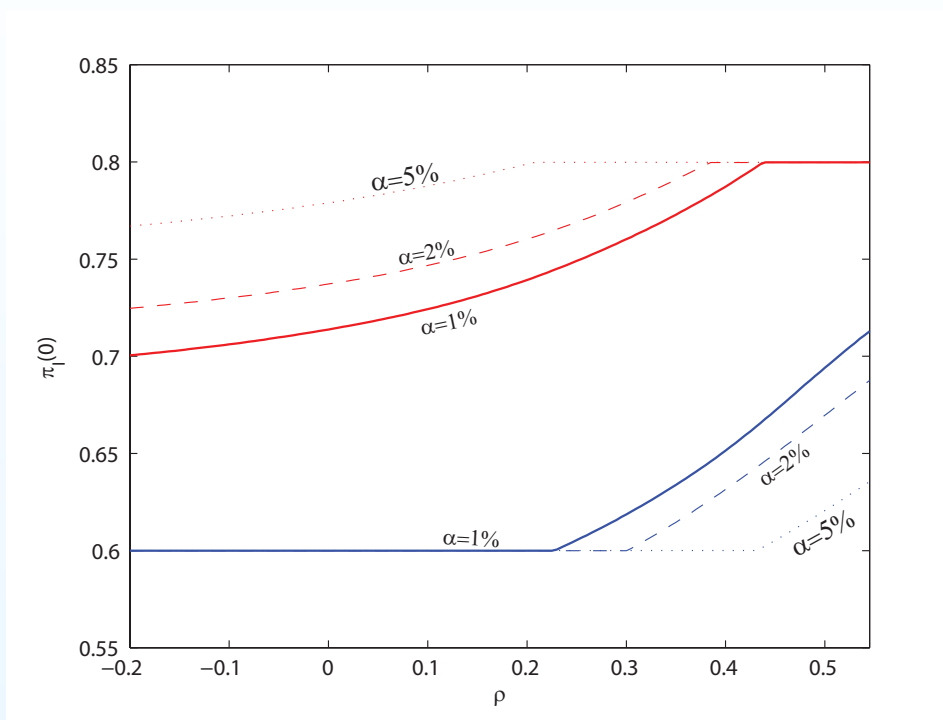
Graphical Illustrations: Optimal Strategy against Time



Parameters: $\gamma = 2$, $T = 5$, $\mu_L = 0.06$, $\sigma_L = 0.20$, $\mu_I = 0.11$, $\sigma_I = 0.25$, $\rho = 0.2$, $\alpha = 0.01$, $\theta = 0.01$, $\underline{b} = 0.60$, and $\bar{b} = 0.80$.

- The lower bound is binding for all time, while the sell boundary reaches the upper bound near maturity.
- The sell strategy is not myopic in the sense that in anticipation of the constraint becoming binding later, it is optimal to change the early trading strategy.

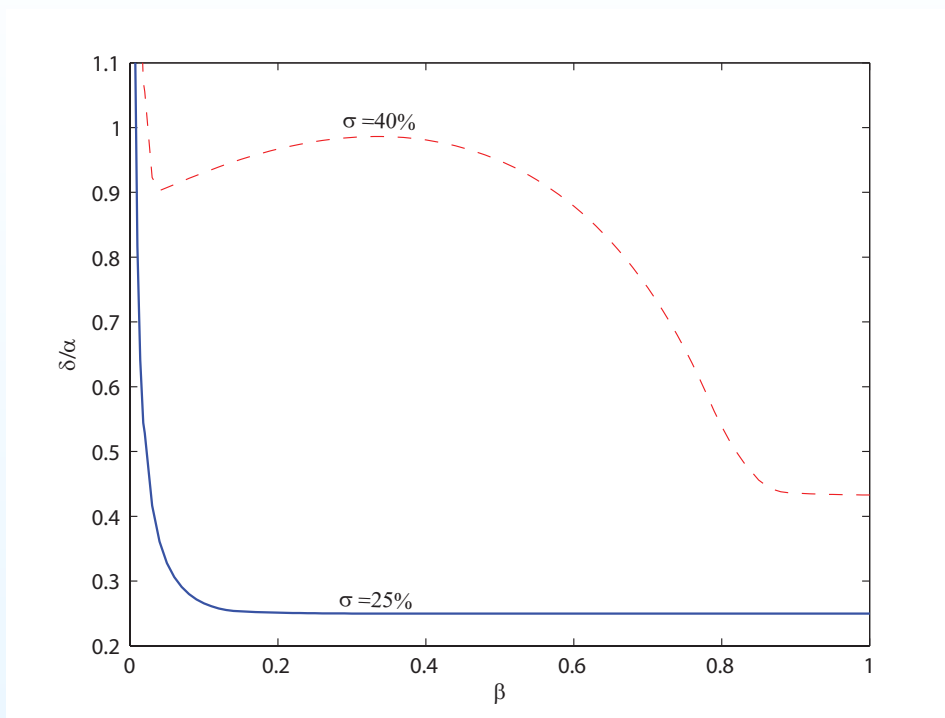
Initial Illiquid Stock Holding against Correlation



Parameters: $\gamma = 2$, $T = 5$, $\mu_L = 0.06$, $\sigma_L = 0.20$, $\mu_I = 0.11$, $\sigma_I = 0.25$, $\theta = \alpha$, $\underline{b} = 0.60$, and $\bar{b} = 0.80$.

- The optimal fraction of wealth in the illiquid asset increases with the correlation coefficient, because of the decrease in the diversification effect of the liquid stock investment.
- The no transaction region widens as the transaction cost rate increases, because the trading in the illiquid asset becomes more costly.

Liquidity Premium against Weights Bandwidth



Parameter values changed: $\underline{b} = 0.7292 - \frac{1}{2}\beta$, and $\bar{b} = 0.7292 + \frac{1}{2}\beta$.

- For very stringent constraints, the liquidity premium (the maximum expected return an investor is willing to exchange for zero transaction cost) can be much greater than what Constantinides (1986) finds, because imposing stringent constraints can force more frequent transactions and also distort the investment strategy.
- The liquidity premium increases with volatility.
- The liquidity premium against β may not be monotonically decreasing ($\sigma = 0.4$).

Conclusions: Summary

- The behaviors of the optimal buy and sell boundary are characterized.
- Both the sell boundary and the buy boundary can be nonmyopic with respect to the portfolio constraints even for a log utility.
- Portfolio constraints can significantly magnify the effect of transaction costs on liquidity premium and can make it a first-order effect.
- Return correlations significantly affect diversification efficiency and optimal trading strategy.