Sharp Stability of Attractive Bose-Einstein Condensates

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§ 1 Background

- **Attractive Bose-Einstein condensate**

Bradley, Sackett, Hulet

Evidence, 1995

Limited condensate number, 1997

- **Model: Gross-Pitaevskii (GP) equation**

$$i\hbar\frac{\partial\tilde{\psi}}{\partial\tilde{t}} = -\frac{\hbar^2}{2m}\triangle\tilde{\psi} + \frac{m\omega^2|\tilde{x}|^2}{2}\tilde{\psi} + \frac{4\pi\hbar^2\tilde{a}}{m}|\tilde{\psi}|^2\tilde{\psi}.$$

By variable rescaling

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + |x|^2\psi + a|\psi|^2\psi. \qquad (0)$$

$\psi := \psi(x,t), x \in R^D, t \geq 0,$ **wave function.**

$$A := \int_{R^D} (|\nabla\psi|^2 + |x|^2|\psi|^2 + \frac{1}{2}a|\psi|^4)dx$$

$$N := \int_{R^D} |\psi|^2 dx.$$

- **Attractive BEC:** $a < 0$**.**

Critical mass number N_c $\Rightarrow$ Stability

Let $\psi(x,0)$ **satisfy** $\int_{R^D} |\psi(x,0)|^2 dx = N_0.$

$\forall\lambda > 0,$ **put** $\psi_\lambda := \lambda^{D/2}\psi(\lambda x, 0).$ $\Rightarrow$

$$N_\lambda = \int_{R^D} |\psi_\lambda|^2 dx = N_0.$$

$$A_\lambda = \int_{R^D} [\lambda^2|\nabla_x\psi(x,0)|^2 + \lambda^{-2}|x|^2|\psi(x,0)|^2 + \frac{1}{2}a\lambda^D|\psi(x,0)|^4]dx.$$

For $D \geq 3, \lambda$ **large** $\Rightarrow$ $A_\lambda < 0.$

- N_0 **small** $\Rightarrow$ **negative energy** $\Rightarrow$ **collapse.**

Contradict to the experiments

§ 2 Sharp Stability

Modified model for $a < 0$

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + |x|^2\psi + a|\psi|^{4/D}\psi. \tag{1}$$

$$\psi(x,0) = \psi_0, \qquad\qquad x \in R^D. \tag{2}$$

$$H := \{u \in H^1(R^D), \int_{R^D} |x|^2|u|^2 dx < \infty\}$$

$$< u, v >_H := \int_{R^D} \nabla u \nabla \bar{v} + u\bar{v} + |x|^2 u\bar{v} dx.$$

H**, Hilbert space,** $||\cdot||_H$.

$$E(\varphi) := \int |\nabla\varphi|^2 + |x|^2|\varphi|^2 + \frac{1}{1+2/D} a|\varphi|^{2+4/D} dx. \tag{3}$$

Lemma 1 *(Local existence, Oh '89; Cazenave '89)*

Let $\psi_0 \in H$*.* $\Rightarrow \exists!$ *a solution* ψ *of the Cauchy problem (1)–(2) in* $C([0,T);H)$ *for some* $T \in (0,\infty]$ *(maximal existence time).* $\psi(t,\cdot)$ *satisfies*

$$\int |\psi|^2 dx = \int |\psi_0|^2 dx,$$

$$E(\psi) = E(\psi_0)$$

for all $t \in [0,T)$*. Furthermore* $T = \infty$ *or else* $T < \infty$ *and*

$$\lim_{t\to T} ||\psi||_H = \infty (collapse).$$

Lemma 2 *(Cazenave '89; Zhang '99)*

Let $\psi_0 \in H$ *and* ψ *be a solution of (1) – (2) in* $C([0,T);H)$*.*

Put $J(t) := \int |x|^2 |\psi|^2 dx$*.* $\Rightarrow$

$$J'(t) = -2\Im \int x\psi \nabla \bar{\psi} dx$$

$$J''(t) = 8E(\psi_0) - 16 \int |x|^2 |\psi|^2 dx.$$

Consider the variational problem

$$d := \inf_{\{u \in H^1(R^D)\}} \frac{(\int |\nabla u|^2 dx)(\int |u|^2 dx)^{2/D}}{(\int |u|^{2+4/D} dx)}, \tag{4}$$

and the nonlinear scalar field equation

$$-\Delta u + \frac{2}{D} u - |u|^{4/D} u = 0, \qquad u \in H^1(R^D) \tag{5}$$

Lemma 3 *(Kwong '89, Weinstein '83)*

$\exists!$ *a positive radially symmetric solution* $Q(x)$ *of (5). Moreover (4) attains at* $Q(x)$ *and*

$$d = \frac{D}{2+D} (\int Q^2 dx)^{2/D}.$$

Theorem 1 ***(Existence of sharp threshold)***

If $\psi_0 \in H$ ***and*** $||\psi_0||_{L^2}^2 < |a|^{-D/2} ||Q||_{L^2}^2$***,***

$\Rightarrow$ ***the solution*** $\psi(x,t)$ ***of (1) – (2) exists globally in time.***

For $\lambda > 0$ ***and*** $|c| \geq 1,$ ***let*** $\psi_0 = c\lambda^{D/2} |a|^{-D/4} Q(\lambda x)$***,***

$\Rightarrow$ $||\psi_0||_{L^2}^2 \geq |a|^{-D/2} ||Q||_{L^2}^2$ ***and the solutions*** $\psi(x,t)$ ***of (1) – (2) collapse in a finite time.***

- **The critical value**

$$N_c = |a|^{-D/2} \int Q^2 dx$$

- **Verify:**

$$N_c = [|a|^{-D/4}||Q||_{L^2}]^2 = (1.148 \times 10^{-2})^{-1} \times 2\pi \times 1.862 \cdots = 1019$$

Define a variational problem for $N > 0$.

$$d_N := \inf_{\{u \in H, \int |u|^2 dx = N\}} E(u). \tag{6}$$

Theorem 2 ***(Existence of soliton)***

If $N < |a|^{-D/2}||Q||_{L^2}^2$***,***

$\Rightarrow$ ***(6) attains at some*** $u \in H$***.***

Denote the set of the minimizers of (6) by S_N**.** $u \in S_N$ $\Rightarrow \exists$ **Lagrange multiplier** $\Lambda \in R$ **such that** u **is a solution of**

$$-\Delta u + |x|^2 u + \Lambda u + au|u|^{4/D} = 0. \qquad (7)$$

$\Rightarrow \psi(t,x) = e^{i\Lambda t} u$ **is a solitary wave of (1) with ground state.**

Theorem 3 ***(Orbital stability)***

If $N < |a|^{-D/2}||Q||_{L^2}^2 \Rightarrow \forall \varepsilon > 0$***,*** $\exists\ \sigma > 0$ ***such that for any*** $\psi_0 \in H$***, if***

$$\inf_{u \in S_N} ||\psi_0 - u||_H < \sigma,$$

then the solution $\psi(t,x)$ ***of (1) – (2) satisfies***

$$\inf_{u \in S_N} ||\psi(t,\cdot) - u(\cdot)||_H < \varepsilon, \quad for \quad all \quad t \geq 0.$$

§ 3 Sharp Threshold for Global Existence

$$i\psi_t = \Delta\psi + |x|^2\psi + a|\psi|^{p-1}\psi \tag{8}$$

$$\psi(x,0) = \psi_0, \qquad\qquad x \in R^D. \tag{9}$$

$$a < 0, \qquad 1 \le p < \frac{D+2}{(D-2)^+} := \begin{cases} \infty, & D = 1,2, \\ \frac{D+2}{D-2}, & D \ge 3. \end{cases}$$

$$E(u) := \frac{1}{2}\int |\nabla u|^2 + |x|^2|u|^2 + \frac{2}{p+1}a|u|^{p+1}dx$$

For any $\mu > 0$, define

$$d_\mu := \inf_{\{u\in H, \int |u|^2dx=\mu\}} E(u).$$

$$S_\mu := \{\text{the} \quad \text{minimizer} \quad u \quad \text{of} \quad d_\mu\}$$

For $u \in S_\mu$ **,** $\exists\ \omega \in R$ **such that**

$$-\Delta u + |x|^2 u + \omega u + au|u|^{p-1} = 0,\ \ u \in H \setminus \{0\}. \tag{10}$$

$\star$ **Standing wave is the solution of equation (8) with the form** $\psi(t,x) = e^{i\omega t}u(x)$**, where** $\omega \in \mathbf{R}$ **is the frequency and** $u(x)$ **is a solution of the nonlinear elliptic equation (10)**

Theorem 4 ***For*** $1 < p < 1 + \frac{4}{D}$***, the standing waves of (8) are orbitally stable, that is, for*** $\forall \varepsilon > 0$***,*** $\exists\ \sigma > 0$ ***such that for any*** $\psi_0 \in H$***, if***

$$\inf_{u \in S_\mu} ||\psi_0 - u||_H < \sigma,$$

then the solution ψ ***of the Cauchy problem (8)-(9) satisfies***

$$\inf_{u \in S_\mu} ||\psi(t,\cdot) - u(\cdot)||_H < \varepsilon, \quad\ \ for \quad all \quad t \geq 0.$$

Let $\omega \in R$ **,** $u \in H$**,**

$$I(u) := \frac{1}{2}\int |\nabla u|^2 + |x|^2|u|^2 + \omega|u|^2 - \frac{2a}{p+1}|u|^{p+1}dx$$

$$S(u) := \int |\nabla u|^2 + |x|^2|u|^2 + \omega|u|^2 + a|u|^{p+1}dx$$

$$Q(u) := \int 2|\nabla u|^2 - 2|x|^2|u|^2 + \frac{p-1}{p+1}Da|u|^{p+1}dx$$

$$M := \{u \in H, S(u) < 0, Q(u) = 0\}$$

$\star$ **Cross-constrained Variational Problem**

$$d_M := \inf_M I(u)$$

$$d_\omega := \inf_{\{u \in H \setminus \{0\}, S(u)=0\}} I(u)$$

$$d := \min\{d_\omega, d_M\}$$

Theorem 5 $d > 0$ ***if and only if*** $1 + \frac{4}{D} \leq p < \frac{D+2}{(D-2)^+}$.

$$K := \{\phi \in H, I(\phi) < d, S(\phi) < 0, Q(\phi) < 0\}$$

Theorem 6 ***Let*** $1 + \frac{4}{D} \leq p < \frac{D+2}{(D-2)^+}$ ***and*** ψ_0 ***satisfy*** $I(\psi_0) < d$***. Then the solution of the Cauchy problem (8) - (9) collapses in a finite time if and only if*** $\psi_0 \in K$ ***.***

$*$ **Stability of standing waves**

$\diamond$ **Zhang(2000):** $p < 1 + \frac{4}{D}$**,** $\exists \omega \in \mathbf{R}$**,** $e^{i\omega t}u$ **is orbitally stable.**

$\diamond$ **Fukuizumi and Ohta (2003):** $p < 1 + \frac{4}{D}$**,** $\omega \to +\infty$**,** $e^{i\omega t}u$ **is orbitally stable.**

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$\diamond$ **Fukuizumi and Ohta (2003):** $p > 1 + \frac{4}{D}$**,** $\omega \to +\infty$**,** $e^{i\omega t}u$ **is unstable.**

In addition,

$\diamond$ **Fukuizumi(2001) :** $p > \frac{D^2+4+4\sqrt{D^2+1}}{D^2} (> 1 + \frac{4}{D})$, $\omega > 0$, $e^{i\omega t}u$ **is unstable .**

$\diamond$ **Fukuizumi and Ohta(2003):** $1 < p < \frac{D+2}{(D-2)^+}$ **,** $\omega \to -\lambda_*$ **and** $\omega > -\lambda_*$**,** $e^{i\omega t}u(x)$ **is stable. Here** $\lambda_* > 0$ **is the first eigenvalue of** $-\triangle + |x|^2$**.**

§4 More Problems

$*$ **Sharp threshold for global existence and blow-up**

$\diamond$ **Zhang, Chen, Shu and Gan, 2000 $\sim$ 2007**

$*$ **Blow up dynamics**

$\diamond$ **Merle and Raphaël, 1989 $\sim$, 2003 $\sim$ 2007;**

$\diamond$ **Zhang, Li, Liu and Zhu, 1994 $\sim$, 2000 $\sim$ 2007.**

§5 References

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