Hyperplane Constrained Continuation Methods for Coupled Nonlinear Schrödinger Equations

Weichung Wang

Department of Mathematics
National Taiwan University

Workshop on Bose-Einstein Condensation: Modeling, Analysis, Computation and Applications
Institute of Mathematical Sciences, National University of Singapore
November 15, 2007

Cooperators

- Yuen-Cheng Kuo

  Department of Applied Mathematics, National University of Kaohsiung

- Wen-Wei Lin

  Department of Mathematics, National Tsing-Hua University

- Shih-Feng Shieh

  Department of Mathematics, National Taiwan Normal University

Outline

Outline

- System of m-coupled nonlinear Schrödinger equations
  - Nonlinear optics for Kerr-like photorefractive media
  - Main numerical issue: ε-solutions
  - A hyperplane-constrained continuation method
  - Theoretical analysis and numerical experiments
- Nonlinear Schrödinger equation
  - Single component rotating Bose-Einstein condensate
  - Main numerical issue: transformation invariant solutions
  - Another hyperplane-constrained continuation method
  - Bistability of solution curves

The m-coupled Nonlinear Schrödinger equations

- The time-independent m-coupled NLSE:

$$\begin{cases} \Delta\phi_j - \lambda_j\phi_j + \mu_j|\phi_j|^2\phi_j + \sum_{i\neq j}\beta_{ij}|\phi_i|^2\phi_j = 0, \\ \phi_j > 0 \quad \text{in } \mathbb{R}^{\mathrm{n}}, \mathrm{j} = 1, \ldots, \mathrm{m}, \\ \phi_j(\boldsymbol{x}) \to 0, \quad \text{as} \quad |\boldsymbol{x}| \to \infty, \end{cases} \tag{1}$$

where $\lambda_j, \mu_j > 0$, $n \leq 3$, and $\beta_{ij} = \beta_{ji}$ for $i \neq j$ are coupling constants.

- 2-component example:

$$\begin{cases} \Delta\phi_1 - \lambda_1\phi_1 + \mu_1|\phi_1|^2\phi_1 + \beta_{21}|\phi_2|^2\phi_1 = 0 \\ \Delta\phi_2 - \lambda_2\phi_2 + \mu_2|\phi_2|^2\phi_2 + \beta_{12}|\phi_1|^2\phi_2 = 0 \end{cases}$$

Physical Model

$$\Delta\phi_j - \lambda_j\phi_j + \mu_j|\phi_j|^2\phi_j + \sum_{i\neq j}\beta_{ij}|\phi_i|^2\phi_j = 0$$

- Nonlinear optics (Kerr-like photorefractive media)

  ϕ_j: the j-th component of the beam

  λ_j: chemical potential

  μ_j: self-focusing in the j-th component of the beam

  β_{ij}: interaction between the beams

  $\beta_{ij} > 0$, the interaction between ϕ_i and ϕ_j is attractive

  $\beta_{ij} < 0$, the interaction between ϕ_i and ϕ_j is repulsive

Ground State Solution for $m = 1$

The weak solution of the decoupled NLSE can be obtained by solving

$$\inf_{\substack{\phi \geq 0 \\ \phi \in H^1(\mathbb{R}^n)}} \frac{\int_{\mathbb{R}^n} |\nabla \phi|^2 + \lambda \int_{\mathbb{R}^n} \phi^2}{\left(\int_{\mathbb{R}^n} \phi^4\right)^{1/2}} \tag{2}$$

Ground State Solution for $m = 1$

or equivalently

$$\inf_{\phi \in \mathcal{N}_1} E(\phi) \tag{3a}$$

where the energy functional on the Nehari manifold are

$$E(\phi) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\phi|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^n} \phi^2 - \frac{\mu}{4}\int_{\mathbb{R}^n} \phi^4 \tag{3b}$$

and

$$\mathcal{N}_1 = \left\{\phi \in H^1(\mathbb{R}^n)|\ \phi \geq 0,\ \phi \not\equiv\ 0, \int_{\mathbb{R}^n} |\nabla\phi|^2 + \lambda\int_{\mathbb{R}^n} \phi^2 = \mu\int_{\mathbb{R}^n} \phi^4\right\}, \tag{3c}$$

respectively.

If ϕ satisfies (3) then ϕ is called a ground state solution.

Ground State Solution for $m \geq 2$

- Consider the minimization problem

$$\inf_{\phi \in \mathcal{N}_m} E(\boldsymbol{\phi}), \tag{4a}$$

where

$$E(\boldsymbol{\phi}) = \sum_{j=1}^{m} \left(\frac{1}{2} \int_{\mathbb{R}^n} |\nabla \phi_j|^2 + \frac{\lambda_j}{2} \int_{\mathbb{R}^n} \phi_j^2 - \frac{\mu_j}{4} \int_{\mathbb{R}^n} \phi_j^4 \right) - \frac{1}{4} \sum_{i \neq j}^{m} \beta_{ij} \int_{\mathbb{R}^n} \phi_i^2 \phi_j^2, \tag{4b}$$

and

$$\mathcal{N}_m = \Big\{ \boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_m) \in (H^1(\mathbb{R}^n))^m | \quad \phi_j \geq 0, \quad \phi_j \not\equiv 0,$$
$$\int_{\mathbb{R}^n} |\nabla \phi_j|^2 + \lambda_j \int_{\mathbb{R}^n} \phi_j^2 = \mu_j \int_{\mathbb{R}^n} \phi_j^4 + \sum_{i \neq j} \beta_{ij} \int_{\mathbb{R}^n} \phi_i^2 \phi_j^2, j = 1, \ldots, m \Big\} \tag{4c}$$

Ground State Solution for $m \geq 2$

Definition of the ground state solution:

If $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_m)$ satisfies the following properties

1. $\phi_j > 0$ for all j and $\boldsymbol{\phi}$ satisfies NLSE.
2. $E(\boldsymbol{\phi}) \leq E(\boldsymbol{\psi})$ for any other solution $\boldsymbol{\psi}$ of NLSE.

Discrete m-coupled Nonlinear Schrödinger Eqs

- We consider the m-coupled DNLSE:

$$\begin{cases} \boldsymbol{A}\boldsymbol{u}_j - \lambda_j \boldsymbol{u}_j + \mu_j \boldsymbol{u}_j^{②} \circ \boldsymbol{u}_j + \sum_{i \neq j}^{m} \beta_{ij} \boldsymbol{u}_i^{②} \circ \boldsymbol{u}_j = \boldsymbol{0}, \\ \boldsymbol{u}_j > 0 \quad \text{for } j = 1, \ldots, m, \end{cases} \tag{5}$$

where $\lambda_j > 0$, $\mu_j > 0$ and $\beta_{ij} = \beta_{ji}$, are coupling constants.

  - $\boldsymbol{A} \in \mathbb{R}^{N \times N}$ corresponding to the operator Δ, $\boldsymbol{u}_j \in \mathbb{R}^N$ is defined by the a approximation of $\phi_j(\boldsymbol{x})$ for $j = 1, \ldots, m$.
  - For $\boldsymbol{u} = (u_1, \ldots, u_N)^\top$, $\boldsymbol{v} = (v_1, \ldots, v_N)^\top \in \mathbb{R}^N$, $\boldsymbol{u} \circ \boldsymbol{v} = (u_1 v_1, \ldots, u_N v_N)^\top$ is the Hadamard product of $\boldsymbol{u}$ & $\boldsymbol{v}$.
  - $\boldsymbol{u}^{ⓡ} = \boldsymbol{u} \circ \cdots \circ \boldsymbol{u}$ denotes the r-time Hadamard product of $\boldsymbol{u}$.

The Discrete Minimization Problem for $m = 1$

- One-component DNLSE

$$\begin{cases} \boldsymbol{A}\boldsymbol{u} - \lambda \boldsymbol{u} + \mu \boldsymbol{u}^{②} \circ \boldsymbol{u} = \boldsymbol{0}, \\ \boldsymbol{u} > 0, \end{cases} \tag{6}$$

  where $\lambda, \mu > 0$ and $\boldsymbol{A}$ is diagonal dominant with positive off-diagonal entries.

- The minimization problem:

$$\inf_{\boldsymbol{u} \geq 0} \widehat{E}(\boldsymbol{u}), \tag{7a}$$

  where

$$\widehat{E}(\boldsymbol{u}) = \frac{-\boldsymbol{u}^\top \boldsymbol{A}\boldsymbol{u} + \lambda \boldsymbol{u}^\top \boldsymbol{u}}{(\boldsymbol{u}^{②\top} \boldsymbol{u}^{②})^{1/2}}. \tag{7b}$$

The Discrete Minimization Problem for $m \geq 2$

- The corresponding discrete minimization problem:

$$\inf_{\boldsymbol{x}\in\mathcal{N}_m} E(\boldsymbol{x}). \tag{8a}$$

where

$$E(\boldsymbol{x}) = \sum_{j=1}^{m}\left(-\frac{1}{2}\boldsymbol{u}_j^\top \boldsymbol{A}\boldsymbol{u}_j + \frac{\lambda_j}{2}\boldsymbol{u}_j^\top \boldsymbol{u}_j - \frac{\mu_j}{4}\boldsymbol{u}_j^{\textcircled{2}\top}\boldsymbol{u}_j^{\textcircled{2}}\right) - \frac{1}{4}\sum_{i\neq j, i=1}^{m}\beta_{ij}\boldsymbol{u}_i^{\textcircled{2}\top}\boldsymbol{u}_j^{\textcircled{2}}, \tag{8b}$$

and

$$\mathcal{N}_m = \Big\{(\boldsymbol{u}_1^\top, \ldots, \boldsymbol{u}_m^\top)^\top \in \mathbb{R}^{Nm} \mid \boldsymbol{u}_j \geq 0,\ \boldsymbol{x} \not\equiv \mathbf{0} \text{ and}$$
$$-\boldsymbol{u}_j^\top \boldsymbol{A}\boldsymbol{u}_j + \lambda_j \boldsymbol{u}_j^\top \boldsymbol{u}_j = \mu_j \boldsymbol{u}_j^{\textcircled{2}\top}\boldsymbol{u}_j^{\textcircled{2}} + \sum_{i\neq j, i=1}^{m}\beta_{ij}\boldsymbol{u}_i^{\textcircled{2}\top}\boldsymbol{u}_j^{\textcircled{2}},\ \ j = 1, \ldots, m\Big\}. \tag{8c}$$

Outline

Iterative Method for One-Component DNLSE I

Recall that the one-component DNLSE is

$$\begin{cases} \boldsymbol{A}\boldsymbol{u} - \lambda \boldsymbol{u} + \mu \boldsymbol{u}^{②} \circ \boldsymbol{u} = \boldsymbol{0}, \\ \boldsymbol{u} > 0, \end{cases} \tag{9}$$

Some notations and facts

- Let

$$\bar{\boldsymbol{A}} = \lambda \boldsymbol{I} - \boldsymbol{A}. \tag{10}$$

Since $\lambda > 0$, then $\bar{\boldsymbol{A}}$ is an irreducible M-matrix and $\bar{\boldsymbol{A}}^{-1}$ is positive definite matrix with positive entries.

Iterative Method for One-Component DNLSE II

- Define the set
$$\mathcal{M} = \left\{ \boldsymbol{u} \in \mathbb{R}^N \middle| \ \|\boldsymbol{u}\|_4 = 1, \boldsymbol{u} \geq 0 \right\}, \tag{11}$$
where $\|\boldsymbol{u}\|_4 = (\boldsymbol{u}^{②\top} \boldsymbol{u}^{②})^{1/4}$.
- If $\boldsymbol{u} \in \mathcal{M}$ then $\bar{\boldsymbol{A}}^{-1} \boldsymbol{u} = (\lambda \boldsymbol{I} - \boldsymbol{A})^{-1} \boldsymbol{u} > 0$.
- Define a mapping $f : \mathcal{M} \to \mathcal{M}$ by
$$f(\boldsymbol{u}) = \frac{\bar{\boldsymbol{A}}^{-1} \boldsymbol{u}^{③}}{\|\bar{\boldsymbol{A}}^{-1} \boldsymbol{u}^{③}\|_4}. \tag{12}$$

The Fixed Point Iterations

Algorithm 2.1 Fixed Point Iteration.

(i) Given $\bar{\boldsymbol{A}} \in \mathbb{R}^{N\times N}$ and $\boldsymbol{u}_0 > 0$ with $\|\boldsymbol{u}_0\|_4 = 1$, let $i = 0$;

(ii) Solve the linear system

$$\bar{\boldsymbol{A}}\boldsymbol{u}_{i+1} = \boldsymbol{u}_i^{③}.$$

Compute $\boldsymbol{u}_{i+1} = \boldsymbol{u}_{i+1}/\|\boldsymbol{u}_{i+1}\|_4$.

(iii) If converges, then $\boldsymbol{u}^* \leftarrow \boldsymbol{u}_{i+1}$, stop; else $i \leftarrow i+1$, go to (ii).

Convergence Analysis of the Fixed Point Iterations

- Existence of fixed point and the resulting solution of the one-component DNLSE
- Globally convergent subsequence
- Globally convergent sequence derived from a mild assumption

Fixed Point and the One-Comp DNLSE Solution

Theorem

The function $f : \mathcal{M} \to \mathcal{M}$ *given in* (12) *has a fixed point* $\boldsymbol{u}^*$ *in* $\mathring{\mathcal{M}}$. *Furthermore, the point*

$$\bar{\boldsymbol{u}}(\mu) = \frac{1}{\mu^{1/2}} \|\bar{\boldsymbol{A}}^{-1} \boldsymbol{u}^{*③}\|_4^{-1/2} \boldsymbol{u}^* \in \mathcal{N}_1, \tag{13}$$

is the solution of $\boldsymbol{A}\boldsymbol{u} - \lambda \boldsymbol{u} + \mu \boldsymbol{u}^{②} \circ \boldsymbol{u} = \boldsymbol{0}$.

Globally Convergent Subsequence I

Theorem

(i) If $\boldsymbol{u} \in \mathcal{M}$ *and* $\boldsymbol{v} = \boldsymbol{f}(\boldsymbol{u})$, *then* $\widehat{E}(\boldsymbol{v}) \leq \widehat{E}(\boldsymbol{u})$, *where* $\widehat{E}(\cdot)$ *is defined as* $\widehat{E}(\boldsymbol{u}) = \frac{-\boldsymbol{u}^\top \boldsymbol{A}\boldsymbol{u} + \lambda \boldsymbol{u}^\top \boldsymbol{u}}{(\boldsymbol{u}^{\textcircled{2}\top} \boldsymbol{u}^{\textcircled{2}})^{1/2}}$. *The equality holds if and only if* $\boldsymbol{u}$ *is a fixed point of* $\boldsymbol{f} : \mathcal{M} \to \mathcal{M}$, *i.e.,* $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$.

(ii) For a sequence $\{\boldsymbol{u}_i\}_{i=0}^{\infty}$ *generated by the Fixed Point Algorithm, there exists a subsequence* $\{\boldsymbol{u}_{n_i}\}_{i=0}^{\infty}$ *such that*

$$\lim_{i \to \infty} \boldsymbol{u}_{n_i} = \boldsymbol{u}^*. \tag{14}$$

Furthermore, $\boldsymbol{u}^*$ *is a fixed point of the function* $\boldsymbol{f}(\boldsymbol{u}) = \frac{\bar{\boldsymbol{A}}^{-1} \boldsymbol{u}^{\textcircled{3}}}{\|\bar{\boldsymbol{A}}^{-1} \boldsymbol{u}^{\textcircled{3}}\|_4}$.

Globally Convergent Subsequence II

Corollary

If the minimization problem (7) has a unique global minimizer $\boldsymbol{u}^* \in \mathcal{M}$, then there exist a neighborhood $R_{\boldsymbol{u}^*}$ of $\boldsymbol{u}^*$ such that the fixed point iteration converges to $\boldsymbol{u}^*$ for any initial vector $\boldsymbol{u}_0 \in R_{\boldsymbol{u}^*}$.

$$\inf_{\boldsymbol{u} \geq 0} \widehat{E}(\boldsymbol{u}), \tag{15a}$$

where

$$\widehat{E}(\boldsymbol{u}) = \frac{-\boldsymbol{u}^\top \boldsymbol{A} \boldsymbol{u} + \lambda \boldsymbol{u}^\top \boldsymbol{u}}{(\boldsymbol{u}^{②\top} \boldsymbol{u}^{②})^{1/2}}. \tag{15b}$$

Globally Convergent Sequence

Theorem

If $\boldsymbol{u}^*$ given in (14) is strictly local minimum of (7), then the sequence $\{\boldsymbol{u}_i\}$ converges to $\boldsymbol{u}^* \in \mathcal{M}$.

Remark

Numerical experience shows that for any arbitrary initial positive vector $\boldsymbol{u}_0$ with $\|\boldsymbol{u}_0\|_4 = 1$, the fixed point iteration converges to the global minimizer of (7).

Outline

Continuation Method for the DNLSE

- Recall that the m-coupled DNLSE is

$$\begin{cases} \boldsymbol{A}\boldsymbol{u}_j - \lambda_j \boldsymbol{u}_j + \mu_j \boldsymbol{u}_j^{②} \circ \boldsymbol{u}_j + \sum_{i \neq j}^{m} \beta_{ij} \boldsymbol{u}_i^{②} \circ \boldsymbol{u}_j = \boldsymbol{0}, \\ \boldsymbol{u}_j > 0 \quad \text{for } j = 1, \ldots, m, \end{cases}$$

  where $\lambda_j > 0$, $\mu_j > 0$ and $\beta_{ij} = \beta_{ji}$, $i \neq j$.

- Let $\beta_{ij} = \beta \delta_{ij}$ (β: continuation parameter) and rewrite as

$$\boldsymbol{G}(\boldsymbol{x}, \beta) = (\boldsymbol{G}_1, \ldots, \boldsymbol{G}_m)(\boldsymbol{x}, \beta) = \boldsymbol{0}, \tag{16}$$

  where $\boldsymbol{x} = (\boldsymbol{u}_1^\top, \ldots, \boldsymbol{u}_m^\top)^\top \in \mathbb{R}^{Nm}$, $\boldsymbol{G} : \mathbb{R}^{Nm} \times \mathbb{R} \to \mathbb{R}^{Nm}$ and

$$\boldsymbol{G}_j(\boldsymbol{x}, \beta) = \boldsymbol{A}\boldsymbol{u}_j - \lambda_j \boldsymbol{u}_j + \mu_j \boldsymbol{u}_j^{②} \circ \boldsymbol{u}_j + \beta \sum_{i \neq j}^{m} \delta_{ij} \boldsymbol{u}_i^{②} \circ \boldsymbol{u}_j, \quad j = 1, \ldots, m. \tag{17}$$

Standard Continuation Method for the DNLSE

The solution curve $\mathcal{C}$ of (16):

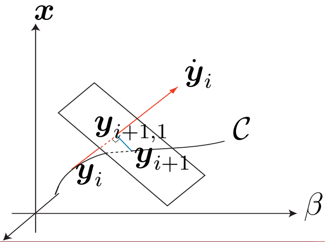
$$\mathcal{C} = \left\{ \boldsymbol{y}(s) = (\boldsymbol{x}(s)^\top, \beta(s))^\top \middle| \, \boldsymbol{G}(\boldsymbol{y}(s)) = \boldsymbol{0}, \; s \in \mathbb{R} \right\}. \tag{18}$$

Assume s is a parametrization via arc length is available on $\mathcal{C}$.

By differentiating with s we have

$$\mathcal{D}\boldsymbol{G}(\boldsymbol{y}(s))\dot{\boldsymbol{y}}(s) \equiv [\boldsymbol{G}_{\boldsymbol{x}}, \boldsymbol{G}_\beta]\dot{\boldsymbol{y}}(s) = 0,$$

where $\dot{\boldsymbol{y}}(s) = (\dot{\boldsymbol{x}}(s)^\top, \dot{\beta}(s))^\top$ is a tangent vector to $\mathcal{C}$ at $\boldsymbol{y}(s)$.



Solution Set of DNLSE for $n = 1$

- We consider 2-coupled NLSE with $n = 1$

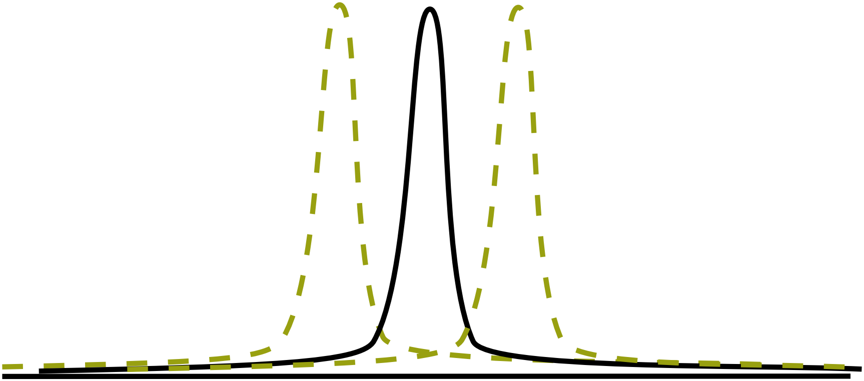
$$\begin{cases} \phi_1^{''} - \lambda_1\phi_1 + \mu_1\phi_1^3 + \beta_{12}\phi_2^2\phi_1 = 0, \\ \phi_2^{''} - \lambda_2\phi_2 + \mu_2\phi_2^3 + \beta_{12}\phi_1^2\phi_2 = 0. \end{cases}$$

- By differentiating with x we have

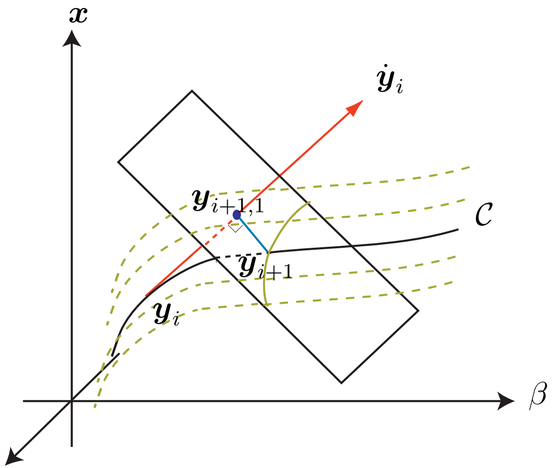
$$\begin{bmatrix} L_1 & 2\beta_{12}\phi_1\phi_2 \\ 2\beta_{12}\phi_1\phi_2 & L_2 \end{bmatrix} \begin{bmatrix} \phi_1^{'} \\ \phi_2^{'} \end{bmatrix} = \mathbf{0}, \tag{19}$$

where $L_1 = \frac{d^2}{dx^2} - \lambda_1 + 3\mu_1\phi_1^2 + \beta_{12}\phi_2^2$ and $L_2 = \frac{d^2}{dx^2} - \lambda_2 + 3\mu_2\phi_2^2 + \beta_{12}\phi_1^2$. From (19) we see that the matrix $\begin{bmatrix} L_1 & 2\beta_{12}\phi_1\phi_2 \\ 2\beta_{12}\phi_1\phi_2 & L_2 \end{bmatrix}$ is singular. It easily seen that the solution set of (1) is one dimensional.

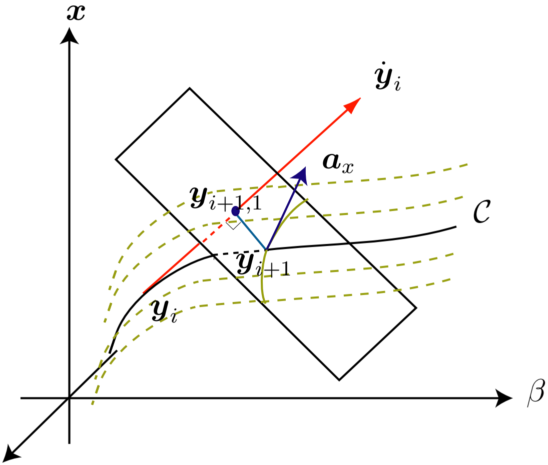
1-D



Numerical Difficulty



Numerical Difficulty



Solution Set of DNLSE for $n = 2$

- We consider the m-coupled DNLSE with domain $[-d, d] \times [-d, d]$. The numerical null space of $\boldsymbol{G}_{\boldsymbol{x}}(\boldsymbol{x}(s), \beta(s))$ is spanned by

$$K_0 = \text{span}\{\boldsymbol{a}_x, \boldsymbol{a}_y\}, \tag{20}$$

where $\boldsymbol{a}_x = \boldsymbol{D}_x \boldsymbol{x}(s)$, $\boldsymbol{a}_y = \boldsymbol{D}_y \boldsymbol{x}(s)$ and

$$\boldsymbol{D}_x = \text{diag}\{D_x, \ldots, D_x\}, \boldsymbol{D}_y = \text{diag}\{D_y, \ldots, D_y\} \in \mathbb{R}^{Nm \times Nm},$$

$D_x, D_y \in \mathbb{R}^{N \times N}$ are discretization matrices of the differential operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, resp..

- Let $\boldsymbol{x}^r$ be translation by x-axis or y-axis from $\boldsymbol{x}$ with $\|\boldsymbol{G}(\boldsymbol{x}^r, \beta)\| < \varepsilon$. Then these solutions are called "*ε-solutions*".

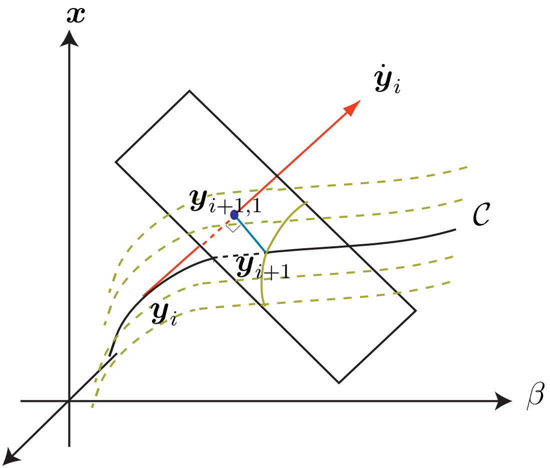
A Comparison

- NLSE
  - Unbounded domain
  - Solutions are translation invariant
- DNLSE
  - Computational bounded domain
  - No translation invariant solutions
  - The ε-solutions (with small residual) exist

Numerical Challenges Due to the ε-solutions

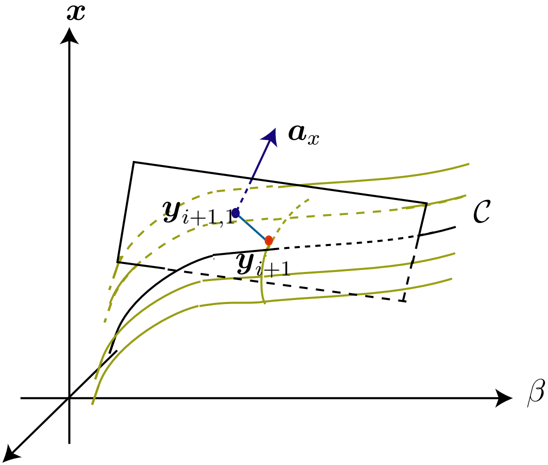
1. Cannot compute a unique prediction direction
2. Newton's correction becomes inaccurate and inefficient (the Jacobian matrix $\boldsymbol{G_x}$ is nearly singular)
3. Detections of bifurcation points are difficult
4. Cannot follow the desired solution curve efficiently (Computed solutions may be random or trapped in the multi-dimensional ε-solution set)

Main Idea



$\boldsymbol{x}$
$\dot{\boldsymbol{y}}_i$
$\boldsymbol{y}_{i+1,1}$
$\mathcal{C}$
$\boldsymbol{y}_{i+1}$
$\boldsymbol{y}_i$
β

Main Idea



$\boldsymbol{x}$
$\boldsymbol{a}_x$
$\boldsymbol{y}_{i+1,1}$
$\mathcal{C}$
$\boldsymbol{y}_{i+1}$
β

Solution Curve

$$\begin{cases} \boldsymbol{G}(\boldsymbol{x}, \beta) = \boldsymbol{0}, \\ \boldsymbol{a}_x^\top \boldsymbol{x} = 0, \\ \boldsymbol{a}_y^\top \boldsymbol{x} = 0, \\ \dot{\boldsymbol{x}}_i^\top \boldsymbol{x} + \dot{\beta}_i \beta = 0. \end{cases}$$

Prediction

Let $\boldsymbol{y}_i = (\boldsymbol{x}_i^\top, p_i)^\top \in \mathbb{R}^{M+1}$ be an approx. point for $\mathcal{C}$. Suppose $\boldsymbol{y}_{i+1,1} = \boldsymbol{y}_i + h_i \dot{\boldsymbol{y}}_i$ is used to predict a new $\boldsymbol{y}_{i+1,1}$, where $\dot{\boldsymbol{y}}_i$ is the tangent vector by solving

$$\begin{bmatrix} \boldsymbol{G}_{\boldsymbol{x}} & \boldsymbol{G}_\beta \\ \boldsymbol{a}_x^\top & 0 \\ \boldsymbol{a}_y^\top & 0 \\ \boldsymbol{c}_i^\top & c_i \end{bmatrix} \dot{\boldsymbol{y}}_i = \begin{bmatrix} \mathbf{0} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{21}$$

with some constant vector $[\boldsymbol{c}_i^\top, c_i]^\top \in \mathbb{R}^{M+1}$.

Correction

$$\begin{cases} \boldsymbol{G}(\boldsymbol{y}) = 0 \\ (\boldsymbol{a}_x^\top, 0)\boldsymbol{y} = \boldsymbol{a}_x^\top \boldsymbol{x}_{i+1,1} \\ (\boldsymbol{a}_y^\top, 0)\boldsymbol{y} = \boldsymbol{a}_y^\top \boldsymbol{x}_{i+1,1} \\ \dot{\boldsymbol{y}}_i^\top \boldsymbol{y} = \dot{\boldsymbol{y}}_i^\top \boldsymbol{y}_{i+1,1} \end{cases}$$

Newton's method is chosen as a corrector,

$$\begin{bmatrix} \boldsymbol{G}_{\boldsymbol{x}}(\boldsymbol{y}_{i+1,l}) & \boldsymbol{G}_\beta(\boldsymbol{y}_{i+1,l}) \\ \boldsymbol{a}_x^\top & 0 \\ \boldsymbol{a}_y^\top & 0 \\ \dot{\boldsymbol{x}}_i^\top & \dot{\beta}_i \end{bmatrix} \boldsymbol{\delta}_l = \begin{bmatrix} -\boldsymbol{G}(\boldsymbol{y}_{i+1,l}) \\ \rho_{x,l} \\ \rho_{y,l} \\ \rho_l \end{bmatrix}, \quad l = 1, 2, \ldots, \tag{22}$$

with $\rho_l = \dot{\boldsymbol{y}}_i^\top(\boldsymbol{y}_{i+1,l} - \boldsymbol{y}_{i+1,1})$, $\rho_{x,l} = \boldsymbol{a}_x^\top(\boldsymbol{x}_{i+1,l} - \boldsymbol{x}_{i+1,1})$ and $\rho_{y,l} = \boldsymbol{a}_y^\top(\boldsymbol{x}_{i+1,l} - \boldsymbol{x}_{i+1,1})$, is solved by $\boldsymbol{y}_{i+1,l+1} = \boldsymbol{y}_{i+1,l} + \boldsymbol{\delta}_l$. If $\{\boldsymbol{y}_{i+1,l}\}$ converges until $l = l_\infty$, we accept $\boldsymbol{y}_{i+1} = \boldsymbol{y}_{i+1,l_\infty}$ as an approx to $\boldsymbol{\mathcal{C}}$.

Outline

Some 3-component NLSE Results I

Let

$$\Sigma = \begin{bmatrix} 1 & |\beta\delta_{12}| & |\beta\delta_{13}| \\ |\beta\delta_{12}| & 1 & |\beta\delta_{23}| \\ |\beta\delta_{13}| & |\beta\delta_{23}| & 1 \end{bmatrix}$$

and $\beta_{ij} = \delta_{ij}\beta$, Lin and Wei (2005) show that

Case 1 (all interactions are repulsive). If $\delta_{12} < 0$, $\delta_{13} < 0$ and $\delta_{23} < 0$, then the ground state solution does not exist.

Case 2 (all interactions are attractive). If $\delta_{12} > 0$, $\delta_{13} > 0$, $\delta_{23} > 0$ and Σ is positive definite, then the ground state solution exists.

Some 3-component NLSE Results II

Case 3 (two repulsive and one attractive interactions). If $\delta_{12} < 0$, $\delta_{13} < 0$, $\delta_{23} > 0$ and Σ is positive definite, then the ground state solution does not exist.

Case 4 (two attractive and one repulsive interactions). If $\delta_{12} > 0$, $\delta_{13} > 0$, $\delta_{23} < 0$, $\beta \ll 1$ and the ground state solution exists, then it must be non-radially symmetric.

The 3-component DNLSE Setting I

- We assume $m = 3,\ \lambda_1 = \lambda_2 = \lambda_3 = \mu_1 = \mu_2 = \mu_3 = 1$.
- The 3-coupled DNLSE $\boldsymbol{G}(\boldsymbol{x}, \beta, \boldsymbol{\delta}) = \boldsymbol{0}$ in (11), where $\boldsymbol{\delta} = (\delta_{12}, \delta_{13}, \delta_{23})$, can be rewritten by

$$\boldsymbol{A}\boldsymbol{u}_1 - \boldsymbol{u}_1 + \boldsymbol{u}_1^{③} + \beta\delta_{12}\boldsymbol{u}_2^{②}\boldsymbol{u}_1 + \beta\delta_{13}\boldsymbol{u}_3^{②}\boldsymbol{u}_1 = \boldsymbol{0}, \tag{1a}$$

$$\boldsymbol{A}\boldsymbol{u}_2 - \boldsymbol{u}_2 + \boldsymbol{u}_2^{③} + \beta\delta_{12}\boldsymbol{u}_1^{②}\boldsymbol{u}_2 + \beta\delta_{23}\boldsymbol{u}_3^{②}\boldsymbol{u}_2 = \boldsymbol{0}, \tag{1b}$$

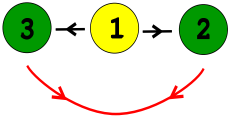
$$\boldsymbol{A}\boldsymbol{u}_3 - \boldsymbol{u}_3 + \boldsymbol{u}_3^{③} + \beta\delta_{13}\boldsymbol{u}_1^{②}\boldsymbol{u}_3 + \beta\delta_{23}\boldsymbol{u}_2^{②}\boldsymbol{u}_3 = \boldsymbol{0}. \tag{1c}$$

The 3-component DNLSE Setting

- Case 1 and 2 are straightforward.
- Case 3 and 4 can be combined by letting

$$\delta_{12} = \delta_{13} = -1, \quad \delta_{23} = 1 \text{ and } \beta \in \mathbb{R}$$

.



- The resulting DNLSE $\boldsymbol{G}(\boldsymbol{x}, \beta) = \mathbf{0}$ in (1) can be rewritten as

$$\boldsymbol{A}\boldsymbol{u}_1 - \boldsymbol{u}_1 + \boldsymbol{u}_1^{\textcircled{3}} - \beta \boldsymbol{u}_2^{\textcircled{2}} \boldsymbol{u}_1 - \beta \boldsymbol{u}_3^{\textcircled{2}} \boldsymbol{u}_1 = \mathbf{0}, \tag{2a}$$

$$\boldsymbol{A}\boldsymbol{u}_2 - \boldsymbol{u}_2 + \boldsymbol{u}_2^{\textcircled{3}} - \beta \boldsymbol{u}_1^{\textcircled{2}} \boldsymbol{u}_2 + \beta \boldsymbol{u}_3^{\textcircled{2}} \boldsymbol{u}_2 = \mathbf{0}, \tag{2b}$$

$$\boldsymbol{A}\boldsymbol{u}_3 - \boldsymbol{u}_3 + \boldsymbol{u}_3^{\textcircled{3}} - \beta \boldsymbol{u}_1^{\textcircled{2}} \boldsymbol{u}_3 + \beta \boldsymbol{u}_2^{\textcircled{2}} \boldsymbol{u}_3 = \mathbf{0}, \tag{2c}$$

Primal Stalk Solution I

Theorem

The primal stalk of 3*-coupled DNLSE* (2) *can be described by*

$$\begin{cases} \boldsymbol{u}_1 = \sqrt{\frac{1+3\beta}{1+\beta-2\beta^2}}\boldsymbol{u}_*, \\ \boldsymbol{u}_2 = \boldsymbol{u}_3 = \sqrt{\frac{1+\beta}{1+\beta-2\beta^2}}\boldsymbol{u}_*, \end{cases} \quad -\frac{1}{3} \leq \beta < 1 \tag{3}$$

where $\boldsymbol{u}_*$ *is a solution of* $\boldsymbol{A}\boldsymbol{u} - \boldsymbol{u} + \boldsymbol{u}^{③} = \boldsymbol{0}$.

Primal Stalk Solution II

Remark

- *If* $\beta \to 1^-$ *then*

$$\boldsymbol{u}_2 = \boldsymbol{u}_3 = \sqrt{\frac{1+\beta}{1+3\beta}}\boldsymbol{u}_1 \to \infty.$$

- *If* $\beta \to -\frac{1}{3}^+$ *then*

$$\boldsymbol{u}_1 \to \boldsymbol{0}, \quad \boldsymbol{u}_2 = \boldsymbol{u}_3 \to \sqrt{\frac{3}{2}}\boldsymbol{u}_*.$$

Bifurcation Analysis

The solution curve $\mathcal{C}$ of (2):

$$\mathcal{C} = \{\boldsymbol{y}(s) = (\boldsymbol{x}^\top(s), \beta(s))^\top | \boldsymbol{G}(\boldsymbol{y}(s)) = \mathbf{0} \text{ is given in (2)}\} \tag{4}$$

Theorem

The primal stalk of $\mathcal{C}$ in (4) given by (3), undergoes at least $N-p$ bifurcation points at $0 < \beta = \beta_q^* < 1$, $q = 1, \ldots, N-p$, where p is the number of nonnegative eigenvalues of $\boldsymbol{A} - \boldsymbol{I} + 3[\![\boldsymbol{u}_*^{②}]\!]$.

Remark

In [2] show that the number of nonnegative eigenvalues of

$$\begin{cases} \Delta\phi - \phi + 3\omega_*^2\phi = \lambda\phi, \\ \phi \in H^2(\mathbb{R}^n), \end{cases}$$

is $n+1$, where ω_* is the unique solution of

$$\begin{cases} \Delta\phi - \phi + \phi^3 = 0, \\ \phi > 0 \textit{ in } \mathbb{R}^n, \\ \omega(x) \to 0 \textit{ as } |x| \to \infty. \end{cases}$$

In square domain ($n=2$), it seem that the number of nonnegative eigenvalues of $\boldsymbol{A} - \boldsymbol{I} + 3[\![\boldsymbol{u}_*^{②}]\!]$ is 3. In numerical test, the number of nonnegative eigenvalues of $\boldsymbol{A} - \boldsymbol{I} + 3[\![\boldsymbol{u}_*^{②}]\!]$ (the eigenvalue bigger than -10^{-3}) is 3.

[2]C.-S. Lin and W.-M. Ni, On the diffusion coefficient of a semilinear Neumann problem, *In Lecture Notes in Mathematics*, **1340**(1988) 160-174.

Outline

Implementation

- Fortran 95 codes
  - Hyperplane-constrained continuation method
  - Interfaces with the packages
  - Eigenvalue solver (ARPACK)
  - Linear system solver (GMRES by CERFACS)
- Linux based workstation
- Automatic bifurcation points detection
- Automatic path following with user defined path following policy
- Restarting from intermediate solutions
- Batch runs

Implementation

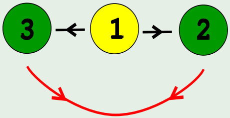
- User defined PDEs
- User defined program parameters
- User defined initial solution
- On-going work
  - Structure aware and efficient eigenvalue solver
  - Structure aware and efficient linear system solver
  - Parallel version (distributed memory or multi-cores)
  - Object version

Simulation 1

Example

- $m = 3$; $\Omega = [-5, 5] \times [-5, 5]$; $\lambda_j = \mu_j = 1$,
- The mesh size h of the grid domain Ω_h is 0.2
- (2 repulsive and 1 attractive) $\delta_{12} = \delta_{13} = -1, \quad \delta_{23} = 1.$
- The solution curve

$$\mathcal{C}^+ = \left\{ (\boldsymbol{x}^\top, \beta)^\top \,\middle|\, \boldsymbol{G}(\boldsymbol{x}, \beta) = \boldsymbol{0} \text{ for } \beta \in \mathbb{R}_+ \right\}. \tag{1}$$



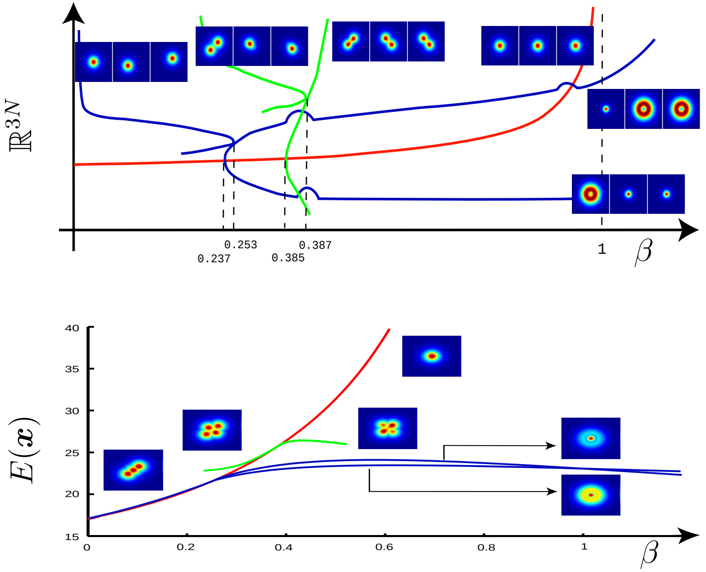


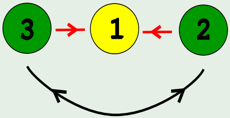
Figure 1. Bifurcation curves and energy curves of DNLSE.

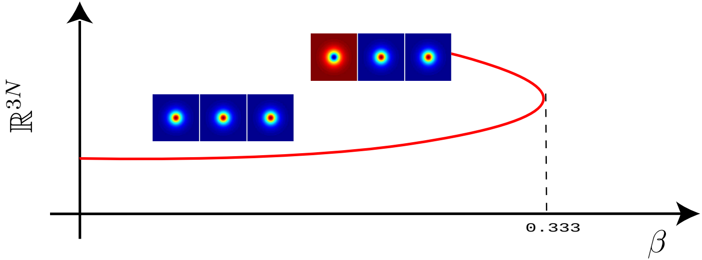
Simulation 2

Example

- $m = 3$; $\Omega = [-5, 5] \times [-5, 5]$; $\lambda_j = \mu_j = 1$,
- The mesh size h of the grid domain Ω_h is 0.2
- (2 attractive and 1 repulsive) $\delta_{12} = \delta_{13} = 1, \quad \delta_{23} = -1.$
- The solution curve

$$\mathcal{C}^- = \left\{ (\boldsymbol{x}^\top, \beta)^\top \middle| \, \boldsymbol{G}(\boldsymbol{x}, \beta) = \boldsymbol{0} \text{ for } \beta \in \mathbb{R}_- \right\}. \tag{1}$$





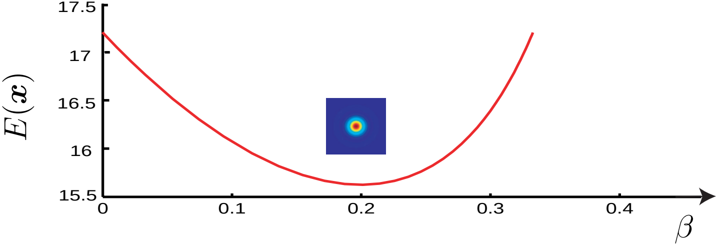


Figure 2. Bifurcation curves and energy curves of DNLSE.

Observations

- The theoretical applicable range of β is $-\frac{1}{3} \leq \beta < 1$.
  - Computed β's keep approaching, but never reaching 1.
  - A turning point is found at $\beta = -0.3333$.
- In Simulation 1, computed energy keeps raising as β increases to 1. This is consistent to the result that if $\beta \to 1^-$ then

$$\boldsymbol{u}_2 = \boldsymbol{u}_3 = \sqrt{\frac{1+\beta}{1+3\beta}}\boldsymbol{u}_1 \to \infty.$$

- Computed solution profiles in Simulation 2 is in line with the result that if $\beta \to -\frac{1}{3}^+$ then

$$\boldsymbol{u}_1 \to \boldsymbol{0}, \quad \boldsymbol{u}_2 = \boldsymbol{u}_3 \to \sqrt{\frac{3}{2}}\boldsymbol{u}_*.$$

Simulation 3

Example

- $m = 3$; $\Omega = [-5, 5] \times [-5, 5]$; $\lambda_j = \mu_j = 1$,
- The mesh size h of the grid domain Ω_h is 0.2
- To search for the non-radially solution whose energy is less than the radially symmetric solutions for small β, where $\delta_{12} = \delta_{13} = 1, \quad \delta_{23} = -1$ (Simulation 2).
- Only radially symmetric solutions are found in Simulation 2.

Procedure

- **Step 1.** We trace the solution curve

$$\mathcal{C}_1 = \left\{ (\boldsymbol{x}^\top, \beta)^\top \mid \boldsymbol{G}(\boldsymbol{x}, \beta) = \boldsymbol{0} \text{ with } \delta_{12} = \delta_{13} = \delta_{23} = 1, \text{ for } 0 \le \beta \le 0.2 \right\}. \tag{2}$$

- **Step 2.** Fix $\beta = 0.2$, then trace the solution curve

$$\mathcal{C}_2 = \left\{ (\boldsymbol{x}^\top, \delta_{23})^\top \mid \boldsymbol{G}(\boldsymbol{x}, \delta_{23}) = \boldsymbol{0} \text{ with } \beta = 0.2, \delta_{12} = \delta_{13} = 1, \text{ for } -1 \le \delta_{23} \le 1 \right\}. \tag{3}$$

- **Step 3.** Fix $\delta_{23} = -1$, then trace the solution curve

$$\mathcal{C}_3 = \left\{ (\boldsymbol{x}^\top, \beta)^\top \mid \boldsymbol{G}(\boldsymbol{x}, \beta) = \boldsymbol{0} \text{ with } \delta_{12} = \delta_{13} = 1 \text{ and } \delta_{23} = -1, \text{ for } \beta \in \mathbb{R} \right\}. \tag{4}$$

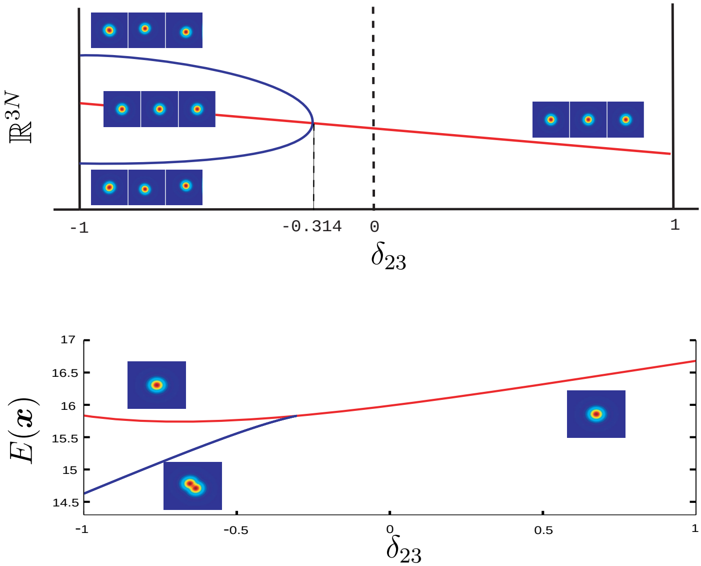


Figure 3. Bifurcation curves and energy curves of DNLSE.

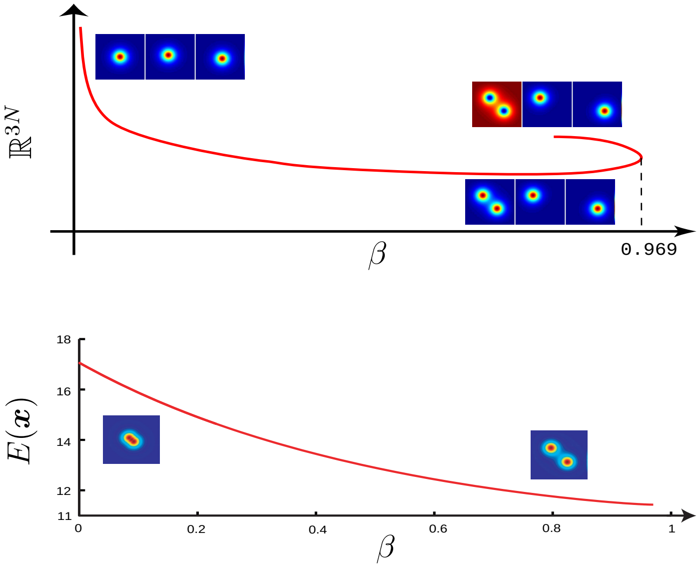


Figure 4. Bifurcation curves and energy curves of DNLSE.

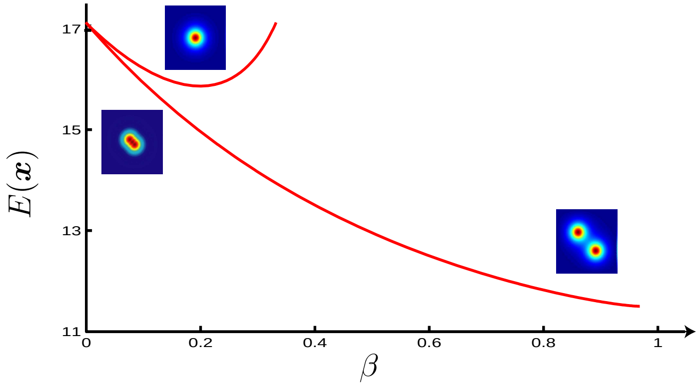


Figure 5. Energy curves of DNLSE.

Motivation and Observations

- Solutions on $\mathcal{C}_1$ are all ground state solutions.
  - Interactions are all attractive($\delta_{12} = \delta_{13} = \delta_{23} = 1$). Solutions tends to gather together.
  - No bifurcation found.
  - Initial solution for $\beta = 0$ is ground state.
- $\mathcal{C}_2$ is a "bridge" connecting $\mathcal{C}_1$ (three attractive) and $\mathcal{C}_3$ (two attractive one repulsive).
- One bifurcation occurs in $\mathcal{C}_2$. Primal stalk solutions lead to the results in Simulation 2. The bifurcation branch leads to lower energy solutions.

Motivation and Observations

- We let β decreases to zero in $\mathcal{C}_3$ to find the target solutions.
- Another type non-radially symmetric solution is found for β increase to 1 in $\mathcal{C}_3$.
- The non-radially symmetric solutions are expected to be ground state, as we start from ground state $(\beta = 0)$ and follow the lower energy path whenever bifurcation occurs.

Outline

One Component Rotating BEC Model

$$-\frac{1}{2}\nabla^2\phi(\boldsymbol{x}) + V(\boldsymbol{x})\phi(\boldsymbol{x}) + \alpha|\phi|^2\phi(\boldsymbol{x}) + \omega\boldsymbol{\imath}\partial_\theta\phi(\boldsymbol{x}) = \lambda\phi(\boldsymbol{x}), \quad (5)$$

for $\boldsymbol{x} \in \Omega \subseteq \mathbb{R}^2$ with

$$\int_\Omega |\phi(\boldsymbol{x})|^2 d\boldsymbol{x} = 1, \quad (6)$$

- Ω is a smooth bounded domain
- $V(\boldsymbol{x}) \geq 0$ is the magnetic trapping potentials
- $\partial_\theta = x\partial_y - y\partial_x$ is the z-component of the angular momentum
- ω is the angular velocity of the rotating laser beam
- λ is the chemical potential

Induced Nonlinear Algebraic Eigenvalue Problem

$$\boldsymbol{A}\boldsymbol{u} + \alpha \boldsymbol{u}^H \circ \boldsymbol{u}^{②} + \omega \boldsymbol{\imath} \mathbf{S} \boldsymbol{u} = \lambda \boldsymbol{u}, \tag{7}$$

$$\boldsymbol{u}^\top \boldsymbol{u} = 1. \tag{8}$$

- $\boldsymbol{A}$: standard central finite difference discretization of $-\frac{1}{2}\nabla^2 + V(\boldsymbol{x})$
- $\mathbf{S}$: discretization matrix corresponding to ∂_θ
- $\boldsymbol{u}^{②} = \boldsymbol{u} \circ \boldsymbol{u}$, and $\circ$ denotes the Hadamard product
- Dirichlet boundary condition

Corresponding Energy Functional

- Energy functional

$$E(\phi) = \int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}V_j|\phi|^2 + \frac{\alpha}{4}|\phi|^4 + \frac{\omega\boldsymbol{\iota}}{2}\phi^*\partial_\theta\phi \right), \qquad (9)$$

where ϕ^* denotes the complex conjugate of ϕ

- Finite dimensional case

$$E(\boldsymbol{u}) = \left(\frac{1}{2}\boldsymbol{u}^H\boldsymbol{A}\boldsymbol{u} + \frac{\alpha}{4}\boldsymbol{u}^{②H}\boldsymbol{u}^{②} \right) + \frac{\omega\boldsymbol{\iota}}{2}\boldsymbol{u}^H\mathbf{S}\boldsymbol{u}. \qquad (10)$$

NAEP rewritten

- The new form

$$\boldsymbol{A}\boldsymbol{u}_1 + \alpha([\![\boldsymbol{u}_1^{②}]\!] + [\![\boldsymbol{u}_2^{②}]\!])\boldsymbol{u}_1 - \omega \mathbf{S}\boldsymbol{u}_2 = \lambda \boldsymbol{u}_1, \quad (11)$$

$$\boldsymbol{A}\boldsymbol{u}_2 + \alpha([\![\boldsymbol{u}_1^{②}]\!] + [\![\boldsymbol{u}_2^{②}]\!])\boldsymbol{u}_2 + \omega \mathbf{S}\boldsymbol{u}_1 = \lambda \boldsymbol{u}_2, \quad (12)$$

with

$$\boldsymbol{u}_1^\top \boldsymbol{u}_1 + \boldsymbol{u}_2^\top \boldsymbol{u}_2 = 1, \quad (13)$$

where $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{\iota}\boldsymbol{u}_2 \in \mathbb{C}^N$ and $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R}^N$

The Parameter Dependent Polynomial System

Let $\omega = \omega_0 + \nu_0 p$ (ω_0, ν_0 are given and p is the continuation parameter), $\tilde{\boldsymbol{u}} = (\boldsymbol{u}_1^\top, \boldsymbol{u}_2^\top)^\top \in \mathbb{R}^{2N}$, and $\boldsymbol{z} = (\tilde{\boldsymbol{u}}^\top, \lambda)^\top \in \mathbb{R}^{2N+1}$.

$$\boldsymbol{G}(\boldsymbol{z}, p) = \mathbf{0}, \tag{14}$$

where $\boldsymbol{G} \equiv (\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{g}) : \mathbb{R}^{2N+1} \times \mathbb{R} \to \mathbb{R}^{2N+1}$ is given by

$$\boldsymbol{G}_1(\boldsymbol{z}, p) = \boldsymbol{A}\boldsymbol{u}_1 + \alpha([\![\boldsymbol{u}_1^{②}]\!] + [\![\boldsymbol{u}_2^{②}]\!])\boldsymbol{u}_1 - \omega \mathbf{S}\boldsymbol{u}_2 - \lambda \boldsymbol{u}_1, \tag{15a}$$

$$\boldsymbol{G}_2(\boldsymbol{z}, p) = \boldsymbol{A}\boldsymbol{u}_2 + \alpha([\![\boldsymbol{u}_1^{②}]\!] + [\![\boldsymbol{u}_2^{②}]\!])\boldsymbol{u}_2 + \omega \mathbf{S}\boldsymbol{u}_1 - \lambda \boldsymbol{u}_2, \tag{15b}$$

$$\boldsymbol{g}(\boldsymbol{z}, p) = \frac{1}{2}(\boldsymbol{u}_1^\top \boldsymbol{u}_1 + \boldsymbol{u}_2^\top \boldsymbol{u}_2 - 1). \tag{15c}$$

Transformation Invariant Solutions

Define $\tilde{\boldsymbol{u}}(\theta) : [0, 2\pi] \to \mathbb{R}^{2N}$ by

$$\tilde{\boldsymbol{u}}(\theta) = \begin{bmatrix} \cos\theta \boldsymbol{u}_1 + \sin\theta \boldsymbol{u}_2 \\ -\sin\theta \boldsymbol{u}_1 + \cos\theta \boldsymbol{u}_2 \end{bmatrix}. \tag{16}$$

- Solution set $\mathcal{C}$ is a two dimensional manifold on $\mathbb{R}^{2N+2}$
- $\boldsymbol{G}(\tilde{\boldsymbol{u}}(\theta), \lambda, p) = \boldsymbol{0}$ for all $\theta \in [0, 2\pi]$
- Same energy: $E(\tilde{\boldsymbol{u}}(0)) = E(\tilde{\boldsymbol{u}}(\theta))$
- Same shape: $|\tilde{\boldsymbol{u}}(0)|^2 = |\tilde{\boldsymbol{u}}(\theta)|^2$

Hyperplane-Constrained Continuation Method

- Consider the quotient solution set

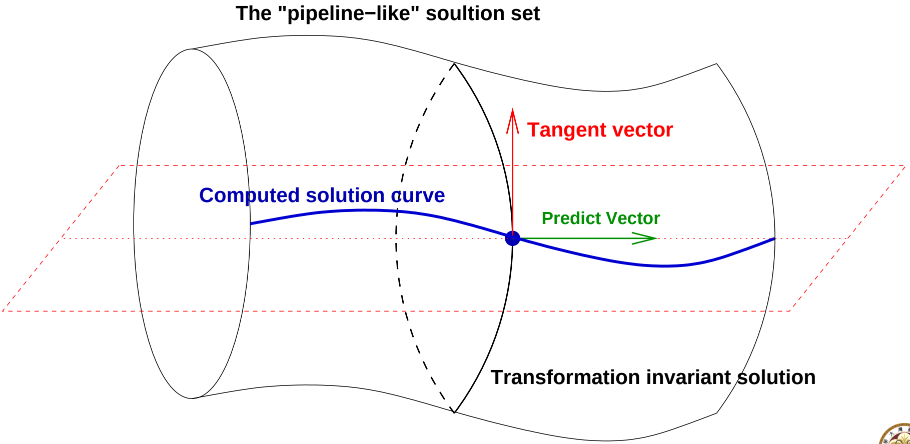
$$\mathcal{C}/\theta = \{\boldsymbol{y}(s) = (\boldsymbol{z}(s)^\top, p(s))^\top | \; \boldsymbol{G}(\boldsymbol{y}(s)) = 0, s \in \mathbb{R}\}. \tag{17}$$

- Compute the tangent vector of $\tilde{\boldsymbol{u}}(\theta)$ at $\theta = 0$,

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta}(0) = (\boldsymbol{u}_2^\top, -\boldsymbol{u}_1^\top)^\top. \tag{18}$$

- Prediction vector $\dot{\boldsymbol{y}}_i = (\dot{\boldsymbol{z}}_i^\top, \dot{p}_i^\top)^\top$ satisfies $\mathcal{D}\boldsymbol{G}(\boldsymbol{y}(s))\dot{\boldsymbol{y}}(s) = \boldsymbol{0}$ and is orthogonal to $\left(\frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta}(0)^\top, 0\right)^\top$
- In correction, add an additional hyperplane constraint, with normal vector $\left(\frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta}(0)^\top, 0\right)^\top$

Visualization of the Idea



An Analysis of Bifurcation

Theorem

Suppose $0 < \alpha < \infty$ *and* $p = \omega$ *in* (8). *Then the solution curve of ground states undergoes at least* n *(*$= N - dim\, \mathcal{N}(\mathbf{S})$*) bifurcation points at finite value* $\omega = \omega_i^*$, $i = 1, \ldots n$. *That is, the Jacobian matrix* $\widetilde{\boldsymbol{G}}_{\boldsymbol{z}}(\boldsymbol{z}, \omega)$ *is singular on* $\mathcal{C}_\theta$ *at* ω_i^*.

An Experimental Result

- Madison, Chevy, Wohlleben, and Dalibard, PRL (84)5, pp. 806–809, 2000

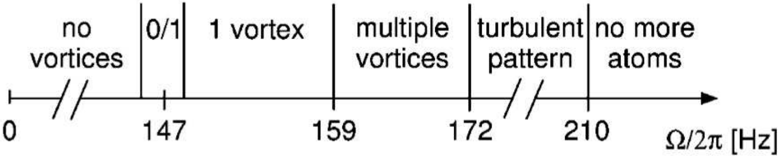
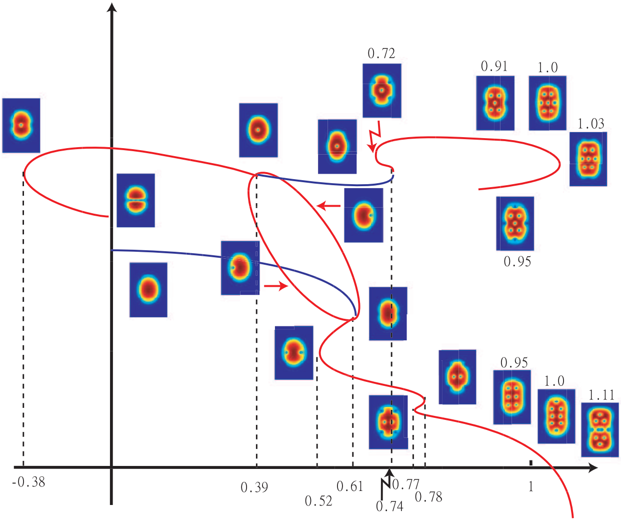


FIG. 2. Edges of the regions of stability for the 0, 1, and multiple vortex configurations. The condensate number is $N_0 = (2.3 \pm 0.6)\ 10^5$ and the temperature below 80 nK.

Preliminary Computational Results



Initial Solutions

- Let $\omega = 0$
- Solve the linear eigenvalue problem $\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{u}$ for $\alpha = 0$
- Take the ith smallest eigenvalue and the corresponding eigenfunction as initial of continuation method
- Follow the solution curve by increasing $\alpha = 0$ to $\alpha = 100$.

Outline

Conclusions

- m-coupled DNLSE
  - Nonlinear optics for Kerr-like photorefractive media
  - A hyperplane-constrained continuation method for ε-solutions
  - Analysis and numerical experiments for 3-coupled DNLSE
  - Bifurcation diagrams and non-radially symmetric ground states
- Single component DNLSE
  - Single component rotating Bose-Einstein condensate
  - A hyperplane-constrained continuation method for transformation invariant solutions
  - Bistability of solution curves

Thank you.