Bose-like condensation in half-Bose half-Fermi statistics and in Fuzzy Bose-Fermi Statistics*

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Abstract : We present new particle statistics intermediate between Bose and Fermi statistics, namely the half-Bose half-Fermi statistics and the Fuzzy Bose-Fermi statistics. The two statistics have Hilbert spaces that are invariant under particle permutation operation, and obeying cluster decomposition. Additionally the Fuzzy Bose-Fermi statistics can have small deviation from Bose statistics and interpolate between the Bose and Fermi statistics. Starting from the grand canonical partition function of both statistics, we could obtained thermodynamics properties for the case of free non interacting particles (ideal gas). In particular, we show a Bose-like condensation for both statistics, in which the critical temperatures are lower than the Bose case.

Keywords : Bose-like condensation, intermediate statistics, quantum statistics

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1 INTRODUCTION

All elementary particles that we know exist right now obey either Bose or Fermi statistics. When there is interaction between particles, and if the interaction is small enough, then perturbation theory can be applied. We know that perturbation method never change the statistics of the particles, because we always use the bosonic or fermionic states as the basis states. When interaction between particles is strong enough, the free particle approximation is no longer valid and bound states could be formed. A bound state of even number of fermions will behave like boson, while a bound state of odd number of fermions will behave like fermion. One may thought that in the transition between free states and bound states, the particle could obey some kind of intermediate statistics.

In handling interactions people has used mean field theory and perturbation theory. Beyond these two method, interaction is difficult to handle. We also know that repulsive interaction between bosons can make bosons behaves as fermions, while an attractive interaction between fermion can make fermions behaves like bosons. In this direction, it could be thought that a system of interacting bosons or fermions may be approximated effectively by a free particle system obeying different kind of statistics other than Bose and Fermi, and replacing effectively their dynamical interaction with 'statistical interaction'. Even though there is no known exact correspondence (in dimension greater than one) between the dynamical interaction and statistical interaction, the simplicity of obtaining exact thermodynamics properties directly through the statistics method provides us with a lot of unexplored possible phenomena that cannot be given by the mean field and perturbative methods. Started from the fractional statistics paper of Haldane's [1], there have been some efforts to use several intermediate statistics and other

*Presented at the workshop on Bose Einstein Condensation (12-16 November 2007), Institute of Mathematical Sciences, National University of Singapore fractional statistics as an effective theory for many quasi-particle system in condensed matter physics. Several recent ones are [2][3][4], see also [5].

In this presentation, first, after some definition and notation, I will explain the statistics. Then I will continue with formulating the grand canonical partition function, of the new statistics. Once the GCPF is known, then almost all thermodynamics properties, including the critical temperature could be easily obtained.

The new statistics that we want to obtained is bassically the simplest statistics that is still invariant under particle permutation operation and obeying cluster decomposition properties — measuring a physical properties of a set isolated of particles should not depend on the existence or non existence of particle elsewhere —, having non negative counting functions, and for one of the case, interpolating between Bose and Fermi statistics. The new statistics obtained are just two of many possibilities inside a class of particle statistics whose Hilbert space is invariant under particle permutation operation and obeys the cluster decomposition properties, a more generalized parastatistical system whose grand canonical partition function have been worked out by Tom Imbo, Randall Espinoza and myself in an unpublished work (but see also [6]). A possible second quantize realization of these statistics has also been considered [7].

2 SOME DEFINITION AND NOTATION

As already known, the symmetry group for a system of identical particle in spatial dimension $d \geq 3$ is the permutation group S_n. For n-particle Hilbert space that is invariant under the action of S_n, the states can be classified according to how they transform under permutation operation. The states will be superselected into irreducible subspace representation of the permutation group. For each number of particle n, these irreducible subspace representation of the permutation group S_n can be labeled by partition of n, denoted with $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_i \geq \lambda_{i+1}$ and $\sum_i \lambda_i = n$. These partitions can be visualized with Young tableau, a left justified array of boxes in which there are λ_i boxes in the i-th row. For example, the Young tableau for $\lambda = (4, 2, 1)$ is given below. The Young tableau visualization are useful because state whose particles being

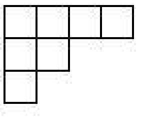

FIGURE 1: Young tableau for $\lambda = (4, 2, 1)$

represented by boxes in the same row can be simultaneously symmetrized while whose particles are represented by boxes in the same column can be simultaneously antisymmetrized. In Bose (Fermi) statistics the Hilbert space has only totally symmetric (one row) or totally antisymmetric states (one column), correspond to the one dimensional irreducible subspaces of S_n.

In forming intermediate statistics between Bose and Fermi, we cannot just form a Hilbert space consist of totally symmetric and totally anti symmetric states (or in other word we cannot just add Bose and Fermi statistics). Because the Hilbert space in this case will not obey cluster decomposition properties[8][9]. This cluster decomposition properties translates into restriction on the allowed λ in the Hilbert space. Hartle, Stolt and Taylor have shown that for permutation invariant statistics to obey cluster decomposition properties, either all state symmetry type has to be included (the case of infinite statistics) or all states whose symmetry type lies inside the (p, q)-envelope of Young tableaux has to be included [8][9]. The (p, q)-envelope is a set of Young tableau whose $p + 1$-th row has no more than q boxes. Bose statistics correspond to the $(1, 0)$-envelope while Fermi statistics correspond to $(0, 1)$-envelope. Therefore if we want to have an intermediate statistics, that has Bose and Fermi states, then all states in the $(1, 1)$-envelope has to be included and there are more states than the totally symmetric and totally

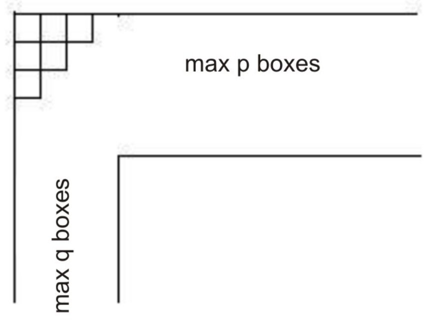

FIGURE 2: (p, q)-envelope

antisymmetric states. There will be mix states also, the one that are not totally symmetric or totally antisymmetric. In general the states in our system has symmetry type denoted with $\lambda = (N-k, \underbrace{1, \ldots, 1}_{k}) \equiv (N-k, 1^k)$. Lets just call this new statistics, the half-Bose half-Fermi statistics. Later on we will impose additional restriction, to get another new statistics.

3 GRAND CANONICAL PARTITION FUNCTION

The thermodynamics properties of any statistics can be obtained once the grand canonical partition function (GCPF) is obtained. The half-Bose half Fermi statistics falls into the class of statistics whose Hilbert space is invariant under particle permutation symmetry, the same like the case of parastatistics. For a system of non interacting particles inside m energy level, the GCPF for parastatistics has been known and it is given by [10],[11],[12]

$$Z(x_1, \ldots, x_m) = \sum_{\lambda \in \Lambda} s_\lambda(x_1, \ldots, x_m). \tag{1}$$

where $x_i = e^{\beta(\mu - E_i)}$, Λ is the set of symmetry type (λ) allowed in the corresponding statistics, and $s_\lambda(x_1, \ldots, x_m)$ is the Schur polynomial in m variables (see for ex. [13][14][15] for detail definition of Schur function and other symmetric polynomial in this paper). It turns out that the derivation for the parastatistics case is general and can also be applied to any statistics that is invariant under particle permutation, including for the half-Bose half-Fermi statistics. Thus, for the half-Bose half Fermi statistics, the GCPF is given by (1), but with Λ is the $(1, 1)$-envelope (denoted as $\Lambda = \lambda \in (1, 1)$).

It can be shown that for the case of $\Lambda \in (1, 0))$ and $(0, 1)$-envelope the sum in (1) reduced into the known forms of the Bose and Fermi GCPF

$$Z_{\text{Bose}}(x_1, \ldots, x_m) = \sum_{\lambda \in (1,0)} s_\lambda(x_1, \ldots, x_m) = \prod_{i=1}^{m} (1 - x_i)^{-1}. \tag{2}$$

$$Z_{\text{Fermi}}(x_1, \ldots, x_m) = \sum_{\lambda \in (0,1)} s_\lambda(x_1, \ldots, x_m) = \prod_{i=1}^{m} (1 + x_i). \tag{3}$$

While for our case, the sum in (1) is not so easy to simplify without using some technique from the symmetric function as follows.

First, the simplification involves Pieri's formula from the theory of symmetric functions [15]:

$$e_k s_\lambda = \sum_{\mu} s_\mu, \tag{4}$$

where $e_k \equiv s_{1^k}$ is the elementary symmetric polynomial, and the μ sum is over all Young tableau obtained by adding k boxes to λ (as a tableau), with no two boxes in the same row. From this formula, the sum over s_λ in the $(1,1)$-envelope can be written as the sum over s_λ in the $(1,0)$-envelope (which is equal to the GCPF for Bose) multiplied by $\sum_{k=0} e_{2k}$.

$$Z_{(1,1)} = \sum_{\lambda\in(1,1)} s_\lambda = \sum_{k=0} e_{2k} \sum_{\lambda\in(1,0)} s_\lambda. \tag{5}$$

Noting that e_k's has the following generating function,

$$\prod_{i=1}^{m}(1+tx_i) = \sum_{k=0}^{m} e_k(x_1,\ldots,x_m)t^k \tag{6}$$

Thus, after a little manipulation, we obtain the GCPF for the half-Bose half-Fermi statistics

$$Z_{(1,1)}(x_1,\ldots,x_m) = \frac{\sum_{k=0} e_{2k}(x_1,\ldots,x_m)}{\prod_{i=1}^{m}(1-x_i)} = \frac{1}{2}\left(1+\frac{\prod_{i=1}^{m} 1+x_i}{\prod_{i=1}^{m} 1-x_i}\right). \tag{7}$$

4 COUNTING FUNCTION

In the last few years, there are some papers that trying to generalize the Bose and Fermi statistics, making it a new statistics that interpolate between Bose and Fermi. The approach toward this direction is usually through the introduction of a new counting function, modifying the Bose and Fermi ones. It is thus important to know the counting function of our new statistics.

The counting function, $w(\nu_1,\ldots,\nu_m)$ which is basically the number of physically distinct linearly independent n-particle states for a given multiplicity ν, can be obtained from the GCPF through

$$Z(x_1,\ldots,x_m) = \sum_\nu w(\nu_1,\ldots,\nu_m)\ m_\nu(x_1,\ldots,x_m). \tag{8}$$

where $m_\nu(x_1,\ldots,x_m)$ is the monomial symmetric polynomial. Using the following relation [13][14]:

$$s_\lambda(x_1,\ldots,x_m) = \sum_\nu K_{\lambda\nu}\ m_\nu(x_1,\ldots,x_m), \tag{9}$$

where $K_{\lambda\nu}$ is the Kostka number(see for ex. [13]), we then have

$$w(\nu_1,\ldots,\nu_m) = \sum_{\lambda\in\Lambda_n} K_{\lambda\nu}. \tag{10}$$

A related counting function $W(n,m)$, gives the number of distinct linearly independent n-particle states that can occupy m energy levels —— regardless of occupation numbers. Clearly we have

$$W(n,m) = \sum_{\nu_1+\cdots+\nu_m=n} w(\nu_1,\ldots,\nu_m). \tag{11}$$

For $\lambda \in (1,1)$-envelope $K_{(n-l,1^l),\nu} = \binom{g-1}{l}$, where g is the number of non-zero ν_i's. Using this, we have

$$w_{(1,1)}(\nu_1,\ldots,\nu_m) = 2^{g-1} \tag{12}$$

and

$$W_{(1,1)}(n,m) = \frac{1}{2}\sum_{k=0}^{n}\binom{k+m-1}{k}\binom{m}{n-k} = m\ {}_2F_1(1-m,1-n;2;2). \tag{13}$$

where ${}_2F_1$ is the generalized hyper geometric function.

5 FUZZY BOSE-FERMI STATISTICS

When looking at the GCPF of the half-Bose half Fermi statistics, we notice that the GCPF does not factorizes into each single energy level GCPF, like the case of Bose and Fermi. This factorization should be natural for any non interacting particle systems. From other aspect, it would be nice if we could have just a small violation from the Bose statistics (or from the Fermi statistics), while at the same time still retain the properties of the half-Bose half Fermi statistics. Or it would be nice also if we could have a statistics interpolating continuously from Bose to Fermi. So here, we will form a new statistics, an extension of the half-Bose half-Fermi statistics, by adding another restriction on the GCPF, requiring it to be factorizes into one energy level GCPF like the case of Bose and Fermi. This factorization will lead to the extensivity of the usual extensive variable of the thermodynamics, although we could possibly also have extensivity without it.

To implement the restriction, we modified the sum in (7) by adding a coefficient c_λ in front each Schur function.

$$Z_{(1,1)}(x_1,\ldots,x_m)=\sum_{\lambda\in(1,1)} c_\lambda s_\lambda(x_1,\ldots,x_m). \tag{14}$$

Previously the $c_\lambda = 1$, which can be understood as the number of states in each irreducible subspace representation denoted by λ. By adding this coefficient, we would like to see other possibilities by allowing other values of c_lambda, including a non integer c_λ. To guarantee extensivity we required that

$$Z_{(1,1)}(x_1,\ldots,x_m)=\sum_{\lambda\in(1,1)} c_\lambda s_\lambda(x_1,\ldots,x_m)=\prod_i^m Z(x_i) \tag{15}$$

But we also have

$$Z(x)=\sum_j c_{(j)}s_{(j)}(x)=\sum_j c(j)x^j, \tag{16}$$

where $c_{(j)}$ is c_λ for $\lambda=(j)$. Putting this back to (15) we hve

$$\begin{aligned}\sum_{\lambda\in(1,1)} c_\lambda\ s_\lambda(x_1,\ldots,x_m) &= \prod_{i=1}^m\sum_{k_i} c_{(k_i)}s_{(k_i)}(x_i)\\ &= \sum_{k_1,\ldots,k_m} c_{(k_1)}\cdots c_{(k_m)}\ x_1^{k_1}\cdots x_m^{k_m}.\end{aligned} \tag{17}$$

By rewriting the sum over k_i's above as a sum over all integers $n\geq 0$ and over all k_i's such that $k_1+\cdots+k_m=n$, we can replace the above relation with

$$\sum_{\lambda\in(1,1)} c_\lambda\ s_\lambda(x_1,\ldots,x_m)=\sum_{\text{all }\nu} c_{(\nu_1)}\cdots c_{(\nu_m)}\ m_\nu(x_1,\ldots,x_m), \tag{18}$$

where we have used the definition of the monomial symmetric polynomial[13]. Using the relation between the Schur polynomials and the monomial symmetric polynomials, (18) can be written as

$$\sum_\nu\sum_{\lambda\in(1,1)} c_\lambda K_{\lambda\nu}\ m_\nu(x_1,\ldots,x_m)=\sum_\nu c_{(\nu_1)}\cdots c_{(\nu_m)}\ m_\nu(x_1,\ldots,x_m). \tag{19}$$

Because for different ν's the $m_\nu(x_1,\ldots,x_m)$'s are linearly independent, we thus have

$$\sum_{\lambda\in(1,1)} K_{\lambda\nu}\ c_\lambda = c_{(\nu_1)}\cdots c_{(\nu_m)}. \tag{20}$$

This is a set of relations between the coefficients c_λ and the $c_{(k)}$'s. From the theory of symmetric functions, this set of relations is equivalent to the following determinant formula (see for example [13][14]),

$$c_\lambda=\det\left(c_{(\lambda_i+j-i)}\right). \tag{21}$$

Thus, once the values of the $c_{(k)}$'s are given, the rest of the c_λ's in $(1,1)$-envelope are determined. This is a reflection of the fact that all information in an extensive GCPF is contained inside the single energy level GCPF.

From (20) for $\nu = (1,1,\ldots,1) \equiv (1^n)$ we have

$$\sum_{\lambda \vdash n} d_\lambda c_\lambda = c_{(1)}^n, \tag{22}$$

where we have used the fact that $K_{\lambda(1^n)} = d_\lambda$. The value of $c_{(1)}$ is normally set to be 1, which means that there exists only one state for one-particle (with any fixed quantum number). Since there is only 1 one-particle state (for each quantum number), then logically $c_{(1)} = 1$. As a consequence of this we have $\sum_{\lambda \vdash n} d_\lambda c_\lambda = 1$. This suggests a probabilistic interpretation of c_λ. The coefficient d_λ in (22) represents the (maximum) number of equivalent irreducible S_n subspaces corresponding to λ. We can also interpret a fractional c_λ as a fractional dimensionality of the Hilbert space or the fractional counting state, in the same spirit of the *fractional exclusion statistics* introduced by Haldane [1]. Nevertheless, the c_λ has to be non-negative. We call the condition that $c_\lambda \geq 0$ as *unitarity condition.*

The unitarity condition is equivalent to the non-negativity of the determinants in (21). It has been shown that this condition is a constraint on the generating function for the $c_{(k)}$'s [13], which for the case of $\lambda \in (1,1)$-envelope, it is written as

$$\sum_k c_{(k)} t^k = \frac{(1+at)}{(1-bt)}, \tag{23}$$

where the a and b are non negative numbers satisfy $a+b=1$ (because $c_{(1)} = 1$). If we choose $t = x$, this generating function is just the single energy level GCPF. Thus the GCPF is then given by

$$Z_{fuzzy}(x_1, x_2, \ldots, x_m) = \prod_{i=1}^{m} \frac{(1+ax_i)}{(1-bx_i)}. \tag{24}$$

The b and a above can be interpreted as the probability that the particles behave like boson or fermion. This GCPF interpolating continuously from boson ($b=1, a=0$) to fermion ($b=0, a=1$), but also obeys cluster decomposition properties and unitarity. Let us called this as Fuzzy Bose-Fermi statistics.

The counting function $w(\nu_1, \ldots, \nu_m)$ for Fuzzy Bose-Fermi statistics can be obtained easily from (15) and (23)

$$W(n,m) = \sum_{k+j=n} \binom{j+m-1}{j} \binom{m}{j} b^k a^j, \tag{25}$$

$$W(n,m) = \frac{(n+m+1)!}{n!(m-1)!} {}_2F_1(1-n, -n; 1-m-n; 1-b), \tag{26}$$

where ${}_2F_1$ is the generalized hypergeometric function.

6 BOSE-LIKE CONDENSATION

Here we will consider some thermodynamic properties of a system of free non interacting particles (ideal gas) that obey half-Bose half Fermi statistics and Fuzzy Bose-Fermi statistics in a d-dimensional space. In particular, we will consider the Bose-like condensation phenomena.

6.1 In half-Bose half-Fermi Statistics

Start from the GCPF given in (7), we can derive many thermodynamics functions. Because of the form of (7), it is useful to introduce the following reduced grand potential

$$\begin{aligned} \tilde{\Phi} &\equiv -\frac{1}{\beta} \log \Big(Z(x_1, \ldots, x_m) - \frac{1}{2} \Big) \\ &= \frac{1}{\beta} \log 2 - \frac{1}{\beta} \sum_{i=1}^{m} \big(\log(1+x_i) - \log(1-x_i) \big), \end{aligned} \tag{27}$$

In term of this the GCPF is

$$Z(x_1, \ldots, x_m) = \exp(-\beta\tilde{\Phi}) + \frac{1}{2}. \tag{28}$$

The average number of particles in each energy level is given by

$$N_k = -\frac{1}{\beta}\frac{\partial}{\partial E_k}\log Z = (1 - \frac{1}{2Z})\frac{2x_k}{1 - x_k^2}. \tag{29}$$

Certainly, it must always hold that $0 \leq N_k \leq N$ with $N = \sum_k N_k$, for all energy levels E_k. In particular, the ground state

$$\frac{2z}{1 - z^2} \geq 0. \tag{30}$$

Therefore, $0 \leq z \leq 1$, which is the same restriction as on the bosonic fugacity. From here, the calculation is straightforward, just like the case of ideal Bose and Fermi gas. In the continuum energy limit (free particles), the reduced potential in (27) becomes

$$\tilde{\Phi} = -\frac{kTV}{\lambda_T^{d/2}}(f_{d/2+1}(z) + g_{d/2+1}(z)) + kT\log 2 - kT\log\frac{1+z}{1-z} \tag{31}$$

with $\lambda_T = \sqrt{2\pi\hbar^2\beta/m}$ is the thermal wavelength. From this the continuum limit for the GCPF can be obtained from

$$Z(T, V, \mu) = \exp(-\beta\tilde{\Phi}) + \frac{1}{2} \tag{32}$$

The average number of particles in the continuum energy limit is

$$\begin{aligned} N(T, V, \mu) &= z\frac{\partial}{\partial z}\log Z(T, V, \mu) \\ &= \Big(1 - \frac{1}{2Z(T, V, \mu)}\Big)\Big(\frac{V}{\lambda_T^{d/2}}(f_{d/2}(z) + g_{d/2}(z)) + \frac{2z}{1 - z^2}\Big) \\ &\equiv N_e + N_0, \end{aligned} \tag{33}$$

where

$$N_0 \equiv \Big(1 - \frac{1}{2Z(T, V, \mu)}\Big)\frac{2z}{1 - z^2} \tag{34}$$

is the average particle number in ground state, and

$$N_e = \Big(1 - \frac{1}{2Z(T, V, \mu)}\Big)\frac{V}{\lambda_T^{d/2}}(f_{d/2}(z) + g_{d/2}(z)) \tag{35}$$

is the average particle number in the excited states. Because $Z(T, V, \mu \to 0) \to \infty$, we have for the critical density and temperature

$$\frac{N_{\text{crit}}}{V} = \lambda_T^{-d}\zeta(d/2)2(1 - \frac{1}{2^{d/2}}), \tag{36}$$

$$T_c = \frac{2\pi\hbar^2}{mk}\Big(\frac{N}{V}\Big)^{2/d}\big(\zeta(d/2)2(1 - \frac{1}{2^{d/2}})\big)^{-2/d}, \tag{37}$$

For spatial dimensions $d \geq 3$, both cases above have higher critical particle density, and lower critical temperature, than the Bose case. A well-known result for Bose statistics that condensation cannot occur in a system with spatial dimension less than three remains true. The internal energy U is also straight forwardly obtained

$$U = \frac{d}{2}\frac{kTV}{\lambda_T^d}\Big(1 - \frac{1}{2Z}\Big)(f_{d/2+1}(z) + g_{d/2+1}(z)) \tag{38}$$

From which, we can also obtained the specific heat at constant volume

$$\frac{C_V}{Nk} = \frac{d(d+2)}{4}\frac{V}{\lambda_T^d}(f_{d/2+1}(z) + g_{d/2+1}(z)), \quad T \leq T_c, \tag{39}$$

$$\frac{C_V}{Nk} = \frac{d}{2}\Big(\frac{f_{d/2+1}(z) + g_{d/2+1}(z)}{f_{d/2}(z) + g_{d/2}(z)} \tag{40}$$

$$+\frac{\partial z}{\partial T}\frac{1}{z}\big(1 + \frac{(f_{d/2+1}(z) + g_{d/2+1}(z))f_{d/2-1}(z) + g_{d/2-1}(z)}{(f_{d/2}(z) + g_{d/2}(z))^2}\big)\Big), \quad T > T_c. \tag{41}$$

6.2 Bose-Like condensation in Fuzzy Bose-Fermi statistics

We start from the GCPF in (24)

$$Z_{fuzzy}(x_1, x_2, \ldots, x_m) = \prod_{i=1}^{m}\frac{(1 + ax_i)}{(1 - bx_i)}. \tag{42}$$

We can calculate the grand potential $\Phi = -kT \log Z$. The fugacity $z = e^{\beta\mu}$ is not fixed and should be restricted so that the particle number in each energy level non negative, particularly for the ground state

$$N_0 = \frac{bz}{1 - bz} + \frac{az}{1 + az}. \tag{43}$$

Thus, we should have a maximum $z \equiv z_c = 1/b$. In the continuum energy limit, we have for the grand potential

$$\Phi = -kT\frac{V}{\lambda_T^d}(f_{d/2+1}(z_1) + g_{d/2+1}(z_2)) - \frac{1}{\beta}\log\frac{1 + z_1}{1 - z_2}, \tag{44}$$

where $z_1 = az$ and $z_2 = bz$. The average particle number is

$$N(T, V, \mu) = \frac{V}{\lambda_T^d}(f_{d/2}(z_1) + g_{d/2}(z_2)) + N_0, \tag{45}$$

where

$$N_e = \frac{V}{\lambda_T}(f_{d/2+1}(z_1) + g_{d/2+1}(z_2)) \tag{46}$$

is for the excited level,

$$N_0 = \frac{z_1}{1 + z_1} + \frac{z_2}{1 - z_2} \tag{47}$$

is for the ground state level (if $z = z_c$, the first part above is negligible). We can see that N_e remains finite at z_c, and thus the system exhibit a Bose-like condensation.

The critical density of the excited states can be obtained in the limit when $z \to z_c$ as

$$\frac{N_{\text{crit}}}{V} = \frac{1}{\lambda_T^d}G_{d/2}(z_c). \tag{48}$$

While the critical temperature is

$$T_c = \frac{2\pi\hbar^2}{mk}\Big(\frac{N}{V}\Big)^{2/d}(G_{d/2}(z_c))^{-2/d}, \tag{49}$$

where $G_n(z)$ is defined as

$$G_n(z) = f_n(z_1) + g_n(z_2) \tag{50}$$

The internal energy U can be obtained also easily

$$U = \frac{kTV}{\lambda_T^d}\frac{d}{2}G_{\frac{d+2}{2}}(z), \tag{51}$$

from which we can obtained the heat capacity at constant volume

$$\frac{C_V}{Nk} = \frac{d(d+2)}{4}\frac{V}{N\lambda_T^d}G_{\frac{d+2}{2}}(z_c), \quad T \leq T_c, \tag{52}$$

$$\frac{C_V}{Nk} = \Big(\frac{d(d+2)}{4}\frac{G_{\frac{d+2}{2}}(z)}{G_{\frac{d}{2}}(z)} - \frac{d^2}{4}\frac{G_{\frac{d}{2}}(z)}{G_{\frac{d-2}{2}}(z)}\Big), \quad T > T_c, \tag{53}$$

7 T_C COMPARISON IN $D = 3$

If we compare the threestatistics, the Bose, the half-Bose half-Fermi and the fuzzy Bose-Fermi statistics, we found out that the two new statistics have critical temperature that are lower than the Bose case. The critical temperature of the three statistics in $d = 3$ are given below For a fixed particle density N/V

1. Bose :

$$T_c = \frac{2\pi\hbar^2}{mk}\Big(\frac{N}{V}\Big)^{2/3}(\zeta(3/2))^{-2/3}, \tag{54}$$

2. Half-Bose half-Fermi :

$$T_c = \frac{2\pi\hbar^2}{mk}\Big(\frac{N}{V}\Big)^{2/3}\big(\zeta(3/2)2(1-\frac{1}{2^{3/2}})\big)^{-2/3}, \tag{55}$$

3. Fuzzy Bose-Fermi ($b \neq 0$):

$$T_c = \frac{2\pi\hbar^2}{mk}\Big(\frac{N}{V}\Big)^{2/3}(f_{3/2}(\frac{1}{b}-1)+\zeta(3/2))^{-2/3}, \tag{56}$$

For the last equation, the $b = 0$ case is equal to the Fermi statistics, and it can be seen that in the limit $b \to 0$ we have $T_c \to 0$. The fact that the new statistics have lower critical temperature is understandable, because here we have antisymmetric and mixed symmetry states. These states will introduce a kind of 'repulsive statistical interaction', thus it takes more lower temperature to condense.

8 CONCLUSION

We have present two intermediate statistics, the half-Bose half-Fermi statistics and the Fuzzy Bose-Fermi statistics. The half-Bose half-Fermi statistics is an intermediate statistics that still obey cluster decomposition property and its Hilbert space is invariant under permutation operation. For free non interacting particles in $d \geq 3$ this statistics shows a Bose-like condensation phenomena with a different T_c than the Bose condensation. The Fuzzy statistics is an intermediate statistics that still retain the properties of half-Bose and half-Fermi statistics but has a parameter that interpolating between Bose and Fermi statistics. This statistics also able to give a small deviation from Bose statistics but still retain a nice properties of Bose statistics, that is unitarity, extensivity and cluster decomposition. A Bose-like condensation also occur for this statistics, with T_c that can have small deviation from the Bose value.

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