New Dynamic Transition Theory and its Applications to Superconductivity

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Bose-Einstein Condensation and Quantized Vortices in Superfluidity and Superconductivity

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I. The Problem and Motivations

For many problems in sciences, we need to understand

- the transitions from one state to another, and
- the stability/robustness of the new states

I. The Problem and Motivations

For many problems in sciences, we need to understand

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This problem is related to bifurcation and stability. Unfortunately, the classical bifurcation and stability theories can only provide some partial information.

Hence we have developed a new dynamic transition theory:

1. T. Ma & S. Wang, *Bifurcation Theory and Applications*, World Scientific Series on Nonlinear Science, Series A - Vol. 53, 2005.

2. T. Ma & S. Wang, *Stability and Bifurcation of Nonlinear Evolution Equations*, Science Press, pp. 413, April, 2007.

3. T. Ma & S. Wang, *Phase Transition Dynamics in Nonlinear Sciences*, in preparation.

II. Dynamic Transitions in Superconductivity

References:

P. Bauman, D. Phillips, Q. Tang, Stable nucleation for the Ginzburg-Landau system with an applied magnetic field, *Arch. Rational Mech. Anal.*, 142 (1998), 1–43

J. Berger, J. Rubinstein, Continuous phase transitions in mesoscopic systems, *Z. Angew. Math. Phys.*, 52 (2001), 347–355

T. Ma, and S. Wang, Bifurcation and stability of superconductivity, *J. Math. Phys.*, 46 (2005), 095112, 31 pp.

T. Ma, and S. Wang, Dynamic Transitions in Superconductivity, 2007.

Time-dependent Ginzburg-Landau (TDGL) equations:

$$\frac{h^2}{2m_sD}\left(\frac{\partial}{\partial t}+\frac{ie_s}{h}\phi\right)\psi+a\psi+b|\psi|^2\psi+\frac{1}{2m_s}(hi\nabla+\frac{e_s}{c}A)^2\psi=0,$$

$$\frac{c}{4\pi}(\mathrm{curl}^2A-\mathrm{curl}H_a)=-\sigma(\frac{1}{c}A_t+\nabla\phi)-\frac{e_s^2}{m_sc}|\psi|^2A-\frac{e_shi}{2m_s}(\psi^*\nabla\psi-\psi\nabla\psi^*)$$

Unknowns:

$\psi:\Omega\to\mathbb{C}$	order parameter,
$A:\Omega\to\mathbb{R}^3$	magnetic potential,
$\phi:\Omega\to\mathbb{R}^1$	electric potential

Lifting: For a given applied field H_a with div $H_a = 0$, let $A = \mathcal{A} + A_a$:

$$\begin{aligned}
&\text{curl } A_a = H_a, \\
&\text{div } A_a = 0, \\
&A_a \cdot n|_{\partial\Omega} = 0.
\end{aligned}$$

Nondim. TDGL:

$$\begin{aligned}
&\psi_t + i\phi\psi = -(i\mu\nabla + A_a)^2\psi + \alpha\psi - 2A_a \cdot \mathcal{A}\psi - 2i\mu\mathcal{A}\cdot\nabla\psi - \mathcal{A}^2\psi - \beta|\psi|^2\psi, \\
&\zeta(\mathcal{A}_t + \mu\nabla\phi) = -\text{curl}^2\mathcal{A} - \gamma A_a|\psi|^2 - \gamma\mathcal{A}|\psi|^2 - \frac{\gamma\mu i}{2}(\psi^*\nabla\psi - \psi\nabla\psi^*), \\
&\text{div}\mathcal{A} = 0
\end{aligned}$$

supplemented with either the Neumann or the Dirichlet or the Robin boundary conditions, depending on the physical properties of the material.

Neumann boundary condition: For the case where Ω is enclosed by an insulator:

$$\frac{\partial\psi}{\partial n} = 0, \quad \mathcal{A}_n = 0, \quad \text{curl}\mathcal{A} \times n = 0 \qquad \text{on } \partial\Omega.$$

Dirichlet boundary condition: For the case where Ω is enclosed by a magnetic material:

$$\psi = 0, \quad \mathcal{A}_n = 0, \quad \text{curl}\mathcal{A} \times n = 0 \qquad \text{on } \partial\Omega.$$

Robin boundary condition: For the case where Ω is enclosed by a normal metal:

$$\frac{\partial\psi}{\partial n} + C\psi = 0, \quad \mathcal{A}_n = 0, \quad \text{curl}\mathcal{A} \times n = 0 \quad \text{on } \partial\Omega.$$

Function Spaces:

$$
\begin{aligned}
&H = L^2(\Omega, \mathbb{C}) \times L^2_{\mathsf{div}}(\Omega, \mathbb{R}^3), \\
&H_1 = H^2_B(\Omega, \mathbb{C}) \times H^2_{\mathsf{div}}(\Omega, \mathbb{R}^3)., \\
&H^2_B(\Omega, \mathbb{C}) = \{\psi \in H^2(\Omega, \mathbb{C}) \quad | \quad \psi \text{ satisfy one of BC's}\}, \\
&L^2_{\mathsf{div}}(\Omega, \mathbb{R}^3) = \{\mathcal{A} \in L^2(\Omega, \mathbb{R}^3) \quad | \quad \mathrm{div}\mathcal{A} = 0, \mathcal{A}_n|_{\partial\Omega} = 0\}, \\
&H^2_{\mathsf{div}}(\Omega, \mathbb{R}^3) = \{\mathcal{A} \in H^2 \cap L^2_{\mathsf{div}}(\Omega, \mathbb{R}^3) \mid \mathrm{curl}\mathcal{A} \times n|_{\partial\Omega} = 0\}.
\end{aligned}
$$

Eigenvalue problems: With one of BC's, the following eigenvalue problem

$$(i\mu\nabla + A_a)^2\psi = \alpha\psi, \qquad x \in \Omega,$$

has **eigenvalues:** $\alpha_1 < \alpha_2 < \cdots, \qquad \lim_{k\to\infty} \alpha_k = \infty,$

and **eigenvectors:** $\left\{e_n \in H^2_B(\Omega, \mathbb{C}) \quad | \quad n = 1, 2, ...\right\}.$

Let the **first eigenvalue** α_1 has multiplicity $2m$ $(m \geq 1)$ with eigenvectors

$$e_{2k-1} = \psi_{k1} + i\psi_{k2}, \quad e_{2k} = -\psi_{k2} + i\psi_{k1}, \quad 1 \leq k \leq m.$$

Nondim. parameters:

$$\alpha = \alpha(T) = \frac{2\sqrt{b} m_s D N_0}{e_s^3 h} \frac{T_c - T}{T_c} \qquad \beta = 2 m_s D / h$$

where T_c is the critical temperature where incipient superconductivity property can be observed.

Physical requirement:

$$\boxed{\alpha_1 < \alpha(0) = \frac{2\sqrt{b} m_s D N_0}{e_s^3 h}}$$

For simplicity, consider the case where $\boxed{m = 1}$. Let $e \in H^2(\Omega, \mathbb{C})$ be a first eigenvector of (2.5). Let

$$\boxed{R = -\frac{\beta}{\gamma} + \frac{2\int_\Omega |\mathrm{curl}\mathcal{A}_0|^2 dx}{\int_\Omega |e|^4 dx}}$$

$$\mathrm{curl}^2\mathcal{A}_0 + \nabla\phi = |e|^2 A_a + \frac{\mu}{2} i(e^*\nabla e - e\nabla e^*),$$

$$\mathrm{div}\mathcal{A}_0 = 0,$$

$$\mathcal{A}_0 \cdot n|_{\partial\Omega} = 0, \quad \mathrm{curl}\mathcal{A}_0 \times n|_{\partial\Omega} = 0.$$

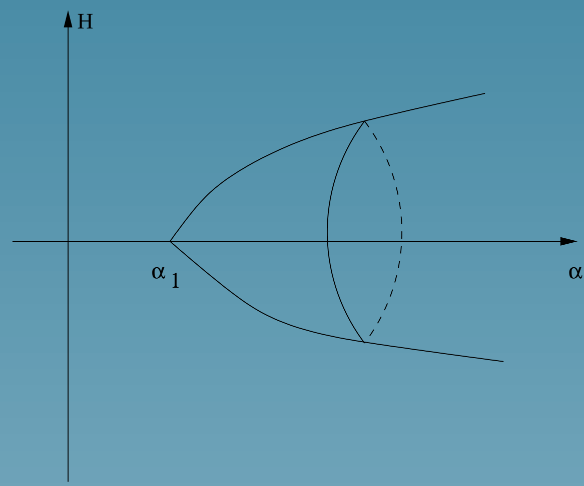
R is computable, and depends on $(A_a,\ \Omega,\ \beta, \gamma, \mu)$.

Theorem [Ma & W., 04] *If $R < 0$, then the following are true:*

(1) If $\alpha \le \alpha_1$, the steady state $(\psi, \mathcal{A}) = 0$ is locally asymptotically stable,

(2) The equations bifurcate from $((\psi, \mathcal{A}), \alpha) = (0, \alpha_1)$ to an attractor $\Sigma_\alpha = S^1$ for $\alpha > \alpha_1$, which consists of steady state solutions.

(3) There is a neighborhood $U \subset H$ of $(\psi, \mathcal{A}) = 0$ s.t. Σ_α attracts $U \setminus \Gamma$ in H, where Γ is the stable manifold of $(\psi, \mathcal{A}) = 0$ with codim. two in H.



(4) For any $(\psi, \mathcal{A}) \in \Sigma_\alpha$ *can be expressed as*

$$\psi = \left|\frac{\alpha - \alpha_1}{R_1}\right|^{1/2} e + o\left(\left|\frac{\alpha - \alpha_1}{R_1}\right|^{1/2}\right),$$

$$\mathrm{curl}^2 \mathcal{A} = -\gamma\left|\frac{\alpha - \alpha_1}{R_1}\right| \cdot \left[|e|^2 A_a + \mu Im(e\nabla e^*)\right] + o\left(\frac{\alpha - \alpha_1}{R_1}\right),$$

$$R_1 = \frac{\gamma R \int_\Omega |e|^4 dx}{\int_\Omega |e|^2 dx},$$

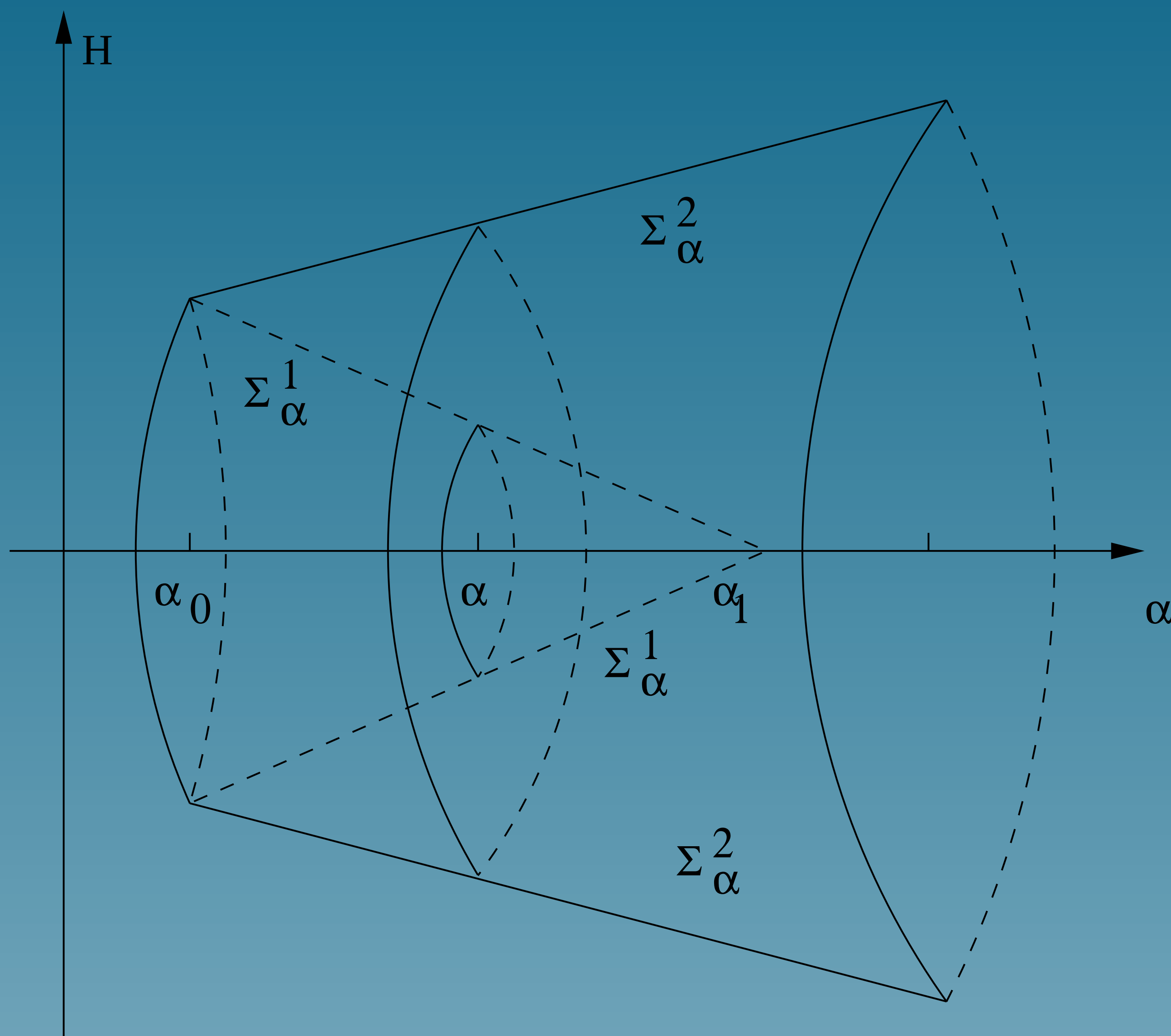
Theorem [Ma & W., 04] *If $R > 0$, then for the TDGL with one of the B.C.s, the following are true:*

(1) The steady state $(\psi, \mathcal{A}) = 0$ is locally asymptotically stable at $\alpha < \alpha_1$, and unstable at $\alpha \geq \alpha_1$,

(2) The equations bifurcate from $(\psi, \mathcal{A}), \alpha) = (0, \alpha_1)$ to an invariant set Σ_α on $\alpha < \alpha_1$, and have no bifurcation on $\alpha > \alpha_1$,

(3) $\Sigma_\alpha = S^1$, consisting of singular points, and

(4) Σ_α has a 2D unstable manifold.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An example

Let

$$\Omega_0 = D_0 \times (0,h) \subset \mathbb{R}^3$$

$$\Omega(L) = \{(Lx', x_3)|\ 0 < L < \infty, x = (x', x_3) \in \Omega_0\}$$

$$H_a = H_a(x') = (0, 0, H(x')) \qquad x' = (x_1, x_2) \in D_0$$

$$\widetilde{H}_a(y) = H_a\left(\frac{y}{L}\right) \qquad y = Lx', x' \in D_0$$

$$A_a = (A_1(x'), A_2(x'), 0) \qquad H_a = \mathsf{curl} A_a$$

$$\widetilde{A}_a = (\widetilde{A}_1(y), \widetilde{A}_2(y), 0) = LA_a(x') \qquad \widetilde{H} = \mathsf{curl} \tilde{A}_a$$

Then we can find the parameter $R = R(L, H)$ defined on $\Omega(L)$:

(1) $$\boxed{R(L,H) = -\kappa^2\mu^2 + 2L(p_3 L^4 + p_2 L^2 + p_1)}$$

where $e(x') = e_{11} + ie_{12}$ be the first eigenfuction of (2), and

$$p_1 = \frac{4\mu^2 \int_{D_0} |\mathrm{curl}\bar{B}_0|^2 dx'}{\int_{D_0} |e|^4 dx'}, \qquad p_2 = \frac{4\mu \int_{D_0} \mathrm{curl}\bar{A}_0 \cdot \mathrm{curl}\bar{B}_0 dx'}{\int_{D_0} |e|^4 dx'},$$

$$p_3 = \frac{\int_{D_0} |\mathrm{curl}\bar{A}_0|^2 dx'}{\int_{D_0} |e|^4 dx'} > \delta > 0 \qquad \forall 0 < L < \infty,$$

$$\mathrm{curl}^2 \bar{A}_0 + \nabla\phi = |e|^2 A_a, \qquad \mathrm{curl}^2 \bar{B}_0 + \nabla\phi = e_{12}\nabla e_{11}$$

(2) $$(i\mu\nabla + L^2 A_a)^2 e = L^2 \alpha e \quad \text{in } D_0, \qquad \frac{\partial e}{\partial n} = 0 \quad \text{on } \partial D_0.$$

Results (Ma-Wang, 07):

Under the assumptions, the following statements hold true:

1. For a given applied field $H_a \neq 0$, there is a critical scale $L_0 > 0$ given by

$$p_3 L_0^5 + p_2 L_0^3 + p_1 L_0 = \frac{1}{2}\kappa^2\mu^2$$

such that

(a) the phase transition in $\Omega(L)$ is continuous if $L < L_0$, and

(b) the transition is jump if $L > L_0$.

2. Fix an L, and let $H_a = (0, 0, H)$ and curl$A_a = H_a$. Then we can find explicit formulas for critical fields H_{c_1}, H_{c_2}, ..., leading to physical predictions the type of transitions as well as type of states (Meissner, mixed, or normal).

III. Dynamic Bifurcation Theory

(3)
$$\begin{aligned} &\frac{du}{dt} = L_\lambda u + G(u, \lambda), \\ &u(0) = u_0. \end{aligned}$$

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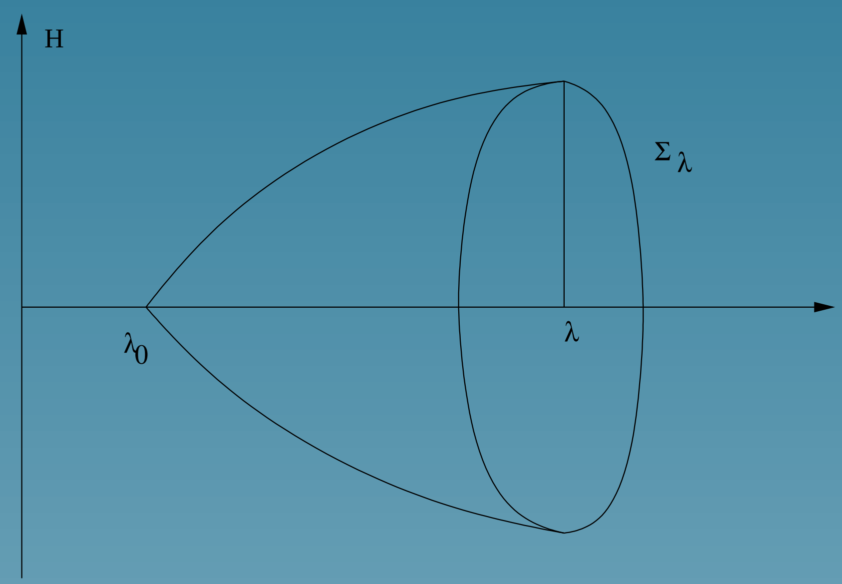
$u : [0, \infty) \to H,$	
$H_1 \hookrightarrow H$	dense and compact,
$-L_\lambda = A - B_\lambda$	a sectorial operator,
$A : H_1 \to H$	a linear homeomorphism,
$B_\lambda : H_1 \to H$	linear compact operators
$G(u, \lambda) = o(\\|u\\|_{H_\alpha})$	$\forall \ \lambda \in \mathbb{R}^1, \quad \alpha < 1.$

Attractor Bifurcation (Ma & W., 04): Equation (3) bifurcates from $(u, \lambda) = (0, \lambda_0)$ to attractors Σ_λ, if there exist attractors $\{\Sigma_{\lambda_n}\}$ of (1) such that

$$0 \notin \Sigma_{\lambda_n}, \qquad \lim_{n\to\infty} \lambda_n = \lambda_0, \qquad \lim_{n\to\infty} \max_{x\in\Sigma_{\lambda_n}} \|x\|_H = 0.$$

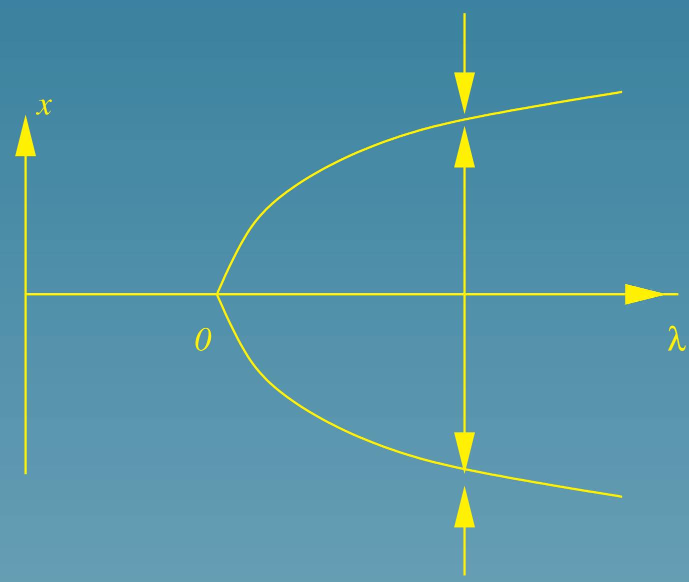


Examples:

For $x \in \mathbb{R}$,

$$\frac{dx}{dt} = \lambda x - x^3 + o(x^3)$$

bifurcates from $(x, \lambda) = (0, 0)$ to an attractor $\Sigma_\lambda = \{x_1^\lambda, x_2^\lambda\}$ for $\lambda > 0$.



Consider:

$$\frac{dx_1}{dt} = \lambda x_1 - x_1^3 + o(|x|^3),$$
$$\frac{dx_2}{dt} = \lambda x_2 - x_2^3 + o(|x|^3)$$

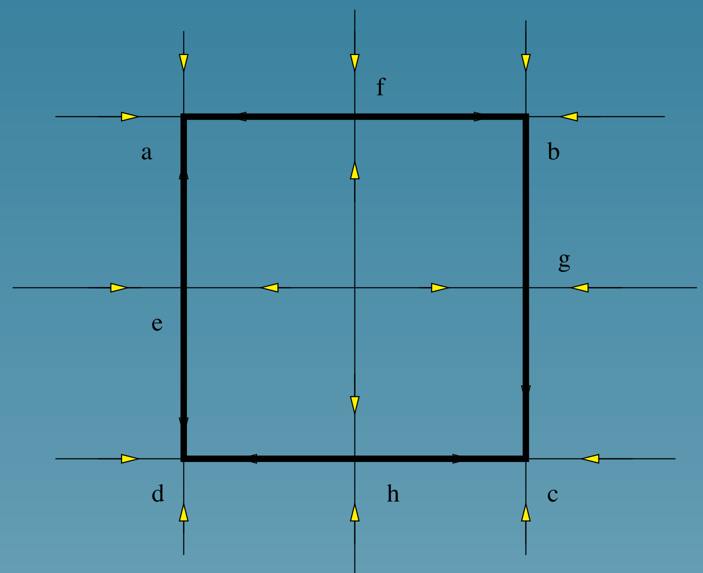


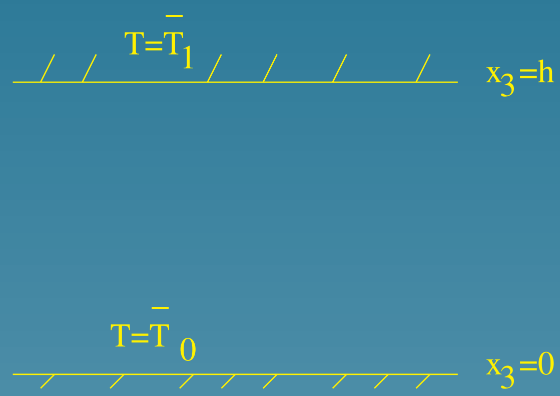
Figure 1:

This bifurcated attractor is as shown in Figure 1, and contains

- exactly 4 nodes (the points a, b, c, and d),
- 4 saddles (the points e, f, g, h), and
- orbits connecting these 8 points.

From the physical transition point of view, as λ crosses 0, the new state after the system undergoes a transition is represented by the **whole bifurcated attractor** Σ_λ, rather than any of the steady states or any of the connecting orbits.

Example (Ma & W., 04): Consider the Bénard convection on a nondimensional domain $\Omega = D \times (0,1)$ with with a set of physical sound BCs. Let Rayleigh number be $\boxed{R = g\alpha(\bar{T}_0 - \bar{T}_1)h^3/(\kappa\nu)}$.



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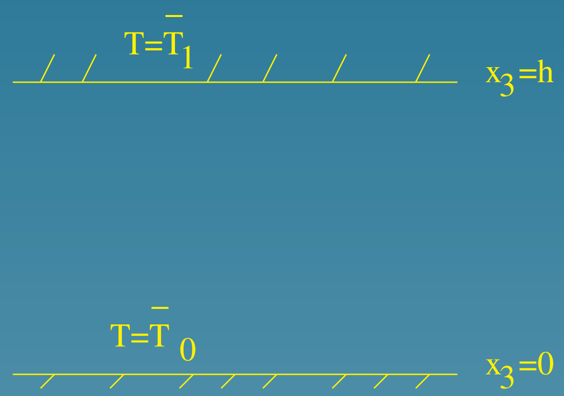

Then as R crosses the first critical Rayleigh number R_c, the basic solution will always bifurcate to an attractor Σ_R, homologic to S^{m-1}, where m is the multiplicity of R_c. In addition, Σ_R attracts every bounded set of $H \setminus \Gamma$, where Γ is the stable manifold of the basic solution.

Example-continued (Ma & W., 06 & 07): Consider the 3D Bénard convection in $\Omega = (0, L_1) \times (0, L_2) \times (0, 1)$ with free top-bottom and periodic horizontal BC's, and with

$$\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2} = \frac{1}{8} \qquad \text{for some } k_1, k_2 \in \mathbb{Z}. \tag{4}$$

Then

$$\Sigma_R = \begin{cases} S^5 & \text{if} \quad L_2 = \sqrt{k^2 - 1} L_1, \quad k = 2, 3, \cdots, \\ S^3 & \text{otherwise.} \end{cases}$$

We remark that S^5 or S^3 contains MANY more solutions than ANY classical bifurcation method can derive.

Note also that there are many problems having no symmetry, which can be handled by our method as well.

Example (Ma & W.): For the 3D Bénard convection in $\Omega = (0, L)^2 \times (0, 1)$ with free BCs and with

$$0 < L^2 < \frac{2 - 2^{1/3}}{2^{1/3} - 1}. \tag{5}$$

The bifurcated attractor Σ_R consists of 1) exactly eight singular points and 2) eight heteroclinic orbits connecting the singular points, as shown in Figure 1, with 4 of them being minimal attractors, and the other 4 saddle points.

Principle of Exchange of Stability (PES):

$\beta_1(\lambda), \beta_2(\lambda), \cdots \in \mathbb{C}$ eigenvalues of L_λ:

$$\text{(6)} \qquad \mathrm{Re}\beta_i(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ > 0 & \text{if } \lambda > \lambda_0 \end{cases} \qquad 1 \leq i \leq m,$$

$$\text{(7)} \qquad \mathrm{Re}\beta_j(\lambda_0) < 0 \qquad m+1 \leq j.$$

$$E_0 = \bigcup_{1 \leq i \leq m} \bigcup_{k \in \mathbb{N}} ker(L_{\lambda_0} - \beta_j(\lambda_0))^k$$

$\boxed{\textbf{Thm (Ma \& W., 04, 05)}}$ Assume (6)-(7) , and $u = 0$ is locally asymptotically stable for (3) at $\lambda = \lambda_0$. Then

- (3) bifurcates from $(u, \lambda) = (0, \lambda_0)$ to an attractor Σ_λ for any $\lambda > \lambda_0$ and near λ_0, with $m - 1 \le \dim \Sigma_\lambda \le m$, which is connected if $m > 1$
- the attractor Σ_λ is homologic to S^{m-1};
- for any $u_\lambda \in \Sigma_\lambda$, $u_\lambda = v_\lambda + o(\|v_\lambda\|), \quad v_\lambda \in E_0$;
- if $u = 0$ is globally stable for (3) at $\lambda = \lambda_0$, then for any bounded open set $U \subset H$ with $0 \in U$ there is an $\varepsilon > 0$ such that when $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, the attractor Σ_λ attracts $U \setminus \Gamma$ in H, where Γ is the stable manifold of $u = 0$ with codim. m.

Remarks

1. General Strategy for Applications:

- Existence of the bifurcated attractor Σ_λ
- Classification of Σ_λ

2. Asymptotic stability at the critical point λ_0:

- based on a general principle motivated by the following example, and used in the Taylor problem below
- based on the center manifold reduction

Example: $\frac{dx}{dt} = \lambda x - x^3$. At $\lambda_0 = 0$, $x = 0$ is locally asymptotically stable:

$$\frac{dx}{dt} = -x^3 \quad \text{has a unique solution} \quad x(t) = \frac{x_0}{\sqrt{2x_0^2 t + 1}} \to 0$$

Note: At $\lambda_0 = 0$, the linearized equation is $\frac{dx}{dt} = 0$ and

$x = 0$ is only **neutrally stable**.

An asymptotic stability theorem at the critical λ_0: Assume that $L_{\lambda_0}: H_1 \to H$ is a symmetric linear completely continuous field, $E_0 = \ker L_{\lambda_0}$, and

$$< G(u, \lambda_0), u >_H = 0 \qquad \forall u \in H_1$$

Thm (Ma -W., 04) Under the assumptions, one and only one of the following assertions holds true:

(1) There exists a sequence of invariant sets $\{\Gamma_n\} \subset E_0$ of (3), such that $0 \notin \Gamma_n$ and

$$\lim_{n\to\infty} \text{dist } (\Gamma_\text{n}, 0) = 0.$$

(2) The trivial equilibrum point $u = 0$ of (3) is locally asymptotically stable under the H-norm.

Furthermore, if (3) has no invariant sets in E_0 except the trivial one $\{0\}$, then $u = 0$ is global asymptotically stable.

Approximation of the center manifold function Φ:

Let the nonlinear operator G be in the following form

$$G(u, \lambda) = G_k(u, \lambda) + o(\|u\|^k),$$

for some integer $k \geq 2$, where G_k is a k-multilinear operator.

Theorem (Ma & Wang, 05). *Under the above assumptions, the center manifold function Φ can be expressed as*

$$\Phi(x, \lambda) = (-L_2^\lambda)^{-1} P_2 G_k(x, \lambda) + o(\|x\|^k) + O(|\mathrm{Re}\beta| \, \|x\|^k),$$

where $x \in E_1^\lambda$, and $\beta = (\beta_1(\lambda), \cdots, \beta_m(\lambda))$.

VI. Dynamic Transition Theory

- The attractor bifurcation theory developed earlier requires that the basic solution $u = 0$ is asymptotically stable at $\lambda = \lambda_0$.
- The main motivation of the new dynamic transition theory is to develop a corresponding theory which covers the general case where the asymptotic stability of the basic state may not be satisfied.

The starting point is the following theorem:

Theorem [MA & W., 2007] *Under the conditions, the system (1) must have a transition from* $(u, \lambda) = (0, \lambda_0)$ *to one of the following three types in a neighborhood* $U \subset X$ *of* $u = 0$*:*

(1) CONTINUOUS TRANSITION: *there exists an open and dense set* $\widetilde{U}_\lambda \subset U$ *such that for any* $\varphi \in \widetilde{U}_\lambda$*, the solution* $u_\lambda(t, \varphi)$ *of (2.1) satisfies*

$$\lim_{\lambda \to \lambda_0} \limsup_{t \to \infty} \|u_\lambda(t, \varphi)\|_X = 0.$$

In particular, the attractor bifurcation of (2.1) at $(0, \lambda_0)$ *is a continuous transition.*

(2) JUMP TRANSITION: *for any* $\lambda_0 < \lambda < \lambda_0 + \varepsilon$ *with some* $\varepsilon > 0$, *there is an open and dense set* $U_\lambda \subset U$ *such that for any* $\varphi \in U_\lambda$,

$$\limsup_{t\to\infty} \|u_\lambda(t,\varphi)\|_X \geq \delta > 0,$$

where $\delta > 0$ *is independent of* λ. *This type of transition is also called the discontinuous transition.*

(3) MIXED TRANSITION: *for any* $\lambda_0 < \lambda < \lambda_0 + \varepsilon$ *with some* $\varepsilon > 0$, U *can be decomposed into two open sets* U_1^λ *and* U_2^λ (U_i^λ *not necessarily connected):* $\bar{U} = \bar{U}_1^\lambda + \bar{U}_2^\lambda$, *such that* $U_1^\lambda \cap U_2^\lambda = \emptyset$, *and*

$$\lim_{\lambda\to\lambda_0} \limsup_{t\to\infty} \|u(t,\varphi)\|_X = 0 \qquad \forall \varphi \in U_1^\lambda,$$

$$\limsup_{t\to\infty} \|u(t,\varphi)\|_X \geq \delta > 0 \qquad \forall \varphi \in U_2^\lambda,$$

where U_1^λ *is called the stable domain, and* U_2^λ *is the unstable domain.*

V. Remarks

We believe the newly developed dynamic transition theory can be useful in many phase transition problems in nonlinear sciences, as evidenced by various problems in statistical physics, classical and geophysical fluid dynamics, ...

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Thank You!