Rapidly rotating Bose-Einstein condensates*

Alexander Fetter, Stanford University

Singapore, November, 2007

1. Physics of one vortex line in harmonic trap
2. Experimental creation and detection of vortices
3. Vortex arrays in mean-field Thomas-Fermi regime
4. Vortex arrays in mean-field quantum-Hall regime
5. Behavior for $\Omega \to \omega_\perp$

*for general references, see [1, 2, 3, 4]

1 Physics of one vortex line in harmonic trap

Assume general axisymmetric trap potential

$$V_{\rm tr}(\boldsymbol{r}) = V_{\rm tr}(r, z) = \tfrac{1}{2}M\left(\omega_\perp^2 r^2 + \omega_z^2 z^2\right)$$

Basic idea (Bogoliubov): for weak interparticle potentials, nearly all particles remain in condensate for $T \ll T_c$

- for ideal gas at $T = 0$, *all* particles are in condensate.
- hence treat occupation of non-condensate as small
- dilute: s-wave scattering length $a_s \ll$ interparticle spacing $n^{-1/3}$
- equivalently, require $na_s^3 \ll 1$
- assume self-consistent condensate wave function $\Psi(\boldsymbol{r})$
- gives nonuniform condensate density $n(\boldsymbol{r}) = |\Psi(\boldsymbol{r})|^2$
- for $T \ll T_c$, normalization requires $N = \int dV\, |\Psi(\boldsymbol{r})|^2$

- assume an energy functional

$$E[\Psi] = \int dV \left[\underbrace{\Psi^* \left(\mathcal{T} + V_{\mathrm{tr}} \right) \Psi}_{\text{harmonic oscillator}} + \underbrace{\tfrac{1}{2} g |\Psi|^4}_{2-\text{body term}} \right],$$

  where $\mathcal{T} = -\hbar^2 \nabla^2 / 2M$ is kinetic energy operator and $g = 4\pi a_s \hbar^2 / M$ is interaction coupling parameter

- balance of kinetic energy $\langle \mathcal{T} \rangle$ and trap energy $\langle V_{\mathrm{tr}} \rangle$ gives mean oscillator length $d_0 = \sqrt{\hbar / M\omega_0}$ where $\omega_0 = \left(\omega_\perp^2 \omega_z \right)^{1/3}$ is geometric mean

- balance of kinetic energy $\langle \mathcal{T} \rangle$ and interaction energy $\langle gn \rangle$ gives healing length

$$\xi = \frac{\hbar}{\sqrt{2Mgn}} = \frac{1}{\sqrt{8\pi a_s n}}$$

- with fixed normalization and μ the chemical potential, variation of $E[\Psi]$ gives Gross-Pitaevskii (GP) eqn

$$\left(\mathcal{T} + V_{\mathrm{tr}} + g|\Psi|^2 \right) \Psi = \mu \Psi$$

- can interpret nonlinear term as a Hartree potential $V_H(\boldsymbol{r}) = gn(\boldsymbol{r})$, giving interaction with nonuniform condensate density

- generalize to time-dependent GP equation

$$i\hbar\frac{\partial\Psi}{\partial t} = (\mathcal{T} + V_{\rm tr} + V_H)\,\Psi$$

- this result implies that stationary solutions have time dependence $\exp(-i\mu t/\hbar)$

Introduce hydrodynamic variables

- write $\Psi(\boldsymbol{r},t) = |\Psi(\boldsymbol{r},t)|\exp\left[iS(\boldsymbol{r},t)\right]$ with phase S
- condensate density is $n(\boldsymbol{r},t) = |\Psi(\boldsymbol{r},t)|^2$
- current is

$$\boldsymbol{j} = \frac{\hbar}{2Mi}\left[\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\right] = |\Psi|^2\,\frac{\hbar\boldsymbol{\nabla}S}{M}$$

- identify last factor as velocity $\boldsymbol{v} = \hbar\boldsymbol{\nabla}S/M$
- note that $\boldsymbol{v}$ is irrotational so $\boldsymbol{\nabla}\times\boldsymbol{v} = 0$

- general property: circulation around contour $\mathcal{C}$ is

$$\oint_{\mathcal{C}} d\boldsymbol{l} \cdot \boldsymbol{v} = \frac{\hbar}{M} \oint_{\mathcal{C}} d\boldsymbol{l} \cdot \boldsymbol{\nabla} S = \frac{\hbar}{M} \left. \Delta S \right|_{\mathcal{C}}$$

  since $\boldsymbol{v} = \hbar \boldsymbol{\nabla} S / M$

- change of phase $\Delta S|_{\mathcal{C}}$ around $\mathcal{C}$ must be integer times 2π since Ψ is single-valued

- hence circulation in BEC is *quantized* in units of $\kappa \equiv 2\pi\hbar/M$

- rewrite time-dependent GP equation in terms of $|\Psi|$ and S

  – imaginary part gives conservation of particles

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\boldsymbol{v}) = 0$$

  – real part gives generalized Bernoulli equation

Presence of harmonic trap yields much richer system than a uniform interacting Bose gas [5]

- trap gives new *energy* scale $\hbar\omega_0$ and new *length* scale $d_0 = \sqrt{\hbar/M\omega_0}$
- assume repulsive interactions with $a_s > 0$
- trap leads to new dimensionless parameter Na_s/d_0
- typical value of ratio of lengths: $a_s/d_0 \sim 10^{-3}$
- but Na_s/d_0 is large for typical $N \sim 10^6$
- in strong repulsive limit ($Na_s/d_0 \gg 1$), condensate expands to mean radius $R_0 \gg d_0$
- neglect radial gradient of Ψ when $R_0 \gg d_0$
- GP equation then simplifies and gives local density

$$\frac{4\pi a_s \hbar^2}{M}|\Psi(r,z)|^2 = \mu - V_{\mathrm{tr}}(r,z)$$

  [called Thomas-Fermi (TF) limit]

- harmonic trap produces quadratic density variation with condensate dimensions $R_j^2 = 2\mu/(M\omega_j^2)$

One vortex line in trapped BEC

First assume bulk condensate with uniform density n and a single straight vortex line along z axis

- Gross and Pitaevskii [6, 7]: take condensate wave function
$$\Psi(\boldsymbol{r}) = \sqrt{n}\, e^{i\phi} f\,(r/\xi)$$
where r and ϕ are two-dimensional polar coordinates
- speed of sound is $s = \sqrt{gn/M}$
- assume $f(0) = 0$ and $f(r/\xi) \to 1$ for $r \gg \xi$
- velocity has circular streamlines with $\boldsymbol{v} = (\hbar/Mr)\,\hat{\boldsymbol{\phi}}$
- this is a quantized vortex line with $\oint d\boldsymbol{l}\cdot\boldsymbol{v} = 2\pi\hbar/M$
- $v \sim s$ when $r \sim \xi$, so vortex core forms by cavitation
- equivalently, centrifugal barrier gives vortex core of radius ξ

Static behavior of a vortex line in axisymmetric trap

$$V_{\mathrm{tr}}(r, z) = \tfrac{1}{2}M\left(\omega_\perp^2 r^2 + \omega_z^2 z^2\right)$$

- If $\omega_z \gg \omega_\perp$, strong axial confinement gives disk-shaped condensate
- If $\omega_\perp \gg \omega_z$, strong radial confinement gives cigar-shaped condensate
- for vortex on axis, condensate wave function is

$$\Psi(\boldsymbol{r}, z) = e^{i\phi}|\Psi(r, z)|$$

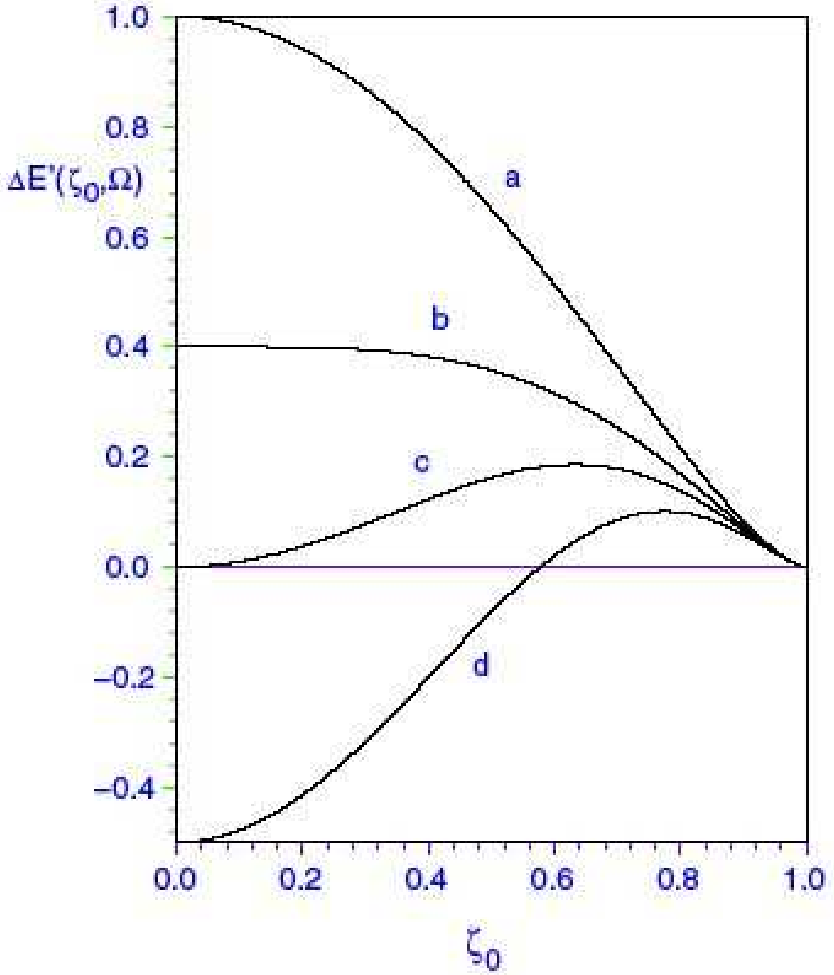
- velocity is $\boldsymbol{v} = (\hbar/Mr)\hat{\boldsymbol{\phi}}$, like uniform condensate
- centrifugal energy again forces wave function to vanish for $r \lesssim \xi$
- density is now toroidal; hole along symmetry axis
- TF limit: separated length scales with

$$\begin{aligned} \xi \text{ (vortex core)} &\ll d_0 \text{ (mean oscillator length)} \\ d_0 \text{ (mean oscillator length)} &\ll R_0 \text{ (mean condensate radius)} \end{aligned}$$

- hence TF density is essentially unchanged by vortex apart from small hole along the vortex core

Energy of rotating TF condensate with one vortex

- use density of vortex-free TF condensate; cut off the logarithmic divergence at core radius ξ
- if condensate is in rotational equilibrium at angular velocity $\boldsymbol{\Omega}$, the appropriate energy functional is [8] $E'[\Psi] = E[\Psi] - \boldsymbol{\Omega} \cdot \boldsymbol{L}[\Psi]$ where $\boldsymbol{L}$ is the angular momentum
- let E'_0 be energy of rotating vortex-free condensate
- let $E'_1(r_0, \Omega)$ be energy of a rotating condensate with straight vortex that is displaced laterally by distance r_0 from symmetry axis
- approximation of straight vortex works best for disk-shaped condensate ($\omega_z \gtrsim \omega_\perp$)
- Difference of these two energies is energy associated with formation of vortex $\Delta E'(r_0, \Omega) = E'_1(r_0, \Omega) - E'_0$
- $\Delta E'(r_0, \Omega)$ depends on position r_0 of vortex and on Ω



Plot $\Delta E'(r_0, \Omega)$ as function of ζ_0 for various fixed Ω [9], where $\zeta_0 = r_0/R_0$ is scaled displacement from center curve (a) is $\Delta E'(r_0, \Omega)$ for $\Omega = 0$

- $\Delta E'(r_0, 0)$ decreases monotonically with increasing ζ_0
- curvature is negative at $\zeta_0 = 0$
- for no dissipation, fixed energy means constant ζ_0
- only allowed motion is uniform precession at fixed r_0
- angular velocity is given by variational Lagrangian method [10, 11, 3] $\dot{\phi}_0 \propto -\partial E(r_0)/\partial r_0$
- precession arises from nonuniform trap potential (*not image vortex*) and nonuniform condensate density
- in presence of weak dissipation, vortex moves to lower energy and slowly spirals outward

As Ω increases, curvature near $\zeta_0 = 0$ decreases

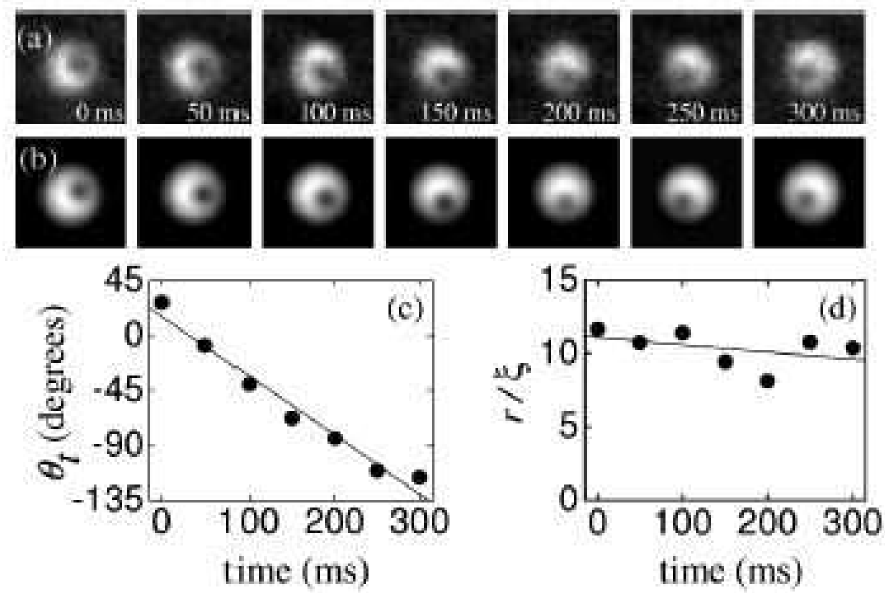
- curve (b) is when curvature near $\zeta_0 = 0$ vanishes
- it corresponds to angular velocity

$$\Omega_m = \frac{3}{2}\frac{\hbar}{MR_\perp^2}\ln\left(\frac{R_\perp}{\xi}\right)$$

- for $\Omega \gtrsim \Omega_m$, energy $\Delta E'(\zeta_0, \Omega)$ has local *minimum* near $\zeta_0 = 0$
- dissipation would now drive vortex back *toward* the symmetry axis
- Ω_m is angular velocity for onset of *metastability*
- vortex at center is *locally* stable for $\Omega > \Omega_m$, but not *globally* stable, since $\Delta E'(0, \Omega_m)$ is positive

2 Experimental creation/detection of vortices in dilute trapped BEC

- first vortex made at JILA (1999) [12]
- use nearly spherical ^{87}Rb condensate containing two different hyperfine components
- use coherent (Rabi) process to control interconversion between two components
- spin up condensate by coupling the two components with a stirring perturbation
- turn off coupling, leaving one component with trapped quantized vortex surrounding nonrotating core of other component
- use selective tuning to make nondestructive image of either component



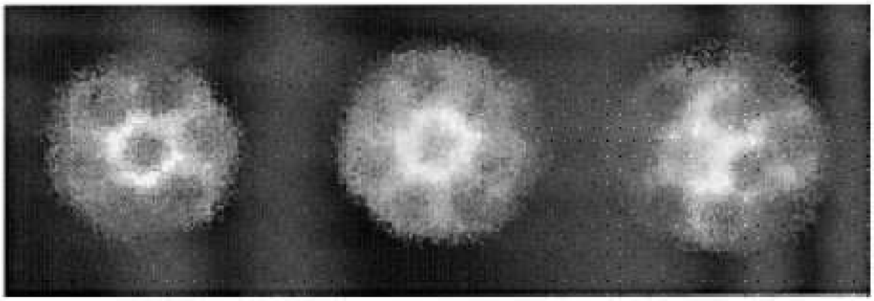
- this vortex with large filled core precesses around trap center
- can also create vortex with (small) empty core [13] that also precesses
  - theory predicts $\dot{\phi}/2\pi \approx 1.58 \pm 0.16$ Hz, and
  - experiment finds $\dot{\phi}/2\pi \approx 1.8 \pm 0.1$ Hz
- see no outward radial motion for ~ 1 s, so dissipation is small on this time scale

École Normale Supérieure (ENS) in Paris studied vortex creation in elongated rotating cigar-shaped condensate with one component [14, 15]

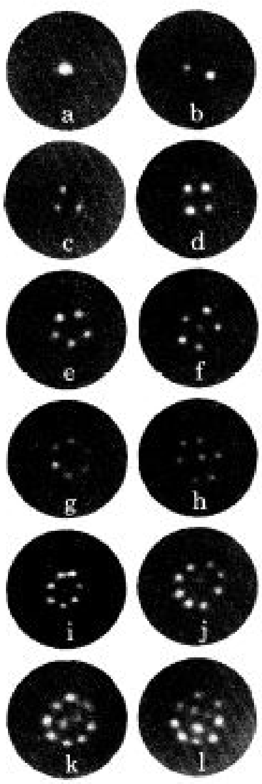
- used off-center toggled rotating laser beam to deform the transverse trap potential and stir the condensate at an applied frequency $\Omega/2\pi \lesssim 200$ Hz

- find vortex appears at a critical frequency $\Omega_c \approx 0.7\omega_\perp$ (detected by expanding the condensate, which now has a disk shape, with vortex core as expanded hole)
- vortex nucleation is dynamical process associated with surface instability (quadrupole oscillation)

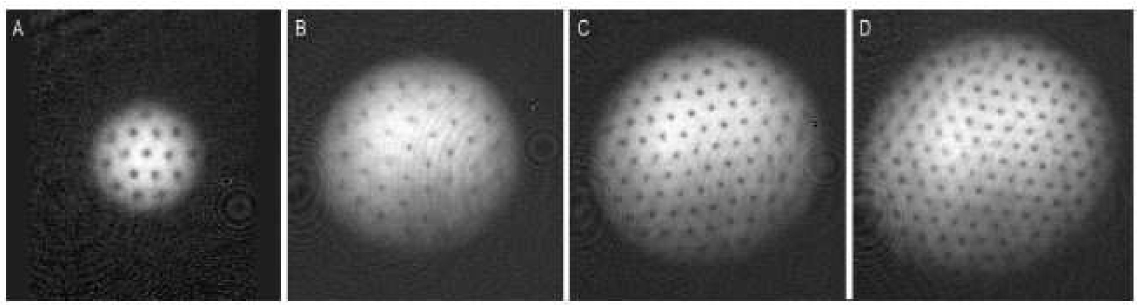
- ENS group observed small vortex arrays of up to 11 vortices (arranged in two concentric rings)



- like patterns predicted and seen in superfluid ^{4}He [16]



- MIT group has prepared considerably larger rotating condensates in less elongated trap
- they have observed triangular vortex lattices with up to 130 vortices [17]



- like Abrikosov lattice of quantized flux lines (which are charged vortices) in type-II superconductors
- JILA group has now made large rotating condensates with several hundred vortices and angular velocity $\Omega/\omega_\perp \approx 0.995$ [18]
- these rapidly rotating systems open many exciting new possibilities (discussed below)

3 Vortex arrays in mean-field Thomas-Fermi regime

Feynman's mean vortex density in a rotating superfluid

- solid-body rotation has $\boldsymbol{v}_{\rm sb} = \boldsymbol{\Omega} \times \boldsymbol{r}$
- $\boldsymbol{v}_{\rm sb}$ has constant vorticity $\boldsymbol{\nabla} \times \boldsymbol{v}_{\rm sb} = 2\boldsymbol{\Omega}$
- each quantized vortex at $\boldsymbol{r}_j$ has localized vorticity

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \frac{2\pi\hbar}{M}\,\delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}_j)\,\hat{\boldsymbol{z}}$$

- assume $\mathcal{N}_v$ vortices uniformly distributed in area $\mathcal{A}$ bounded by contour $\mathcal{C}$
- circulation around $\mathcal{C}$ is $\mathcal{N}_v \times 2\pi\hbar/M$
- but circulation in $\mathcal{A}$ is also $2\Omega\mathcal{A}$
- hence vortex density is $n_v = \mathcal{N}_v/\mathcal{A} = M\Omega/\pi\hbar$
- area per vortex $1/n_v$ is $\pi\hbar/M\Omega \equiv \pi l^2$ which defines radius $l = \sqrt{\hbar/M\Omega}$ of circular cell
- intervortex spacing $\sim 2l$ decreases like $1/\sqrt{\Omega}$
- analogous to quantized flux lines (charged vortices) in type-II superconductors

As Ω increases, the mean vortex density $n_v = M\Omega/\pi\hbar$ increases linearly following the Feynman relation

- in addition, centrifugal forces expand the condensate radially, so that the area $\pi R_\perp^2$ also increases
- hence the number of vortices $\mathcal{N}_v = n_v \pi R_\perp^2 = M\Omega R_\perp^2/\hbar$ increases faster than linearly with Ω
- conservation of particles implies that the condensate also shrinks axially
- TF approximation assumes that interaction energy $\langle g|\Psi|^4\rangle$ and trap energy $\langle V_{\rm tr}|\Psi|^2\rangle$ are large relative to kinetic energy for density variations $(\hbar^2/M)\langle(\boldsymbol{\nabla}|\Psi|)^2\rangle$
- radial expansion of rotating condensate means that central density eventually becomes small

Quantitative description of rotating TF condensate

Kinetic energy of condensate involves

$$\frac{\hbar^2}{2M}\int dV\,|\boldsymbol{\nabla}\Psi|^2 = \underbrace{\int dV\,\frac{1}{2}Mv^2|\Psi|^2}_{\text{superflow energy}} + \underbrace{\frac{\hbar^2}{2M}\int dV\,\left(\boldsymbol{\nabla}|\Psi|\right)^2}_{\text{density variation}}$$

where $\Psi = \exp(iS)|\Psi|$ and $\boldsymbol{v} = \hbar\boldsymbol{\nabla}S/M$ is flow velocity

- generalized TF approximation: retain the energy of superflow but ignore the energy from density variation
- this approximation will fail eventually when vortex lattice becomes dense and cores start to overlap
- in rotating frame, generalized TF energy functional is

$$E'[\Psi] = \int dV \quad \Big[\left(\tfrac{1}{2}Mv^2 + V_{\text{tr}} - M\boldsymbol{\Omega}\cdot\boldsymbol{r}\times\boldsymbol{v}\right)|\Psi|^2 + \tfrac{1}{2}g|\Psi|^4\Big]$$

- here, $\boldsymbol{v}$ is flow velocity generated by all the vortices

For $\boldsymbol{\Omega}$ along z, can complete square and rewrite $E'[\Psi]$ as

$$E'[\Psi] = \int dV \left[\frac{1}{2}M\Big(\underbrace{\boldsymbol{v}-\boldsymbol{\Omega}\times\boldsymbol{r}}_{\boldsymbol{v}-\boldsymbol{v}_{\rm sb}}\Big)^2|\Psi|^2 + \frac{1}{2}M\omega_z^2 z^2|\Psi|^2 + \frac{1}{2}M\left(\omega_\perp^2-\Omega^2\right)r^2|\Psi|^2 + \frac{1}{2}g|\Psi|^4\right]$$

- in the rotating frame, the dominant effect of the dense vortex array is that spatially averaged flow velocity $\boldsymbol{v}$ is close to $\boldsymbol{\Omega}\times\boldsymbol{r} = \boldsymbol{v}_{\rm sb}$
- hence can ignore first term in $E'[\Psi]$, giving

$$E'[\Psi] \approx \int dV \left[\frac{1}{2}M\omega_z^2 z^2|\Psi|^2 + \frac{1}{2}M\left(\omega_\perp^2-\Omega^2\right)r^2|\Psi|^2 + \frac{1}{2}g|\Psi|^4\right]$$

- E' now looks exactly like TF energy for nonrotating condensate but with a *reduced* radial trap frequency $\omega_\perp^2 \to \omega_\perp^2 - \Omega^2$

Hence TF wave function depends explicitly on Ω through the altered radial trap frequency $\omega_\perp^2 \to \omega_\perp^2 - \Omega^2$

$$|\Psi(r,z)|^2 = n(0)\left(1 - \frac{r^2}{R_\perp^2} - \frac{z^2}{R_z^2}\right)$$

where

$$R_\perp^2 = \frac{2\mu}{M(\omega_\perp^2 - \Omega^2)} \quad \text{and} \quad R_z^2 = \frac{2\mu}{M\omega_z^2}$$

- for pure harmonic trap, must have $\Omega < \omega_\perp$ to retain radial confinement
- normalization $\int dV |\Psi|^2 = N$ shows that

  $$\frac{\mu(\Omega)}{\mu(0)} = \left(1 - \frac{\Omega^2}{\omega_\perp^2}\right)^{2/5}$$

  in three dimensions
- central density given by $n(0) = \mu(\Omega)/g$
- $n(0)$ decreases with increasing Ω because of reduced radial confinement

- TF formulas for condensate radii show that

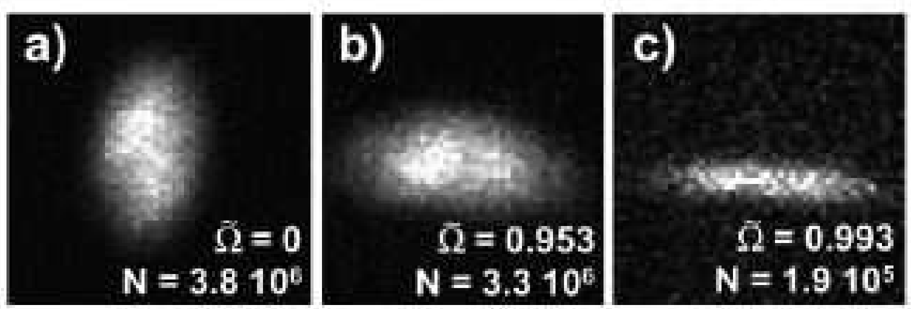
$$\frac{R_z(\Omega)}{R_z(0)} = \left(1 - \frac{\Omega^2}{\omega_\perp^2}\right)^{1/5}, \quad \frac{R_\perp(\Omega)}{R_\perp(0)} = \left(1 - \frac{\Omega^2}{\omega_\perp^2}\right)^{-3/10}$$

confirming axial shrinkage and radial expansion

- for nonzero Ω, aspect ratio changes

$$\frac{R_z(\Omega)}{R_\perp(\Omega)} = \frac{\sqrt{\omega_\perp^2 - \Omega^2}}{\omega_z}$$

- this last effect provides an important diagnostic tool to determine actual angular velocity Ω [19, 18]



- measured aspect ratio [18] indicates that $\Omega/\omega_\perp$ can become as large as ≈ 0.993

How uniform is the vortex array?

The analysis of the TF density profile $|\Psi_{TF}|^2 = n_{TF}$ in the rotating condensate assumed that the flow velocity $\boldsymbol{v}$ was precisely the solid-body value $\boldsymbol{v}_{\rm sb} = \boldsymbol{\Omega} \times \boldsymbol{r}$

- this led to the cancellation of the contribution

$$\int dV \, (\boldsymbol{v} - \boldsymbol{\Omega} \times \boldsymbol{r})^2 \, n_{TF}$$

  in the TF energy functional

- a more careful study [20] shows that there is a small nonuniformity in the vortex lattice

- specifically, each regular vortex lattice position vector $\boldsymbol{r}_j$ experiences a small displacement field $\boldsymbol{u}(\boldsymbol{r})$, so that $\boldsymbol{r}_j \to \boldsymbol{r}_j + \boldsymbol{u}(\boldsymbol{r}_j)$

- as a result, the two-dimensional vortex density changes to

$$n_v(\boldsymbol{r}) \approx \overline{n_v} \, (1 - \boldsymbol{\nabla} \cdot \boldsymbol{u})$$

  where $\overline{n_v} = M\Omega/\pi\hbar$ is the uniform Feynman value

- variation with respect to $\boldsymbol{u}$ yields an Euler-Lagrange equation that can be solved to give

$$\boldsymbol{u}(\boldsymbol{r}) \approx \frac{\bar{l}^2}{4R_\perp^2} \ln\left(\frac{\bar{l}^2}{\xi^2}\right) \frac{\boldsymbol{r}}{1 - r^2/R_\perp^2}$$

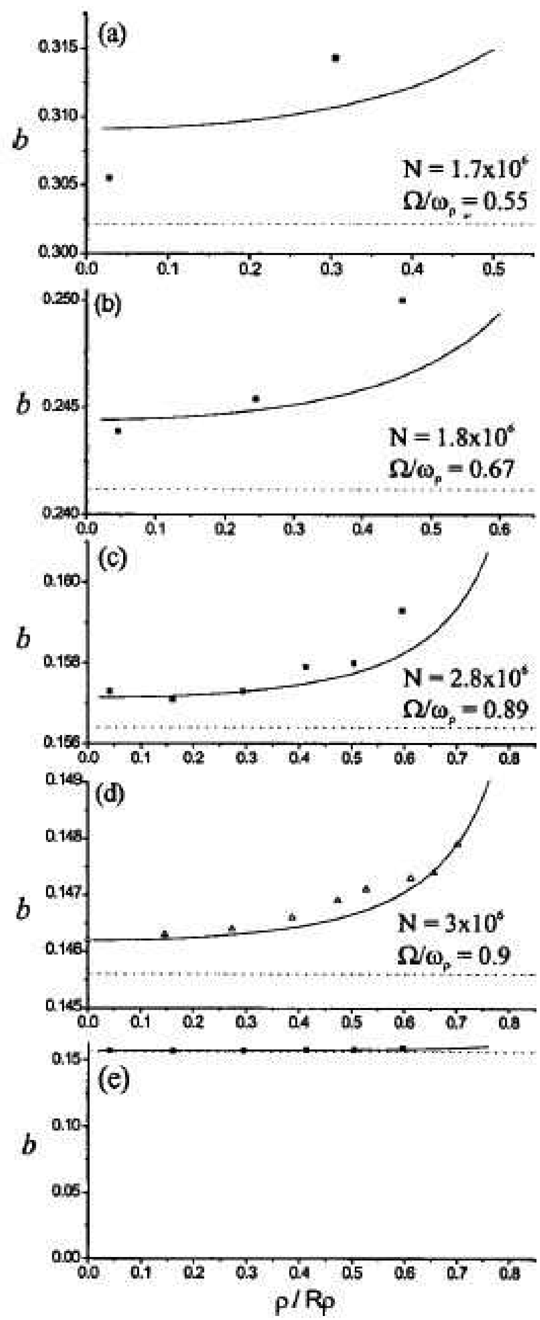
  where $\pi\bar{l}^2 = 1/\overline{n_v}$ can be taken as the area of a circular vortex cell inside the slowly varying logarithm

- the deformation of the regular vortex lattice is purely radial (as expected from symmetry)
- $R_\perp^2/\bar{l}^2$ is the number of vortices $\mathcal{N}_v$ in the rotating condensate, so that the nonuniform distortion is small, of order $1/\mathcal{N}_v$ (at most a few %), even though the TF number density n_{TF} changes dramatically near edge
- correspondingly, the vortex density becomes

$$n_v(r) \approx \overline{n_v} - \frac{1}{2\pi R_\perp^2} \ln\left(\frac{\bar{l}^2}{\xi^2}\right) \frac{1}{\left(1 - r^2/R_\perp^2\right)^2}$$

  (the correction is again of order $1/\mathcal{N}_v$)

• recent JILA experiments [21] confirm these predicted small distortions for relatively dense vortex lattices [note suppressed zero for Figs. (a)-(d); Fig. (e) shows that effect is small]



Tkachenko oscillations of the vortex lattice

Tkachenko (1966) [22] studied equilibrium arrangement of a rotating vortex array as model for superfluid ^{4}He

- assumed two-dimensional incompressible fluid with straight vortices
- showed that a triangular lattice has lowest energy in rotating frame
- small perturbations about equilibrium positions had unusual collective motion in which vortices undergo nearly transverse wave of lattice distortions (like two-dimensional transverse "phonons" in vortex lattice, but with no change in fluid density)
- for long wavelengths (small k), Tkachenko found a linear dispersion relation $\omega_k \approx c_T k$
- speed of Tkachenko wave $c_T = \sqrt{\frac{1}{4}\hbar\Omega/M} = \frac{1}{2}\hbar/M\bar{l}$, where $\bar{l} = \sqrt{\hbar/M\Omega}$ is radius of circular vortex cell

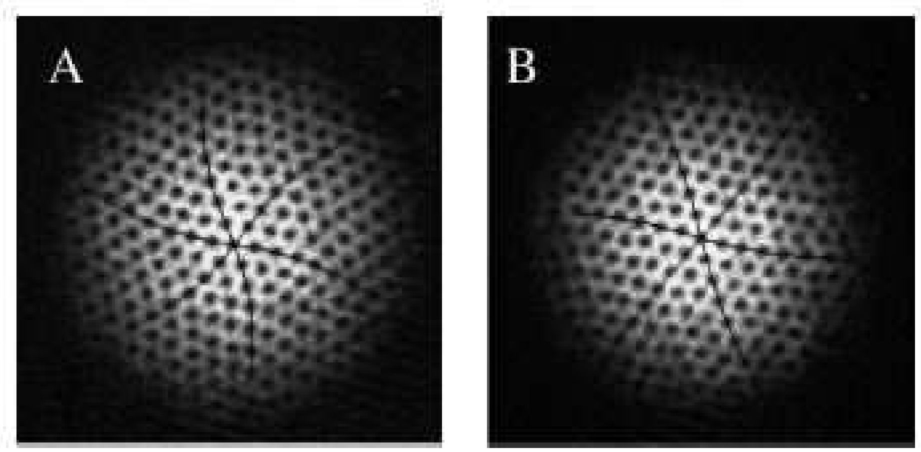
In a rotating two-dimensional gas, the compressibility becomes important, as shown by Sonin [23, 24] and Baym [25]

- let the speed of sound in the compressible gas be c_s
- coupling between the vortices and the compressible fluid leads to generalized dispersion relation

$$\omega^2 = c_T^2 \, \frac{c_s^2 k^4}{4\Omega^2 + c_s^2 k^2}$$

- if $c_s k \gg \Omega$, recover Tkachenko's result $\omega = c_T k$ (short-wavelength incompressible limit)
- but if $c_s k \ll \Omega$ (long wavelength), mode becomes *soft* with $\omega \propto k^2$
- Sonin [24] obtains dynamical equations for waves in a nonuniform condensate, along with appropriate boundary conditions at the outer surface
- Baym [25] uses theory for uniform condensate plus approximate boundary conditions from Anglin and Crescimanno [26]

- rough agreement with JILA experiments [27] on low-lying Tkachenko modes in rapidly rotating BEC (up to $\Omega/\omega_\perp \approx 0.975$)



- A is 1/4 period after weak perturbation (note deformation of lines of vortices)
- B is 3/4 period after weak perturbation (note reversed deformation of lines of vortices)

4 Vortex arrays in mean-field quantum-Hall regime

Lowest-Landau-Level (quantum-Hall) behavior

When the vortex cores overlap, kinetic energy associated with density variation around each vortex core becomes important

- hence the TF approximation breaks down (it ignores this kinetic energy from density variations)
- return to full GP energy $E'[\Psi]$ in the rotating frame.
- in this limit of rapid rotations ($\Omega \lesssim \omega_\perp$), Ho [28] incorporated kinetic energy *exactly*
- condensate expands and is effectively two dimensional
- for simplicity, treat a two-dimensional condensate that is *uniform* in the z direction over a length Z
- condensate wave function $\Psi(\boldsymbol{r}, z)$ can be written as $\sqrt{N/Z}\,\psi(\boldsymbol{r})$, where $\psi(\boldsymbol{r})$ is a two-dimensional wave function with unit normalization $\int d^2r\,|\psi|^2 = 1$

General two-dimensional energy functional in rotating frame becomes

$$E'[\psi] = \int d^2r\,\psi^* \left(\underbrace{\frac{p^2}{2M} + \frac{1}{2}M\omega_\perp^2 r^2 - \Omega L_z}_{\text{one}-\text{body oscillator } \mathcal{H}_0} + \underbrace{\frac{1}{2}g_{2D}|\psi|^2}_{\text{interaction}} \right) \psi,$$

where $\boldsymbol{p} = -i\hbar\boldsymbol{\nabla}$, $L_z = \hat{\boldsymbol{z}} \cdot \boldsymbol{r} \times \boldsymbol{p}$, and $g_{2D} = Ng/Z$

One-body oscillator hamiltonian in rotating frame $\mathcal{H}_0$ is exactly soluble and has eigenvalues [29]

$$\epsilon_{nm} = \hbar\left[\omega_\perp + n\left(\omega_\perp + \Omega\right) + m\left(\omega_\perp - \Omega\right)\right]$$

where n and m are non-negative integers

- in limit $\Omega \to \omega_\perp$, these eigenvalues are essentially independent of m (massive degeneracy)
- n becomes the Landau level index
- lowest Landau level with $n = 0$ is separated from higher states by gap $\sim 2\hbar\omega_\perp$

Large radial expansion means small central density $n(0)$, so that interaction energy $gn(0)$ eventually becomes small compared to gap $2\hbar\omega_\perp$

Hence focus on "lowest Landau level" (LLL), with $n = 0$ and general non-negative $m \geq 0$

- ground-state wave function is Gaussian $\psi_{00} \propto e^{-r^2/2d_\perp^2}$
- general LLL eigenfunctions have a very simple form

$$\psi_{0m}(\boldsymbol{r}) \propto r^m e^{im\phi}\, e^{-r^2/2d_\perp^2}$$

- here, $d_\perp = \sqrt{\hbar/M\omega_\perp}$ is analogous to the "magnetic length" in the Landau problem
- in terms of a complex variable $\zeta \equiv x + iy$, these LLL eigenfunctions become

$$\psi_{0m} \propto \zeta^m\, e^{-r^2/2d_\perp^2} \propto \zeta^m \psi_{00}$$

  with $m \geq 0$ (note that $\zeta = r\, e^{i\phi}$ when expressed in two-dimensional polar coordinates)
- apart from ground-state Gaussian ψ_{00}, this is just ζ^m (a non-negative power of the complex variable)

- assume that the GP wave function is a finite linear combination of these LLL eigenfunctions

$$\psi_{LLL}(\boldsymbol{r}) = \sum_{m\geq 0} c_m \psi_{0m}(\boldsymbol{r}) = f(\zeta)\, e^{-r^2/2d_\perp^2}$$

  where $f(\zeta) = \sum_{m\geq 0} c_m\, \zeta^m$ is an *analytic function* of the complex variable ζ

- specifically, $f(\zeta)$ is a complex polynomial and thus can be factorized as $f(\zeta) = \prod_j (\zeta - \zeta_j)$ apart from overall constant

- $f(\zeta)$ vanishes at each of the points $\{\zeta_j\}$, which are the positions of the zeros of ψ_{LLL}

- in addition, phase of wave function increases by 2π whenever ζ moves around any of these zeros $\{\zeta_j\}$

- we conclude that the LLL trial solution has singly quantized vortices located at positions of zeros $\{\zeta_j\}$

- basic conclusion: vortices are *nodes* in the condensate wave function

- spatial variation of number density $n(\boldsymbol{r}) = |\psi_{LLL}(\boldsymbol{r})|^2$ is determined by spacing of the vortices
- core size is comparable with the intervortex spacing $\bar{l} = \sqrt{\hbar/M\Omega}$ which is simply $d_\perp$ in the limit $\Omega \approx \omega_\perp$
- unlike TF approximation at lower Ω, wave function ψ_{LLL} automatically includes *all* the kinetic energy
- since LLL wave functions play a crucial role in the quantum Hall effect (two-dimensional electrons in a strong magnetic field), this LLL regime has been called "mean-field quantum-Hall" limit [30]
- note that we are still in the regime governed by GP equation, so there is still a BEC
- corresponding many-body ground state is simply a Hartree product with each particle in *same* one-body solution $\psi_{LLL}(\boldsymbol{r})$, namely

$$\Psi_{GP}(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) \propto \prod_{n=1}^{N} \psi_{LLL}(\boldsymbol{r}_n)$$

- this is *coherent (superfluid)* state, since a single GP state ψ_{LLL} has *macroscopic* occupation

Take this LLL trial function seriously

- for *any* LLL state ψ_{LLL} that is linear combination of ψ_{0m}, can show that (use oscillator units with $\omega_\perp$ and $d_\perp$ for energy and length) [28, 30, 31]
$$\int d^2r\, r^2\, |\psi_{LLL}|^2 = 1 + \int d^2r\, \psi^*_{LLL} L_z \psi_{LLL}$$
namely $M\omega_\perp \langle r^2\rangle = \hbar + \langle L_z\rangle$ in dimensional units

- dimensionless energy functional becomes
$$E'[\psi_{LLL}] = \Omega + \int d^2r\ \left[(1-\Omega)\, r^2 |\psi_{LLL}|^2 + \tfrac{1}{2} g_{2D} |\psi_{LLL}|^4\right]$$

- unrestricted variation with respect to $|\psi|^2$ would lead to inverted parabola
$$|\psi|^2 = n(r) = \frac{2}{\pi R_0^2}\left(1 - \frac{r^2}{R_0^2}\right)$$
where $\pi R_0^4 = 2g_{2D}/(1-\Omega)$ fixes condensate radius

- looks like earlier TF profile, but here include all kinetic energy explicitly

- these results ignore vortices and violate form of ψ_{LLL}

- to include effect of vortices, study logarithm of the particle density for any LLL state
- use ψ_{LLL} to find

$$\ln n_{LLL}(\boldsymbol{r}) = -\frac{r^2}{d_\perp^2} + 2\sum_j \ln|\boldsymbol{r} - \boldsymbol{r}_j|$$

where $\boldsymbol{r}_j$ is the position of the jth vortex

- apply two-dimensional Laplacian: use standard result $\nabla^2 \ln|\boldsymbol{r} - \boldsymbol{r}_j| = 2\pi\delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}_j)$ to obtain

$$\nabla^2 \ln n_{LLL}(\boldsymbol{r}) = -\frac{4}{d_\perp^2} + 4\pi\sum_j \delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}_j)$$

- here, sum over delta functions is precisely the *vortex* density $n_v(\boldsymbol{r})$
- this result relates *particle* density $n_{LLL}(\boldsymbol{r})$ in LLL approximation to *vortex* density $n_v(\boldsymbol{r})$ [28, 30, 31]

$$n_v(\boldsymbol{r}) = \frac{M\omega_\perp}{\pi\hbar} + \frac{1}{4\pi}\nabla^2 \ln n_{LLL}(\boldsymbol{r})$$

- if vortex lattice is exactly uniform (so n_v is constant), then density profile is strictly Gaussian, with $n_{LLL}(\boldsymbol{r}) \propto \exp(-r^2/\sigma^2)$ and $\sigma^{-2} = M\omega_\perp/\hbar - \pi n_v = M(\omega_\perp - \Omega)/\hbar$

- note that $\sigma^2 \gg d_\perp^2$, so that this resulting Gaussian is much bigger than original ground stat

- to better minimize the energy, mean density profile $\overline{n}_{LLL}$ should approximate inverted parabolic shape $\overline{n}_{LLL}(\boldsymbol{r}) \propto 1 - r^2/R_\perp^2$

- then find *nonuniform* vortex density with

$$n_v(r) \approx \frac{1}{\pi d_\perp^2} - \frac{1}{\pi R_\perp^2} \frac{1}{\left(1 - r^2/R_\perp^2\right)^2}$$

  similar to result at lower Ω [20] (in both cases, small correction term is of order $\sim \mathcal{N}_v^{-1}$)

- independently, numerical work by Cooper *et al.* [32] shows that allowing the vortices in the LLL to deviate from the triangular array near the outer edge lowers the energy

5 Behavior for $\Omega \to \omega_\perp$

What happens beyond the "mean-field quantum Hall" regime is still subject to vigorous debate

Predict *quantum phase transition* from *coherent BEC* states to various *correlated many-body* states

- define the ratio $\nu \equiv N/\mathcal{N}_v$ = the number of atoms per vortex
- because of similarities to a two-dimensional electron gas in a strong magnetic field, ν is called the "filling fraction" [33, 34]
- current experiments [18] have $N \sim 10^5$ and $\mathcal{N}_v \sim$ several hundred, so $\nu \sim$ a few hundred
- numerical studies [34] for small number of vortices ($\mathcal{N}_v \lesssim 8$) and variable N indicate that the coherent GP state is favored for $\nu \gtrsim 6 - 8$
- softening of Tkachenko spectrum for rapid rotation can induce melting of vortex lattice at similar filling fraction ν [35, 36]

- in either scenario, BEC and the associated coherent state disappears at a quantum phase transition for $\nu \sim$ 5-10
- replaced by qualitatively different correlated states that are effectively incompressible vortex liquids
- for smaller ν, there is a sequence of *highly correlated* states similar to some known from the quantum Hall effect
- in particular, theorists have proposed *boson* version of the *Laughlin* state [34] (here $z_n = x_n + iy_n$ refers to nth particle)

$$\Psi_{\text{Lau}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) \propto \prod_{n<n'}^{N} (z_n - z_{n'})^2 \exp\left(-\sum_{n=1}^{N} \frac{|z_n|^2}{2d_\perp^2} \right)$$

Note: these correlated many-body states are *qualitatively different* from the coherent GP state

- the GP state $\Psi_{GP}(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) \propto \prod_n \psi(\boldsymbol{r}_n)$ is the Hartree product of N factors of *same* one-body function $\psi(\boldsymbol{r})$
- in the Laughlin state $\Psi_{\mathrm{Lau}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N)$, the 2-body product $\prod_{n<n'}(z_n - z_{n'})^2$ involves $N(N-1)/2$ factors for all possible *pairs* of particles
- note that this correlated wave function vanishes when any two particles approach each other
- thus it minimizes total interaction energy for short-range repulsive potentials, which is the source of the correlations
- for large N, the correlated Laughlin state is much more difficult to use

What is physics of this phase transition?
Why does it occur for relatively large $\nu \sim 10$?

- for small N and large L (namely small ν), exact ground states have correlated form [33]
- different symmetry of ground states for small and large ν requires a quantum phase transition at some intermediate critical ν_c
- For N particles in two dimensions, there are $2N$ degrees of freedom
- each vortex has *one* collective degree of freedom
- hence $\mathcal{N}_v$ vortices have $\mathcal{N}_v$ collective degrees of freedom
- usually treated as additive, but in fact only $2N - \mathcal{N}_v$ particle degrees of freedom remain
- for $N \gg \mathcal{N}_v$ (namely large ν), depletion of particle degrees is unimportant
- eventually, when $\mathcal{N}_v$ is a finite fraction of N (namely $\nu \sim 10$), original description fails and get transition to new correlated ground state

How to reach the highly correlated regime?

- need to reduce the ratio $\nu = N/\mathcal{N}_v$ (number of atoms per vortex)
- very challenging since need small $N \lesssim 100$ and rapid rotation $\Omega \gtrsim 0.999\,\omega_\perp$
- one possibility is to use one-dimensional array of small pancake condensates trapped in optical lattice
- need to rotate each condensate to a relatively high angular velocity
- several experimental groups working on this option

References

[1] F. Dalfovo, S. Giorgini, L. P. Pitaevskii and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).

[2] A. L. Fetter, in *Bose-Einstein Condensation in Atomic Gases*, edited by M. Inguscio, S. Stringari and C. E. Wieman (IOP Press, Amsterdam, 1999), p. 201.

[3] A. L. Fetter and A. A. Svidzinsky, J. Phys.: Condens. Matter **13**, R135 (2001).

[4] A. L. Fetter, J. Low Temp. Phys. **129**, 263 (2002).

[5] G. Baym and C. J. Pethick, Phys. Rev. Lett. **76**, 6 (1996).

[6] E. P. Gross, Nuovo Cimento **20**, 454 (1961).

[7] L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **40**, 646 (1961) [Sov. Phys. JETP **13**, 451 (1961)].

[8] L. D. Landau and E. M. Lifshitz, *Mechanics*, Pergamon Press, Oxford (1960); E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics*, Pergamon Press, Oxford (1980).

[9] A. A. Svidzinsky and A. L. Fetter, Phys. Rev. Lett. **84**, 5919 (2000).

[10] E. Lundh and P. Ao, Phys. Rev. A **61**, 063612 (2000).

[11] S. A. McGee and M. J. Holland, Phys. Rev. A **63**, 043608 (2001).

[12] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman and E. A. Cornell, Phys. Rev. Lett. **83**, 2498 (1999).

[13] B. P. Anderson, P. C. Haljan, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **85**, 2857 (2000).

[14] K. W. Madison, F. Chevy, W. Wohllenben and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000).

[15] K. W. Madison, F. Chevy, W. Wohllenben and J. Dalibard, J. Mod. Opt. **47**, 2725 (2000).

[16] E. J. Yarmchuk, M. J. V. Gordon, and R. E. Packard, Phys. Rev. Lett. **79**, 214 (1979).

[17] J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle, Science **292**, 476 (2001).

[18] V. Schweikhard, I. Coddington, P. Engels, V.P. Mogendorff, and E. A. Cornell, Phys. Rev. Lett. **92**, 040404 (2004).

[19] P. C. Haljan, I. Coddington, P. Engels, and E. A. Cornell, Phys. Rev. Lett. **87**, 210403 (2001).

[20] D. E. Sheehy and L. Radzihovsky, Phys. Rev. A **70**, 051602(R) (2004); Phys. Rev. A **70**, 063620 (2004) .

[21] I. Coddington, P. C. Haljan, P. Engels, V. Schweikhard, S. Tung, and E. A. Cornell, Phys. Rev. A **70**, 063607 (2004).

[22] V. K. Tkachenko, Zh. Eksp. Teor. Fiz. **49**, 1875 (1965) [Sov. Phys. JETP **22**, 1282 (1966)]; Zh. Eksp. Teor. Fiz. **50**, 1573 (1966) [Sov. Phys. JETP **23**, 1049 (1966)].

[23] E. B. Sonin, Rev. Mod. Phys. **59**, 87 (1987).

[24] E. B. Sonin, Phys. Rev. A **71**, 011603(R) (2005).

[25] G. Baym, Phys. Rev. Lett. **91**, 110402 (2003).

[26] J. R. Anglin and M. Crescimanno, cond-mat/0210063.

[27] I. Coddington, P. Engels, V. Schweikhard and E. A. Cornell, Phys. Rev. Lett. **91**, 100402 (2003).

[28] T.-L. Ho, Phys. Rev. Lett. **87**, 060403 (2001).

[29] C. Cohen-Tannoudji, B. Diu and F. Laloë, *Quantum Mechanics* (J. Wiley & Sons, New York, 1977), Volume I, pp. 742-764.

[30] G. Watanabe, G. Baym, and C. J. Pethick, Phys. Rev. Lett. **93**, 190401 (2004).

[31] A. Aftalion, X. Blanc, and J. Dalibard, Phys. Rev. A **71**, 023611 (2005).

[32] N. R. Cooper, S. Komineas, and N. Read, Phys. Rev. A **70**, 033604 (2004).

[33] N. K. Wilkin and M. J. F. Gunn, Phys. Rev. Lett. **84**, 6 (2000).

[34] N. R. Cooper, N. K. Wilkin, and M. J. F. Gunn, Phys. Rev. Lett. **87**, 120405 (2001).

[35] J. Sinova, C. B. Hanna, and A. H. MacDonald, Phys. Rev. Lett. **89**, 030403 (2002).

[36] G. Baym, Phys. Rev. A **69**, 043618 (2004).