Workshop on Bose Einstein Condensation
IMS - NUS Singapore

Bose-like condensation in half-Bose half-Fermi statistics and in Fuzzy Bose-Fermi Statistics

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Part of a more general work done by Myself, Randall Espinoza and Tom D. Imbo (Phys. Dept. Univ. of Illinois at Chicago) about Generalized Particle Statistics.

Abstract

We present two statistics intermediate between Bose and Fermi statistics, namely the half-Bose half-Fermi statistics and the Fuzzy Bose-Fermi statistics. The two statistics have Hilbert spaces that is invariant under particle permutation operation, and obeying cluster decomposition. Additionally the Fuzzy Bose-Fermi statistics can have small deviation from Bose statistics and interpolate between the Bose and Fermi statistics. Starting from the grand canonical partition function of both statistics, we could obtained thermodynamics properties for the case of free non interacting particles (ideal gas). In particular, we show a Bose-like condensation for both statistics, in which the critical temperatures are different than the Bose case.

Keywords :Bose-like condensation, Intermediate statistics, quantum statistics

Introduction

- All elementary particles that exist right now obey either Bose or Fermi statistics.
- Interaction between particles, and if the interaction is small enough, then use perturbation theory. But perturbation method never change the statistics.
- When interaction is strong enough, free particle approximation is no longer valid and bound states could be formed. The bound state could be treated as particle which may have different statistics than its composition.
- One may thought that in the transition between free states and bound states, the particle could be treated by some kind of intermediate statistics.

- In handling interaction between particles, people have used mean field theory and perturbation theory. Beyond these two methods, interaction is difficult to handle.
- Non interacting particles could also have 'interaction' due to their statistics.
- Is it possible a system of interacting bosons or fermions could be approximated effectively by a free particle system obeying different kind of statistics other than Bose and Fermi (?). No known exact correspondence.
- Since Haldane's fractional statistics paper ['Fractional Statistics' In Arbitrary Dimensions: A Generalization Of The Pauli Principle', Phys. Rev. Lett. **67**, 937 (1991)], there have been some efforts to use several intermediate statistics and other exotic statistics as an effective theory for many quasi-particle system in condensed matter physics. Recent ones: [Phys. Rev. **A75**, 042111 (2007)], [J.Phys. **A37** 2527-2536 (2004)], [arXiv:cond-mat/0610400v2], see also [A.

Khare, *Fractional Statistics and Quantum Theory*, (World Scientific Publ., Singapore, 2005)].

What I am going to do here

- Describe two (new) statistics
- Find their grand canonical parition function (GCPF)
- Show simple Bose-like condensation phenomena for free non interacting identical particle system (ideal gas) obeying the new statistics. Also some other thermodynamics properties.

We consider two statistics, that we call the half-Bose half-Fermi statistics and the Fuzzy Bose-Fermi statistics.

The simplest statistics having both Bose and Fermi states (but also have more besides those two) whose Hilbert space is invariant under particle permutation and obeys the cluster decomposition properties — that is, measuring any physical properties of a set of "isolated" particles should not depend on the existence or nonexistence of particles elsewhere.

The Statistics

Some definition and notation

- Hilbert space invariant under permutation operation $\longrightarrow$ the states can be classified according to how they transform under permutation operation. (n-particle states will be superselected into irreducible subspace representation (IRREPS) of the permutation group S_n).
- The IRREPS of S_n are labeled by partition of n, denoted with $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_i \geq \lambda_{i+1}$ and $\sum_i \lambda_i = n$.
- The labels of S_n IRREPS $\leftrightarrow$ Young tableau, a left justified array of boxes, with λ_i boxes in the i-th row.
- State(s) in IRREPS $\lambda \longrightarrow$ State with symmetry type λ

For example, the Young tableau for $\lambda = (4, 3, 2)$ is

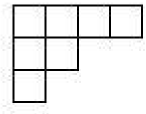

Figure 1: Young tableau for $\lambda = (4, 2, 1)$

Defining the statistics

- Determining the statistics $\longrightarrow$ determining the states symmetry type (determining the Hilbert space). But there should be some constraint .

- For example we cannot just form a Hilbert space consist of totally symmetric and totally anti symmetric states (or in other word we cannot just add Bose and Fermi statistics).

- Cluster decomposition properties $\longrightarrow$ Constraint on the allowed set of state symmetry types (constraint on λ).

- Hartle, Stolt and Taylor [Phys. Rev. D **2**, 1759 (1970)],[Phys. Rev. D **1**, 2226 (1970)] have shown that for permutation invariant statistics to obey cluster decomposition properties $\longrightarrow$

  1. All λ has to be included.

2. Or all λ inside the (p, q)-envelope has to be included.

(p, q)-envelope is a set of Young tableau whose $p+1$-th row has no more than q boxes.

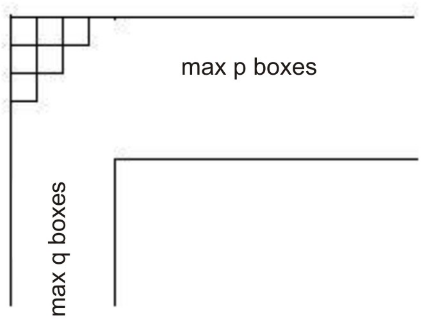

Figure 2: (p, q)-envelope

Thus, the simplest statistics that has both totally symmetric (Bose) and totally antisymmetric (Fermi) states has to include all states λ in $(1,1)$-envelope, or $\lambda = (N-k, \underbrace{1,\ldots,1}_{k}) \equiv (N-k, 1^k)$.

Meaning, we have to include mixed symmetry, not just the Bose and Fermi.

Lets just call this new statistics as the **half-Bose half-Fermi statistics.**

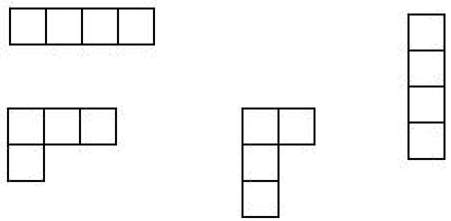

Figure 3: λ for 4 particles

Grand Canonical Partition Function

- Almost all thermodynamics properties are easy to derive once we get the grand canonical partition function (GCPF).
- The GCPF for non interacting particles in m energy levels obeying parastatistics has been known and can be written formally as [A. P. Polychronakos, 'Path Integrals and Parastatistics', Nucl. Phys. B **474**, 529 (1996)], [arXiv:hep-th/9605064], [arXiv:hep-th/9509150]

$$Z(x_1, \ldots, x_m) = \sum_{\lambda \in \Lambda} s_\lambda(x_1, \ldots, x_m). \tag{1}$$

where $x_i = e^{\beta(\mu - E_i)}$, Λ is the set of symmetry type (λ) allowed in the corresponding parastatistics.

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- $s_\lambda(x_1, \ldots, x_m)$ is the Schur polynomial in m variables defined as

$$s_\lambda(x_1, \ldots, x_m) = \begin{cases} \frac{|x_j^{m+\lambda_i - i}|}{\Delta(x_1, \ldots, x_m)} & \text{for } \lambda_{m+1} = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

with $\Delta(x_1, \ldots, x_m) = \prod_{i<j}^{m}(x_i - x_j)$ being the Vandermonde polynomial.

- But it turns out that their result also applied for any statistics whose (n-particle) Hilbert space is invariant under S_n.
- The half-Bose half Fermi statistics, the GCPF is given by (1) with Λ is the $(1,1)$-envelope, denoted as $\Lambda = \lambda \in (1,1)$.
- $$Z(x_1, \ldots, x_m) = \sum_{\lambda \in \Lambda} s_\lambda(x_1, \ldots, x_m). \qquad (3)$$
- But the sum in its current forms is useless unless if we can simplify it. (Like the case of Bose and Fermi)

Simplification of (3)

- The calculation involves Pieri's formula (from the theory of symmetric functions, see for ex. [W. Fulton, *Young Tableaux*, (Cambridge Univ. Press, New York, 1991)]):

$$e_k s_\lambda = \sum_\mu s_\mu, \tag{4}$$

  $e_k \equiv s_{(1^k)}$ is the elementary symmetric polynomial, and the μ sum is over all Young tableau obtained by adding k boxes to λ (as a tableau), with no two boxes in the same row.

- Illustration

- The sum over s_λ in $(1,1)$-envelope can be written as the sum over s_λ in the $(1,0)$-envelope (which is equal to the GCPF for Bose) multiplied by $\sum_{k=0} e_{2k}$.

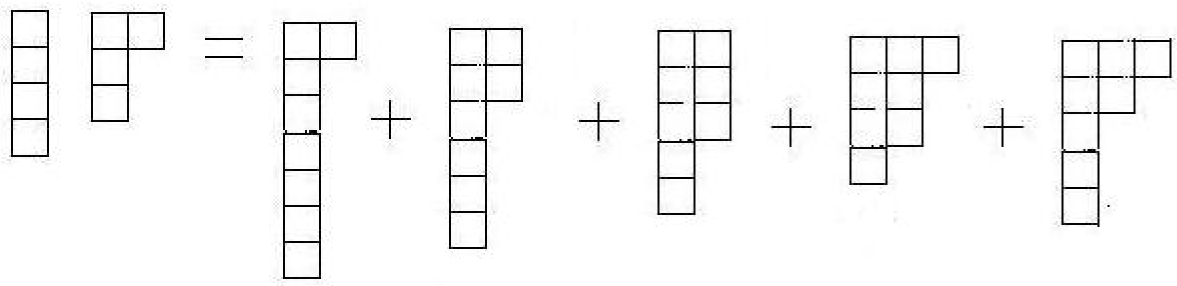

Figure 4: Example of Pieri's formula

- Illustration
- Thus,

$$Z_{(1,1)}(x_1, \ldots, x_m) = \frac{\sum_{k=0} e_{2k}(x_1, \ldots, x_m)}{\prod_{i=1}^{m}(1 - x_i)} = \frac{1}{2}\left(1 + \frac{\prod_{i=1}^{m} 1 + x_i}{\prod_{i=1}^{m} 1 - x_i}\right). \tag{5}$$

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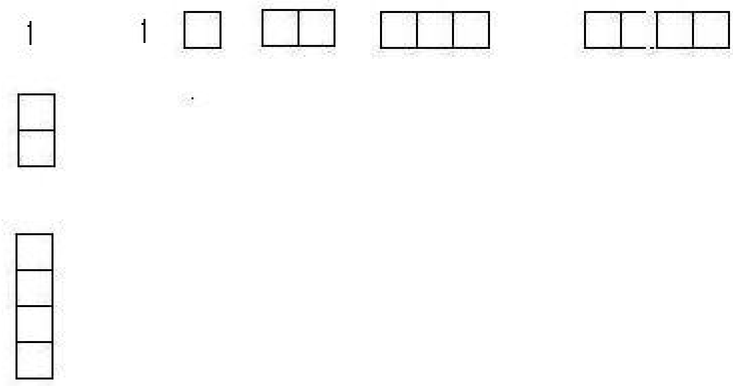

Figure 5: Rewriting the (1,1)-envelope

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A side issue: Counting function

The counting function, $w(\nu_1, \ldots, \nu_m) \longrightarrow$ the number of physically distinct linearly independent n-particle states for a given multiplicity ν,

- Can be obtained from the GCPF

$$Z(x_1, \ldots, x_m) = \sum_{\nu} w(\nu_1, \ldots, \nu_m)\; m_\nu(x_1, \ldots, x_m). \qquad (6)$$

  $m_\nu(x_1, \ldots, x_m)$ is the monomial symmetric polynomial.

- Using the following relation [I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, (Clarendon Press, Oxford 1995)][W. Fulton, J. Harris, *Representation Theory*, (Springer, New York, 1991)]

$$s_\lambda(x_1, \ldots, x_m) = \sum_{\nu} K_{\lambda\nu}\; m_\nu(x_1, \ldots, x_m), \qquad (7)$$

$K_{\lambda\nu}$ is Kostka number

- We then have

$$w(\nu_1, \ldots, \nu_m) = \sum_{\lambda \in \Lambda_n} K_{\lambda\nu}. \tag{8}$$

- A related counting function $W(n,m)$, the number of distinct linearly independent n-particle states that can occupy m energy levels —— regardless of occupation numbers.

$$W(n,m) = \sum_{\nu_1+\cdots+\nu_m=n} w(\nu_1, \ldots, \nu_m). \tag{9}$$

- For $\lambda \in (1,1)$-envelope $K_{(n-l,1^l),\nu} = \binom{g-1}{l}$, where g is the number of non-zero ν_i's. Using this, we have

$$w_{(1,1)}(\nu_1, \ldots, \nu_m) = 2^{g-1} \tag{10}$$

•

$$W_{(1,1)}(n,m) = \frac{1}{2}\sum_{k=0}^{n}\binom{k+m-1}{k}\binom{m}{n-k} \tag{11}$$

$$= m\ {}_2F_1(1-m, 1-n; 2; 2). \tag{12}$$

where ${}_2F_1$ is the generalized hyper geometric function.

Another new statistics

- How about something that interpolate between Bose and Fermi statistics? But still obey cluster decomposition.
- In this direction, there are several attemmpt for example, the Haldane's fractional statistics. Defining the interpolating counting function.

$$W(n,m) = \frac{[m-(g-1)(n-1)]!}{n![m-g(n-1)-1]!}, \tag{13}$$

for $0 \leq g \leq 1$, ($g = 0$ Bose) ($g = 1$ Fermi). (Note that if the argument of the factorial above is fractional, it should be understood as a gamma function).

- But not just interpolate, the GCPF also has to factorizes into one energy level GCPF ($\rightarrow$ non interacting particles). If the Bose and Fermi GCPF's factorizes, in the intermediate it should also factorizes.

$$Z(x_1, \ldots, x_m) = \prod_{i}^{m} Z(x_i) \tag{14}$$

Fractional Counting

- We approach the problem by modifying the sum for the GCPF in (5) by adding c_λ in front of each Schur function.

$$Z_{(1,1)}(x_1, \ldots, x_m) = \sum_{\lambda \in (1,1)} c_\lambda s_\lambda(x_1, \ldots, x_m) \tag{15}$$

- Previously $c_\lambda = 1$, $\longrightarrow$ The number of states in each IRREPS λ (counting the states).
- Imposing the constraint in (14)

$$Z_{(1,1)}(x_1, \ldots, x_m) = \sum_{\lambda \in (1,1)} c_\lambda s_\lambda(x_1, \ldots, x_m) = \prod_{i}^{m} Z(x_i) \tag{16}$$

- But we also have

$$Z(x) = \sum_j c_{(j)} s_{(j)}(x) = \sum_j c_{(j)} x^j, \tag{17}$$

- Putting this back to (16) we have

$$\begin{aligned} \sum_{\lambda\in(1,1)} c_\lambda \; s_\lambda(x_1,\ldots,x_m) &= \prod_{i=1}^{m} \sum_{k_i} c_{(k_i)} s_{(k_i)}(x_i) \\ &= \sum_{k_1,\ldots,k_m} c_{(k_1)} \cdots c_{(k_m)} \; x_1^{k_1} \cdots x_m^{k_m}. \end{aligned} \tag{18}$$

- By rewriting the sum over k_i's above as a sum over all integers $n \geq 0$ and over all k_i's such that $k_1 + \cdots + k_m = n$, we can replace the above relation with

$$\sum_{\lambda\in(1,1)} c_\lambda \; s_\lambda(x_1,\ldots,x_m) = \sum_{\text{all } \nu} c_{(\nu_1)} \cdots c_{(\nu_m)} \; m_\nu(x_1,\ldots,x_m), \tag{19}$$

m_ν is the monomial symmetric polynomial [I. G. Macdonald].

- Using the relation between the Schur polynomials and the monomial symmetric polynomials, (19) can be written as

$$\sum_{\nu} \sum_{\lambda \in (1,1)} c_\lambda K_{\lambda\nu}\, m_\nu(x_1, \ldots, x_m) = \sum_{\nu} c_{(\nu_1)} \cdots c_{(\nu_m)}\, m_\nu(x_1, \ldots, x_m). \tag{20}$$

- We thus have

$$\sum_{\lambda \in (1,1)} K_{\lambda\nu}\, c_\lambda = c_{(\nu_1)} \cdots c_{(\nu_m)}. \tag{21}$$

- From the theory of symmetric functions, this set of relations is equivalent to the following determinant formula (see for example [I. G. Macdonald][W. Fulton, J. Harris],

$$c_\lambda = \det \left(c_{(\lambda_i + j - i)} \right). \tag{22}$$

- Once the values of the $c_{(k)}$'s are given, all the c_λ's are determined.
- A reflection of the fact that all information in the GCPF is contained inside the single energy level GCPF.

- From (21) for $\nu = (1, 1, \ldots, 1) \equiv (1^n)$ we have

$$\sum_{\lambda \vdash n} d_\lambda c_\lambda = c^n_{(1)}, \tag{23}$$

where we have used the fact that $K_{\lambda(1^n)} = d_\lambda$ [W. Fulton, J. Harris].

d_λ is the dimension of IRREPS λ

The meaning of c_λ's

- The value of $c_{(1)}$ should be set to 1, because there exists only one state for one-particle (with any fixed quantum number).
- As a consequence of this we have $\sum_{\lambda \vdash n} d_\lambda c_\lambda = 1$.
- Suggests a probabilistic interpretation of c_λ.
- Because d_λ always integer then c_λ must be fractional $\longrightarrow$ fractional counting.
- Fractional dimensionality of Hilbert (sub)space (?)

Another Condition

- The c_λ has to be non-negative (either it is a probability, a fractional counting or fractional dimension). Lets call this condition as *unitarity condition*.
- Unitarity condition $\leftrightarrow$ non-negativity of the determinants in (22).
- It has been shown that this condition is a constraint on the generating function for the $c_{(k)}$'s (see [I. G. Macdonald]), which for the case of $\lambda \in (1,1)$-envelope, it can be written as

$$\sum_k c_{(k)} t^k = \frac{(1+at)}{(1-bt)}, \tag{24}$$

- The a and b are non negative numbers satisfying $a+b=1$ (because $c_{(1)}=1$).

- If we choose $t = x$ in (24), this generating function is just the single energy level GCPF.
- Thus the GCPF is then given by

$$Z_{fuzzy}(x_1, x_2, \ldots, x_m) = \prod_{i=1}^{m} \frac{(1 + a x_i)}{(1 - b x_i)}. \tag{25}$$

  The b and a above can be interpreted as probabilities that the particles behave like boson or fermion.

- Note that this GCPF interpolating continuously from boson ($b = 1, a = 0$) to fermion ($b = 0, a = 1$)
- But also obeys cluster decomposition properties and unitarity.
- Let us called this as Fuzzy Bose-Fermi statistics.

Side Issue: Fuzzy Counting Function

- The counting function $w(\nu_1, \ldots, \nu_m)$ for Fuzzy Bose-Fermi statistics can be obtained easily from (16) and (24)

$$W(n,m) = \sum_{k+j=n} \binom{j+m-1}{j} \binom{m}{j} b^k a^j, \qquad (26)$$

$$W(n,m) = \frac{(n+m+1)!}{n!(m-1)!} {}_2F_1(1-n, -n; 1-m-n; 1-b), \quad (27)$$

where ${}_2F_1$ is the generalized hypergeometric function.

- As a comparison, the Haldane's statistics: the single energy level counting function for this theory is given by $w(n) = \frac{[1-(g-1)(n-1)]!}{n![1-g(n-1)-1]!}$, which we see is negative if $2k-1 < g(n-1) < 2k$ for $k = 1, 2, \ldots$. Only when $g = 0, 1$ is $W(n,m)$ in (13) become non-negative for all n. Thus Haldane's statistics does not obey unitarity condition.

Cluster coefficient

- Defined $S(k)$ through the single energy level GCPF by

$$\log Z(x) = \sum_{k=1} \frac{S(k)}{k} x^k. \tag{28}$$

- For Fuzzy Bose-Fermi statistics, in the continuum limit $S(k)$ is related to the cluster coefficient b_k by $b_k = S(k)/k$. For simplicity, we will often refer to $S(k)$ itself as a cluster coefficient.

- $$S(k) = b^k + (-1)^{k+1} a^k. \tag{29}$$

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Bose-like Condensation

Consider a system of ideal gas obeying the half-Bose half Fermi statistics and the Fuzzy Bose-Fermi statistics in a d-dimensional spatial space.

In half-Bose half-Fermi Statistics

Start from the GCPF

$$Z_{(1,1)}(x_1, \ldots, x_m) = \frac{\sum_{k=0} e_{2k}(x_1, \ldots, x_m)}{\prod_{i=1}^{m}(1 - x_i)} = \frac{1}{2}\left(1 + \frac{\prod_{i=1}^{m} 1 + x_i}{\prod_{i=1}^{m} 1 - x_i}\right). \tag{30}$$

Defined the following reduced grand potential

$$\begin{aligned}\tilde{\Phi} \equiv & -\frac{1}{\beta}\log\Big(Z(x_1,\ldots,x_m)-\frac{1}{2}\Big) \\ =&\frac{1}{\beta}\log 2-\frac{1}{\beta}\sum_{i=1}^{m}\big(\log(1+x_i)-\log(1-x_i)\big),\end{aligned} \tag{31}$$

from which

$$Z(x_1,\ldots,x_m)=\exp(-\beta\tilde{\Phi})+\frac{1}{2}. \tag{32}$$

The average number of particles in each energy level

$$N_k=-\frac{1}{\beta}\frac{\partial}{\partial E_k}\log Z=(1-\frac{1}{2Z})\frac{2x_k}{1-x_k^2}. \tag{33}$$

Certainly, it must always hold that $0\leq N_k\leq N$ with $N=\sum_k N_k$, for all energy levels E_k.

In particular, the ground state

$$\frac{2z}{1-z^2}\geq 0. \tag{34}$$

Therefore, $0 \leq z \leq 1$, which is the same as the restriction on the bosonic fugacity.

The GCPF in the continuum energy limit

- In the continuum energy limit, the reduced potential in (31) becomes

$$\tilde{\Phi} = -\frac{kTV}{\lambda_T^{d/2}}(f_{d/2+1}(z)+g_{d/2+1}(z))+kT\log 2-kT\log\frac{1+z}{1-z} \quad (35)$$

with $\lambda_T = \sqrt{2\pi\hbar^2\beta/m}$ is the thermal wavelength.

- From this the continuum limit for the GCPF can be obtained from

$$Z(T,V,\mu) = \exp(-\beta\tilde{\Phi}) + \frac{1}{2} \quad (36)$$

- The average number of particles

$$
\begin{aligned}
N(T,V,\mu) &= z\frac{\partial}{\partial z}\log Z(T,V,\mu) \\
&= \left(1-\frac{1}{2Z(T,V,\mu)}\right)\left(\frac{V}{\lambda_T^{d/2}}(f_{d/2}(z)+g_{d/2}(z))+\frac{2z}{1-z^2}\right) \\
&\equiv N_e + N_0,
\end{aligned}
\tag{37}
$$

where

$$
N_0 \equiv \left(1-\frac{1}{2Z(T,V,\mu)}\right)\frac{2z}{1-z^2} \tag{38}
$$

is the average particle number in ground state, and

$$
N_e = \left(1-\frac{1}{2Z(T,V,\mu)}\right)\frac{V}{\lambda_T^{d/2}}(f_{d/2}(z)+g_{d/2}(z)) \tag{39}
$$

- Because $Z(T,V,\mu\to 0)\to\infty$ in this limit, we have for the critical density and temperature

$$\frac{N_{\text{crit}}}{V} = \lambda_T^{-d}\zeta(d/2)2(1-\frac{1}{2^{d/2}}), \tag{40}$$

$$T_c = \frac{2\pi\hbar^2}{mk}\left(\frac{N}{V}\right)^{2/d}\left(\zeta(d/2)2(1-\frac{1}{2^{d/2}})\right)^{-2/d}, \tag{41}$$

For spatial dimensions $d \geq 3$, both cases above have higher critical particle densities, and lower critical temperatures, than the Bose case. A well-known result for Bose statistics that condensation cannot occur in a system with spatial dimension less than three remains true.

- The internal energy U

$$U = \frac{d}{2}\frac{kTV}{\lambda_T^d}\left(1-\frac{1}{2Z}\right)(f_{d/2+1}(z)+g_{d/2+1}(z)) \tag{42}$$

- The specific heat at constant volume

$$\frac{C_V}{Nk} = \frac{d(d+2)}{4}\frac{V}{\lambda_T^d}(f_{d/2+1}(z)+g_{d/2+1}(z)), \quad T \leq T_c, \tag{43}$$

For $T > T_c$,

$$\frac{C_V}{Nk} = \frac{d}{2}\Big(\frac{f_{d/2+1}(z) + g_{d/2+1}(z)}{f_{d/2}(z) + g_{d/2}(z)} \tag{44}$$

$$+ \frac{\partial z}{\partial T}\frac{1}{z}(1 + \frac{(f_{d/2+1}(z) + g_{d/2+1}(z))f_{d/2-1}(z) + g_{d/2-1}(z)}{(f_{d/2}(z) + g_{d/2}(z))^2})\Big). \tag{45}$$

Bose-Like condensation in Fuzzy Bose-Fermi statistics

Start from the GCPF

$$Z_{fuzzy}(x_1, x_2, \ldots, x_m) = \prod_{i=1}^{m} \frac{(1 + ax_i)}{(1 - bx_i)}. \tag{46}$$

- The fugacity $z = e^{\beta\mu}$ is not fixed and should be restricted according to the following condition for the particle number:

$$N = \sum_i N_i = -\beta z \frac{\partial \Phi}{\partial z}, \tag{47}$$

- For ground state

$$N_0 = \frac{bz}{1-bz} + \frac{az}{1+az}. \tag{48}$$

Thus, if $b \neq 0$ a finite $z_c > 1$ exists, and this Fuzzy Bose-Fermi statistics will exhibit a Bose-like condensation. From (48) we have $z_c = 1/b$.

- The grand canonical potential is given by

$$\Phi = -kT\frac{V}{\lambda_T^d}(f_{d/2+1}(z_1) + g_{d/2+1}(z_2)) - \frac{1}{\beta}\log\frac{1+z_1}{1-z_2}, \tag{49}$$

where $z_1 = az$ and $z_2 = bz$.

- The average particle number

$$N(T,V,\mu) = \frac{V}{\lambda_T^d}(f_{d/2}(z_1) + g_{d/2}(z_2)) + N_0, \tag{50}$$

where

$$N_e = \frac{V}{\lambda_T}(f_{d/2+1}(z_1) + g_{d/2+1}(z_2)) \tag{51}$$

is for the excited level,

$$N_0 = \frac{z_1}{1+z_1} + \frac{z_2}{1-z_2} \tag{52}$$

is for the ground state level (if $z = z_c$, the first part above is negligible).

- N_e remains finite at z_c, and thus the system exhibit a Bose-like condensation.
- The critical density of the excited states can be obtained in the limit when $z \to z_c$ as

$$\frac{N_{\rm crit}}{V} = \frac{1}{\lambda_T^d} G_{d/2}(z_c). \tag{53}$$

- Critical Temperature

$$T_c = \frac{2\pi\hbar^2}{mk}\left(\frac{N}{V}\right)^{2/d} (G_{d/2}(z_c))^{-2/d}, \tag{54}$$

where $G_n(z)$ is defined as

$$G_n(z) = f_n(z_1) + g_n(z_2) = \sum_{k=1}^{\infty} \frac{S(k)z^k}{k^n}. \tag{55}$$

and $S(k) = b^k + (-1)^{k+1}a^k$

- The internal energy U

$$U = \frac{kTV}{\lambda_T^d}\frac{d}{2}G_{\frac{d+2}{2}}(z), \tag{56}$$

- The heat capacity at constant volume C_V.

$$\frac{C_V}{Nk} = \frac{d(d+2)}{4}\frac{V}{N\lambda_T^d}G_{\frac{d+2}{2}}(z_c), \quad T \leq T_c, \tag{57}$$

$$\frac{C_V}{Nk} = (\frac{d(d+2)}{4}\frac{G_{\frac{d+2}{2}}(z)}{G_{\frac{d}{2}}(z)} - \frac{d^2}{4}\frac{G_{\frac{d}{2}}(z)}{G_{\frac{d-2}{2}}(z)}), \quad T > T_c, \tag{58}$$

T_c comparison in $d = 3$

For a fixed particle density N/V

- Bose :

$$T_c = \frac{2\pi\hbar^2}{mk}\left(\frac{N}{V}\right)^{2/3}(\zeta(3/2))^{-2/3}, \tag{59}$$

- Half-Bose half-Fermi :

$$T_c = \frac{2\pi\hbar^2}{mk}\left(\frac{N}{V}\right)^{2/3}\left(\zeta(3/2)2(1-\frac{1}{2^{3/2}})\right)^{-2/3}, \tag{60}$$

- Fuzzy Bose-Fermi ($b \neq 0$):

$$T_c = \frac{2\pi\hbar^2}{mk}\left(\frac{N}{V}\right)^{2/3}(f_{3/2}(\frac{1}{b}-1)+\zeta(3/2))^{-2/3}, \tag{61}$$

Conclusion

We have present two intermediate statistics, the half-Bose half-Fermi statistics and the Fuzzy Bose-Fermi statistics. The half-Bose half-Fermi statistics is an intermediate statistics that still obey cluster decomposition property and its Hilbert space is invariant under permutation operation. For free non interacting particles in $d \geq 3$ this statistics shows a Bose-like condensation phenomena with a lower T_c than the Bose case. The Fuzzy statistics is an intermediate statistics that still retain the properties of half-Bose and half-Fermi statistics but has a parameter that interpolating between Bose and Fermi statistics. This statistics also able to give a small deviation from Bose statistics but still retain a nice properties of Bose statistics, that is unitarity, extensivity and cluster decomposition. A Bose-like condensation also occur for this statistics, with T_c that can have small deviation from the Bose case.

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