STRONG CONFINEMENT FOR THE GROSS-PITAEVSKII EQUATION :

A MATHEMATICAL FRAMEWORK USING ALMOST PERIODICITY

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Bose-Einstein Condensation and Quantized Vortices in Superfluidity and Superconductivity

Motivation of this work

Physical context : description of the transport of a Bose-Einstein condensate strongly confined along one or two directions, via the Gross-Pitaevskii equation (mean field theory).

Objective : justify a model in reduced dimensions thanks to asymptotics analysis.

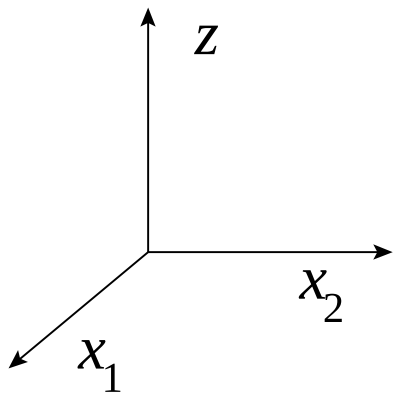
$$3\mathrm{d} \text{ NLS equation} \quad \xrightarrow{\varepsilon \to 0} \quad 2\mathrm{d} \text{ or } 1\mathrm{d} \text{ model}$$

Outline of the presentation

1. Scaling and heuristics

Consider a quantum gas described by the GP equation and subject to a strong anisotropic harmonic potential with $0 < \varepsilon \ll 1$:

$$i\partial_t \Psi = -\Delta\Psi - \Omega L_z \Psi + \underbrace{\left(|x|^2 + \frac{z^2}{\varepsilon^\alpha} \right)}_{\text{confinement potential}} \Psi + \delta\, \varepsilon^\beta \, |\Psi|^2 \Psi$$



The space variables denoted (x_1, x_2, z) do not play the same roles :

– strong confinement in direction $z \in \mathbb{R}$

– transport allowed in directions $x = (x_1, x_2) \in \mathbb{R}^2$

(Disk-shape confinement in the talk. But cigar-shape also works by setting $z \in \mathbb{R}^2$, $x \in \mathbb{R}$)

Rescaling the equation

We have in mind that $\sqrt{\varepsilon} \ll 1$ is the size of the condensate in direction z :

$$\Psi(t,x,z) = \frac{1}{\varepsilon^{1/4}}\psi\left(t,x,\frac{z}{\sqrt{\varepsilon}}\right)$$

Then

$$i\partial_t\psi = \mathcal{H}_x\psi + \frac{1}{\varepsilon}(-\partial_z^2 + \varepsilon^{2-\alpha}\, z^2)\psi + \delta\, \varepsilon^{\beta-1/2}\, |\psi|^2\psi$$

with

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Tuning of the parameters : let us choose $\alpha = 2, \quad \beta = 1/2$.

Formal asymptotics as $\varepsilon \to 0$

$$i\partial_t \psi = \mathcal{H}_x \psi + \frac{1}{\varepsilon} \mathcal{H}_z \psi + \delta \left|\psi\right|^2 \psi \qquad \text{with} \quad \mathcal{H}_z = -\partial_z^2 + z^2$$

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At the order ε^{-1} : we have $i\partial_t \psi = \frac{1}{\varepsilon} \mathcal{H}_z \psi$

This is a spectral problem : introduce the eigenpairs $(E_p, \chi_p(z))_{p\in\mathbb{N}}$ of $\mathcal{H}_z$

$$\mathcal{H}_z\, \chi_p = E_p\, \chi_p \qquad (p \in \mathbb{N})$$

then for all z one can decompose ψ on this complete orthonormal basis :

$$\psi = \sum_{p=0}^{\infty} \phi_p\, e^{-itE_p/\varepsilon}\, \chi_p(z)$$

Modulation of the coefficients w.r.t. the other variables (t, x) :

$$\psi(t,x,z) = \sum_{p=0}^{\infty} \phi_p(t,x)\, e^{-itE_p/\varepsilon}\, \chi_p(z)$$

Finally, the initial problem is equivalent to the system of coupled bidimensional GP equations :

$$i\partial_t \phi_p = \mathcal{H}_x \phi_p + \sum_{q,r,s} \gamma_{p,q,r,s}\, e^{-it(E_q+E_r-E_s-E_p)/\varepsilon}\, \phi_q\, \phi_r\, \overline{\phi_s}$$

where $\gamma_{p,q,r,s} = \delta \int \chi_p\, \chi_q\, \chi_r\, \chi_s\, dz$

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At the order ε^0 : pass to the limit in the red terms $\Longrightarrow$ 0 or 1

The formal limiting model

$$\psi(t,x,z) = \sum_{p=0}^{\infty} \phi_p(t,x)\, e^{-itE_p/\varepsilon}\, \chi_p(z)$$

where the ϕ_p's solve an infinite system of coupled 2D Schrödinger equations

$$i\partial_t \phi_p = \mathcal{H}_x \phi_p + \sum_{(q,r,s)\in\Lambda_p} \gamma_{p,q,r,s}\ \phi_q\, \phi_r\, \overline{\phi_s}$$

$$\Lambda_p = \Big\{ (q,r,s) \in \mathbb{N}^3 : \quad E_p + E_s = E_q + E_r \Big\}$$

Major difficulty : well-posedness of the limiting model and summation of series for initial data not prepared on one single mode.

Two former works :

[Ben Abdallah-F.M.-Schmeiser-Weishäupl 2005]

Rigourous convergence proof in the special case $\psi(t=0) = \phi_0\, \chi_0$ by simple energy arguments.

[Bao-Markowich-Schmeiser-Weishäupl 2005]

Numerics for the limiting model in the general case, analysis of a truncated model.

Here : we prove the convergence in the general case.

General framework

$$i\partial_t \psi = \mathcal{H}_x \psi + \frac{1}{\varepsilon} \mathcal{H}_z \psi + F(|\psi|^2)\psi$$

$$\mathcal{H}_z = -\partial_z^2 + V_c(z)$$

with

$$V_c(z) \to +\infty \text{ as } z \to \pm\infty, \quad V_c \geq 0$$

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Key point : do not project on the χ_p's !

Filtering procedure

Set

$$\psi(t,x,z) = e^{-it\mathcal{H}_z/\varepsilon}\,\Phi(t,x,z).$$

Then the initial problem

$$i\partial_t\psi = \mathcal{H}_x\psi + \frac{1}{\varepsilon}\mathcal{H}_z\psi + F(|\psi|^2)\psi$$

is equivalent to

$$i\partial_t\Phi = \mathcal{H}_x\Phi + \widetilde{F}\left(\frac{t}{\varepsilon},\Phi(t)\right)$$

with the nonlinearity

$$\widetilde{F}(\tau,u) = e^{i\tau\mathcal{H}_z}\,F\left(\left|e^{-i\tau\mathcal{H}_z}u\right|^2\right)e^{-i\tau\mathcal{H}_z}u\,.$$

ODE-like phenomenon : decoupling of the fast and slow time scales

Prototype :

$$y'_\varepsilon = a y_\varepsilon + f\left(\frac{t}{\varepsilon}, y_\varepsilon(t)\right)$$

Theorem : If the time average $f_{\text{av}}(\cdot) = \lim\limits_{T\to\infty} \frac{1}{T}\int_0^T f(\tau,\cdot)\,d\tau$ exists

Then $y_\varepsilon \longrightarrow y$ as $\varepsilon \longrightarrow 0$ and the limiting equation is

$$y' = a y + f_{\text{av}}(y)$$

2. Reformulation of the limiting model

ODE-like phenomenon : decoupling of the fast and slow time scales

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Then $y_\varepsilon \longrightarrow y$ as $\varepsilon \longrightarrow 0$ and the limiting equation is

$$y' = ay + f_{\text{av}}(y)$$

References

ODEs : Sanders-Verhulst 1985, Bidegaray-Castella-Degond 2004 (*systems*)

PDEs : Schochet 1994 (*hyperbolic systems*), Grenier 1997 (*fluid mechanics*)

Candidate for our limiting model as $\varepsilon \to 0$:

$$i\partial_t \Phi = \mathcal{H}_x \Phi + F_{\mathrm{av}}(\Phi(t))$$

with

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Remarks :

- unknown $\Phi(t, x, z)$ with 3d space variables but 2d dynamics
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Steps of the proof :

– design a functional framework to define properly this time average

– adapt to our PDEs the ODE proof of fast/slow time scales decoupling

We need to average with respect to τ the function

$$\widetilde{F}(\tau, u) = e^{i\tau\mathcal{H}_z} F\left(\left|e^{-i\tau\mathcal{H}_z}u\right|^2\right) e^{-i\tau\mathcal{H}_z}u$$

where $\mathcal{H}_z$ has a discrete spectrum, with no uniform gap assumption.

Two crucial mathematical tools :

- Adapted functional spaces : a Sobolev scale adapted to the two operators $\mathcal{H}_z = -\partial_z^2 + V_c(z)$ and $\mathcal{H}_x = -\Delta_x + |x|^2 - \Omega L_z$
- Existence of average in time : Hilbert–valued almost periodic functions

Sobolev scale adapted to $\mathcal{H}_z$ and $\mathcal{H}_x$

$$B^m = \left\{ u : \quad \|u\|_{B^m}^2 = \|u\|_{L^2}^2 + \|\mathcal{H}_x^{m/2} u\|_{L^2}^2 + \|\mathcal{H}_z^{m/2} u\|_{L^2}^2 < \infty \right\}$$

where $m \in \mathbb{N}$

Interest : $e^{-it\mathcal{H}_z/\varepsilon}$ commutes with $\mathcal{H}_x$ and $\mathcal{H}_z$ $\Longrightarrow$ uniform estimates

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Difficulty : in order to manipulate such spaces, there is a need to identify this norm. One can prove that it is equivalent to the more convenient norm

$$\|u\|_{H^m}^2 + \||x|^m u\|_{L^2}^2 + \|V_c(z)^{m/2} u\|_{L^2}^2$$

Proof in the case without rotation ($\Omega = 0$ and $\mathcal{H}_x = -\Delta_x + |x|^2$)

Not obvious !

It requires adapted Weyl-Hörmander pseudodifferential calculus.

Ref. : Helffer 1984, Bony-Chemin 1994.

Extension to condensates in rotation

It works under the condition $\Omega < 1$, which ensures that $-\Omega L_z$ is strictly bounded by $-\Delta_x + |x|^2$.

Hilbert–valued almost periodic functions

Definition

$AP(\mathbb{R}, B^m)$ is the closure of the set of trigonometric polynomials

$$\sum_{k=1}^{K} e^{i\lambda_k \tau} u_k, \qquad \lambda_k \in \mathbb{R}, \quad u_k \in B^m$$

with respect to the $L^\infty(\mathbb{R}, B^m)$ norm.

Properties of $AP(\mathbb{R}, B^m)$

Time average

Let $\Theta(\tau) \in AP(\mathbb{R}, B^m)$. Then the following strong limit exists in B^m :

$$\Theta_{\mathrm{av}} := \lim_{T\to\infty} \frac{1}{T} \int_0^T \Theta(\tau)\, d\tau.$$

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Stability

(i) Let $\Theta(\tau) \in AP(\mathbb{R}, B^m)$. Then the following function also belongs to B^m :

$$e^{i\tau\mathcal{H}_z}\,\Theta(\tau)$$

discrete spectrum of $\mathcal{H}_z$!

(ii) If $m \geq 2$ the space $\Theta(\tau) \in AP(\mathbb{R}, B^m)$ is an algebra.

Take $m \geq 2$, $F \in C^\infty$ and $u(x,z) \in B^m$. Then the function

$$\widetilde{F}(\tau, u) = e^{i\tau\mathcal{H}_z}\, F\left(\left|e^{-i\tau\mathcal{H}_z} u\right|^2\right) e^{-i\tau\mathcal{H}_z} u$$

belongs to $AP(\mathbb{R}, B^m)$.

Hence one can define the long time average

$$F_{\mathrm{av}}(u) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \widetilde{F}(\tau, u)\, d\tau\,,$$

the function $u \mapsto F_{\mathrm{av}}(u)$ is locally Lipschitz continuous on B^m and it satisfies the tame estimate

$$\|F_{\mathrm{av}}(u)\|_{B^m} \leq C_F\left(\|u\|_{L^\infty}\right) \|u\|_{B^m}\,.$$

Main result

- If $0 \leq \Omega < 1$, the limiting model as $\varepsilon \to 0$

$$i\partial_t \Phi = \mathcal{H}_x \Phi + F_{\text{av}}(\Phi(t))$$

is well-posed on B^m locally in time.

- For the initial model

$$i\partial_t \Phi = \mathcal{H}_x \Phi + \widetilde{F}\left(\frac{t}{\varepsilon}, \Phi(t)\right),$$

the fast and slow time scales can be decoupled and one can prove convergence as $\varepsilon \to 0$ to the limit model in B^m, locally in time.

- Only the discrete spectrum of $\mathcal{H}_z$ is used (no small divisor assumption).

5. Final remarks

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- The life time of the solution of the initial model can be as close to the life time of those of the limiting model as we want, provided ε is small enough.

5. Final remarks

- Only the discrete spectrum of $\mathcal{H}_z$ is used (no small divisor assumption).
- The life time of the solution of the initial model can be as close to the life time of those of the limiting model as we want, provided ε is small enough.
- The limiting models enjoys the following conservation laws :

$$\|\Phi(t)\|_{L^2} = \text{const}, \qquad \|(\mathcal{H}_z^{m/2})\Phi(t)\|_{L^2} = \text{const},$$

$$\|(\mathcal{H}_x)^{1/2}\Phi(t)\|_{L^2}^2 + \int G_{\text{av}}(\Phi(t))\, dx dz = \text{const},$$

where $G_{\text{av}}(\Theta) := \lim_{T\to\infty} \int_0^T G\left(\left|e^{-i\tau\mathcal{H}_z}\Theta\right|^2\right) d\tau, \qquad G' = F,$

Proof in the framework of the energy space : see talk of N. Ben Abdallah.