Institute for Mathematical Sciences

National University of Singapore

BRAIDS: summer school and workshop

Tutorial: Braids - definitions and braid groups

June 4-8, 2007

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Introduction.

The braid groups B_n were introduced by E. Artin eighty years ago [1], although their significance to mathematics was possibly realized a century earlier by Gauss, as evidenced by sketches of braids in his notebooks. The braid groups provide a very attractive blending of geometry and algebra, and have applications in a wide variety of areas of mathematics, physics, and recently in polymer chemistry and molecular biology.

Despite their ripe old age, and an enormous amount of attention paid to them by mathematicians and physicists, these groups still provide us with big surprises. The subject was given a major boost when V. Jones discovered, in the mid 1980's, new representations of the braid groups, which led to the famous Jones polynomial of knot theory. In this minicourse, I will touch on that discovery, and more recent ones, regarding this wonderful family of groups, and raise a few questions about them which are still open. Along the way, I will also discuss some properties of these groups which deserve to be better known. Among my personal favorites is the orderability of braid groups, which implies that they have some special algebraic properties. That will be the subject of my last lecture.

Due to time restrictions, I will not be able to discuss many interesting aspects of braid theory, such as the conjugacy problem as solved by Garside, dynamics of braids, applications to cryptography and connections with homotopy groups of spheres. Some of these issues will be considered in lectures by others in this summer school. Also because of time limitation, many proofs will be omitted in my presentation. Some proofs will be left as exercises for the students, often with hints or references.

The emphasis of my presentation is that there are many ways of looking at braids, and each point of view provides new information about these groups. Though much of the motivation for studying braids comes from knot theory, my emphasis will be on the algebraic aspects of braid groups.

Prerequisites

Students will be assumed to have background in basic topology and group theory. No other specialized expertise is necessary.

Suggested reading

• Joan Birman and Tara Brendle, Braids, A Survey, in *Handbook of Knot Theory* (ed. W. Menasco and M. Thistlethwaite), Elsevier, 2005.

Available electronically at:

http://www.math.columbia.edu/ jb/Handbook-21.pdf

• Joan Birman, *Braids, links and mapping class groups*, Annals of Mathematics Studies, Princeton University Press, 1974.

This classic is still a useful reference.

• Gerhard Burde and Heiner Zieschang, *Knots*, De Gruyter Studies in Mathematics 5, second edition, 2003.

This contains a very readable chapter on braid groups.

• Patrick Dehornoy, Ivan Dynnikov, Dale Rolfsen, and Bert Wiest, *Why are braids orderable?* Soc. Math. France series Panoramas et Syntheses 14(2002).

Available electronically at:

http://perso.univ-rennes1.fr/bertold.wiest/bouquin.pdf

• Vaughan F. R. Jones, *Subfactors and Knots*, CBMS Regional Conference series in Mathematics, Amer. Math. Soc., 1991.

This gem focusses on the connection between operator algebras and knot theory, via the braid groups.

• Kunio Murasugi and Bohdan Kurpita, *A study of braids*, Mathematics and its applications, Springer, 2006.

This rather expensive book does not seem to be available online.

Also see research papers and other books in the bibliography at the end of the notes.

1 Braids as strings, dances or symbols:

One of the interesting things about the braid groups B_n is they can be defined in so many ways, each providing a unique insight. I will briefly describe some of these definitions, punctuated by facts about B_n which are revealed from the various points of view. You can check the excellent references [1], [2], [6], [9], [12], [17] for more information.

Definition 1: Braids as strings in 3-D. This is the usual and visually appealing picture. An n-braid is a collection of n strings in (x, y, t)-space, which are disjoint and monotone in (say) the t direction.

We require that the endpoints of the strings are at fixed points, say the points $(k, 0, 0)$ and $(k, 0, 1)$, $k = 1, \ldots, n$. We also regard two braids to be equivalent (informally, we will say they are equal) if one can be deformed to the other through the family of braids, with endpoints fixed throughout the deformation.

Although most authors draw braids vertically, I prefer to view them horizontally with the t-axis running from left to right. My reason is that we do algebra (as in writing) from left to right, and the multiplication of braids is accomplished by concatenation (which is made more explicit in the second definition below), which we take to be in the same order as the product. In the vertical point of view, some authors view the product $\alpha\beta$ of two braids α and β with α above β, while others would put β on top of α. The horizontal convention eliminates this ambiguity.

There is, of course, a strong connection between braids and knots. A braid β defines a knot or link $\hat{\beta}$, its closure, by connecting the endpoints in a standard way, without introducing further interaction between the strings.
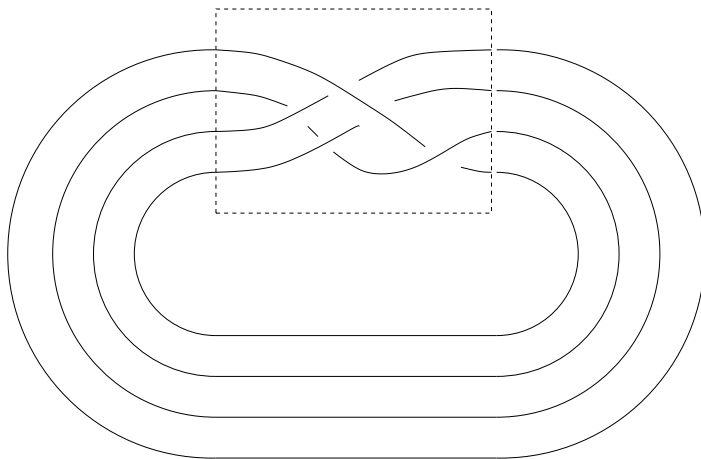

Figure 1: The closure of a braid.

Equivalent braids give rise to equivalent links, but different braids may give rise to

the same knot or link. We will discuss how to deal with this ambiguity later. B_n will denote the group formed by n-braids with concatenation as the product, which we will soon verify has the properties of a group, namely an associative multiplication, with identity element and inverses.

Definition 2: Braids as particle dances. If we take t as a time variable, then a braid can be considered to be the time history of particles moving in the (x, y) plane, or if you prefer, the complex plane, with the usual notation $x + y\sqrt{-1}$.

This gives the view of a braid as a dance of noncolliding particles in $\mathbb{C}$, beginning and ending at the integer points $\{1, \ldots, n\}$. We think of the particles moving in trajectories

$$\beta(t) = (\beta_1(t), \ldots \beta_n(t)), \quad \beta_i(t) \in \mathbb{C},$$

where t runs from 0 to 1, and $\beta_j(t) \neq \beta_k(t)$ when $j \neq k$. A *braid* is then such a time history, or dance, of noncolliding particles in the plane which end at the spots they began, but possibly permuted. Equivalence of braids roughly reflects the fact that choreography does not specify precise positions of the dancers, but rather their relative positions. However, notice that if particle j and $j + 1$ interchange places, rotating clockwise, and then do the same move, but counterclockwise, then their dance is equivalent to just standing still!

The product of braids can be regarded as one dance following the other, both at double speed. Formally, if α and β are braids, we define their product $\alpha\beta$ (in the notation of this definition) to be the braid which is $\alpha(2t)$ for $0 \leq t \leq 1/2$ and $\beta(2t-1)$ for $1/2 \leq t \leq 1$.

Exercise 1.1 *Verify that the product is associative, up to equivalence. The identity braid is to stand still, and each dance has an inverse; doing the dance in reverse. More formally, if β is a braid, as defined in this section, define $\bar{\beta}(t) = \beta(1 - t)$. Write a formula for the product $\gamma = \beta\bar{\beta}$ and verify that there is a continuous deformation from γ to the identity braid. [Hint: let s be the deformation parameter, and consider the dance which is to perform b up till time $s/2$, then stand still until time $1 - s/2$, then do the dance $\bar{\beta}$ in the remaining time.]*

A braid β defines a permutation $i \to \beta_i(1)$ which is a well-defined element of the permutation group Σ_n. This is a homomorphism $B_n \to \Sigma_n$ with kernel, by definition, the subgroup $P_n < B_n$ of *pure* braids. P_n is sometimes called the *colored* braid group, as the particles can be regarded as having identities, or colors. P_n is of course normal in B_n, of index $n!$, and there is an exact sequence

$$1 \to P_n \to B_n \to \Sigma_n \to 1.$$

Exercise 1.2 *Show that any braid is eqiuvalent to a piecewise-linear braid. Moreover, one may assume that under the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$ given by $p(x, y, t) = (y, t)$,*

there are only a finite number of singularities of p restricted to the (image of the) given braid. One may assume these are all double points, where a pair of strings intersect transversely, and at different t-values.
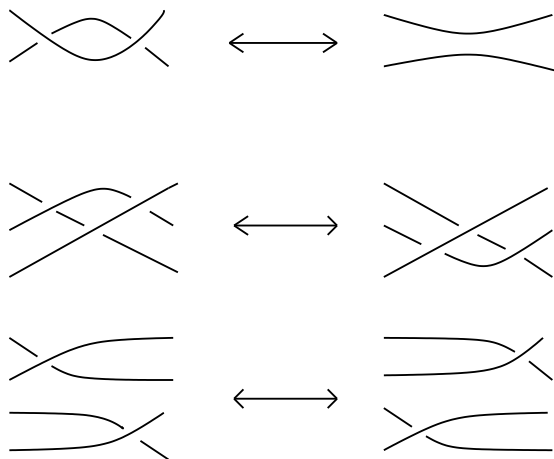

Figure 2: Equivalent braids.

Exercise 1.3 *Viewing the projection in the (y, t) - plane, with the t-axis as horizontal, for any fixed value of t at which a crossing does not occur, label the strings $1, 2, \ldots, n$ counting from the bottom to the top in the y direction. Let σ_i for $i = 1, \ldots, n-1$ denote the braid with all strings horizontal, except that the string $i + 1$ crosses over the i string, resulting in the permutation $i \leftrightarrow i+1$ in the labels of the strings. (Think of it as a right-hand screw motion of these two strings. Argue that any braid is equivalent to a product of these "generators" of the braid group. [In Figure 1, for example, the braid could be written either as $\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2\sigma_1^{-1}$ or $\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2\sigma_1^{-1}$ (or indeed many other ways).]*

Finally, note that the inverse of σ_i is the braid with all strings horizontal, except that the string i crosses over the $i+1$ string. The first sketch in Figure 2 shows that these are indeed inverse to each other.

Definition 3: The algebraic braid group. B_n can be regarded algebraically as the group presented with generators $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i is the braid described in the previous exercise. These generators are subject to the relations (as suggested in Figure 2:

$$\sigma_i\sigma_j = \sigma_j\sigma_i, |i - j| > 1,$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, |i - j| = 1.$$

It was proved by Artin that these are a complete set of relations to abstractly define B_n. This means that any relation among the σ_j can be deduced from the above relations.

Exercise 1.4 *Verify that, in B_n, all the generators σ_j are conjugate to each other. Show the total degree $\deg(\beta)$ which is the sum of the exponents which appear in an expression of β in the σ_j, is well-defined, i.e. independent of the expression. Verify that the abelianization of B_n, for $n \geq 2$, is the infinite cyclic group $\mathbb{Z}$, and $\deg : B_n \to \mathbb{Z}$ is equivalent to the abelianization homomorphism. Thus the commutator subgroup of B_n consists of all words of total degree zero.*

Exercise 1.5 *In B_3, consider the special element $\Delta = \Delta_3 = \sigma_1\sigma_2\sigma_1$, which consists of a clockwise "half-twist" of the three strings. Verify that $\Delta\sigma_1 = \sigma_2\Delta$ and $\Delta\sigma_2 = \sigma_1\Delta$. Conclude that Δ^2 commutes with both generators of B_3 and therefore is in the centre of B_3. (It actually generates the centre, but we will not prove this.)*

Show that in B_3 there are two braids α and β such that $\alpha \neq \beta$ but $\alpha^3 = \beta^3$. What about squares?

Exercise 1.6 *For $n \geq 3$, define $\Delta = \Delta_n = (\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$. Draw this braid and convince yourself that it is a half-twist and that $\Delta\sigma_j\Delta^{-1} = \sigma_{n-j}$ for all $j \in \{1, \ldots, n-1\}$. Conclude that Δ^2 is central in B_n.*

We can take a whole countable set of generators $\sigma_1, \sigma_2, \ldots$ subject to the above relations, which defines the infinite braid group B_∞. If we consider the (non-normal) subgroup generated by $\sigma_1, \ldots, \sigma_{n-1}$, these algebraically define B_n. Notice that this convention gives "natural" inclusions $B_n \subset B_{n+1}$ and $P_n \subset P_{n+1}$.

Exercise 1.7 *Does B_∞ have a nontrivial centre?*

Going the other way, if one forgets the last string of an $n + 1$-braid the result is an n-braid. But strictly speaking, this is only a well defined homomorphism for pure braids, (or at best for the subgroup of braids in which the string beginning at the point $n + 1$ also ends there). Later, we will have a use for this forgetful map

$$f : P_{n+1} \to P_n$$

It is easy to see that f is a left inverse of the inclusion, or in other words a retraction in the category of groups.

Artin Combing. We now have the ingredients for the combing technique, by which Artin solved the word problem for pure braid groups, and therefore for the full braid groups. Recall that the word problem is to decide when two words in the generators of a given group presentation actually represent the same group element, or equivalently, when a given word actually represents the identity in the group.

Given a word in the σ_j, first work out its corresponding permutation. If the permutation is nontrivial, so is the group element it represents. If trivial, the braid is a pure braid, and we have reduced the problem to the word problem for P_n.

Let β be a pure n-braid and $f(\beta)$ the pure $n-1$ braid obtained by forgetting the last string, but then by inclusion, regard $f(\beta)$ in P_n. Then β and $f(\beta)$ can be visualized as the same braid, except the last string has been changed in $f(b)$ so as to have no interaction with the other strings. Let K be the kernel of f. Then it is easy to verify that $f(\beta)^{-1}\beta \in K$ and the map

$$\beta \rightarrow (f(\beta), f(\beta)^{-1}\beta)$$

maps P_n bijectively onto the cartesian product $P_{n-1} \times K$. However, the multiplicative structure is that of a *semidirect* product, as happens whenever we have a split exact sequence of groups, in this case

$$1 \rightarrow K \rightarrow P_n \rightarrow P_{n-1} \rightarrow 1.$$

Also notice that every element of K can be represented by a braid in which the first $n-1$ strands go straight across. In this way we identify K with the fundamental group of the complement of the points $\{1, \ldots, n-1\}$ in the plane, which is a free group: $K \cong F_{n-1}$. This process can then be iterated on P_{n-1} to obtain the Artin normal form: P_n is an iterated semidirect product of free groups $F_1, F_2, \ldots, F_{n-1}$.

Exercise 1.8 *Draw the braid* $\alpha_j = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_{j+1}\sigma_j^2\sigma_{j+1}^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}$ *and verify that* $\alpha_1, \ldots, \alpha_{n-1}$ *represent a free basis for the group* K *above.*

To solve the word problem in P_n– to decide whether a braid β expressed as a word in the generators is trivial in the group – construct the unique expansion

$$\beta = \beta_1\beta_2\cdots\beta_{n-1}$$

from the Artin combing, with $\beta_j \in F_j$. The word problem in free groups being very easy (two words represent the same group element iff they become equal when all cancellations of xx^{-1} or $x^{-1}x$ are made) we have solved the word problem for P_n.

Theorem 1.9 *There is an algorithmic solution to the word problem in* P_n *and* B_n.

For later reference, we will call the vector

$$(\beta_1, \beta_2, \cdots, \beta_{n-1})$$

the "Artin coordinates" of β.

Exercise 1.10 *Show that the abelianization of* P_n *is free abelian of rank* $\binom{n}{2}$.

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It is worth mentioning, without details, a theorem of W. Thurston. An infinite group is called *automatic* if is well-modelled (in a well-defined technical sense which I won't elaborate here) by a finite-state automaton. The standard reference is [11].

Theorem 1.11 (Thurston) B_n *is automatic.*

This implies, for example, that the word problem can be solved by an algorithm which is quadratic in the length of the input.

Exercise 1.12 *What is the complexity of the Artin combing algorithm?*

Exercise 1.13 *Show that K is not normal in B_n, but is normalized by the subgroup B_{n-1}. Moreover, the action of B_{n-1} on K by conjugation is essentially the same as the Artin presentation, given in the next section.*

2   Mapping class groups and braids

Definition 4: B_n as a mapping class group. Going back to the second definition, imagine the particles are in a sort of planar jello and pull their surroundings with them as they dance about. Topologically speaking, the motion of the particles extends to a continuous family of homeomorphisms of the plane (or of a disk, fixed on the boundary). This describes an equivalence between B_n and the mapping class of D_n, the disk D with n punctures (marked points). That is, B_n can be considered as the group of homeomorphisms of D_n fixing ∂D and permuting the punctures, modulo isotopy fixing $\partial D \cup \{1, \ldots, n\}$.

This definition enables us to define a broader class of braid groups, that is, *surface braid groups*. Just replace the disk D in the above discussion with a connected surface Σ, with or without boundary. And let $MCG(\Sigma, n)$ be the group of (orientation preserving) homeomorphisms of σ which take some n fixed points interior to Σ to themselves (possibly permuting them), and fixed on $\partial \Sigma$ if the boundary is nonempty, modulo isotopy fixed on the boundary and the n given points. This is independent of the choice of the n points, and one refers to $MCG(\Sigma, n)$ as the n-string braid group on Σ.

Definition 5: B_n as a group of automorphisms. A mapping class $[h]$, where $h : D_n \to D_n$ gives rise to an automorphism $h_* : F_n \to F_n$ of free groups. Using the interpretation of braids as mapping classes, this defines a homomorphism

$$B_n \to Aut(F_n),$$

which Artin showed to be faithful, i. e. injective.

The generator σ_i acts as

$$x_i \to x_i x_{i+1} x_i^{-1}$$
$$x_{i+1} \to x_i$$
$$x_j \to x_j, j \neq i, i+1. \tag{1}$$

Theorem 2.1 (Artin) *Under the identification described above, B_n is the set of automorphisms $h \in Aut(F_n)$ of the form*

$$h(x_i) = w_j^{-1} x_j w_j,$$

where w_j are words in F_n, and satisfying $h(x_1 \cdots x_n) = x_1 \cdots x_n$.

This point of view gives further insight into the group-theoretic properties of braid groups. A group G is *residually finite* if for every $g \in G$ there is a homomorphism $h : G \to F$ onto a finite group F such that $h(g)$ is not the identity. In other words, any element other than the identity can be proved nontrivial by looking at some homomorphism of G to a finite group.

Exercise 2.2 *Show that subgroups of residually finite groups are residually finite.*

Exercise 2.3 *Show that the automorphism group of any finitely generated residually finite group is itself residually finite. [This is a result of Baumslag. You may want to consult [21] for a proof, and further discussion on the subject.]*

Exercise 2.4 *Show that finitely-generated abelian groups are residually finite. The matrices $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ generate a free group in the modular group $SL(2, \mathbb{Z})$ which is the set of automorphisms of $\mathbb{Z}^2$. Thus $SL(2, \mathbb{Z})$ is residually finite and so is the rank 2 free group. Conclude that a free group of any finite or countable rank is residually finite.*

From these observations, we see that $Aut(F_n)$ is residually finite and conclude the following.

Theorem 2.5 B_n *is residually finite.*

A group G is said to be Hopfian if it is not isomorphic with any nontrivial quotient group. In other words, if $G \to H$ is any surjective homomorphism with nontrivial kernel, the $H \not\cong G$.t

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Exercise 2.6 *A finitely-generated residually finite group is Hopfian.*

It follows immediately that

Theorem 2.7 B_n *is Hopfian.*

The modular group. There is an interesting connection between the braid group B_3 and two-by-two integral matrices. Consider the matrices

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

One can easily check that $S^2 = -I = (ST)^3$. These matrices generate cyclic subgroups of $SL(2,\mathbb{Z})$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$, respectively (we use the abbrviation $\mathbb{Z}_n = \mathbb{Z}/(n\mathbb{Z})$). It is well-known that, using the generators S and ST, there is an amalgamated free product structure

$$SL(2,\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6.$$

The *modular group* $PSL(2,\mathbb{Z})$ is the quotient of $SL(2,\mathbb{Z})$ by its center $\{\pm I\}$. One can also regard $PSL(2,\mathbb{Z})$ as the group of Möbius transformations of the complex plane

$$z \mapsto \frac{az+b}{cz+d} \quad \text{corresponding to} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z}).$$

The modular group has the structure of a free product $\mathbb{Z}_2 * \mathbb{Z}_3$, with (the cosets of) S and ST generating the respective factors.

Exercise 2.8 *Show that the braid group* $B_3 \cong \langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ *also has the presentation* $\langle x, y | x^2 = y^3 \rangle$*, by finding appropriate expressions of* x, y *in terms of* σ_1, σ_2*, and vice-versa, so that the transformations respect the relations and are mutual inverses.*

Verify that the element $x^2 (= y^3)$ *is central in* B_3 *and in fact generates the center of* B_3*.*

Show that B_3 *modulo its center is isomorphic with* $PSL(2,\mathbb{Z})$*.*

Definition 6: B_n as a fundamental group. In complex n-space $\mathbb{C}^n$ consider the big diagonal

$$\Delta = \{(z_1, \ldots, z_n); \quad z_i = z_j, \quad \text{some} \quad i < j\} \subset \mathbb{C}^n.$$

Using the basepoint $(1, 2, \ldots, n)$, we see that

$$P_n = \pi_1(\mathbb{C}^n \setminus \Delta).$$

In other words, pure braid groups are fundamental groups of complements of a special sort of complex *hyperplane arrangement*, itself a deep and complicated subject.

To get the full braid group we need to take the fundamental group of the *configuration space*, of orbits of the obvious action of Σ_n upon $\mathbb{C}^n \setminus \Delta$. Thus

$$B_n = \pi_1((\mathbb{C}^n \setminus \Delta)/\Sigma_n).$$

Notice that since the singularities have been removed, the projection

$$\mathbb{C}^n \setminus \Delta \longrightarrow (\mathbb{C}^n \setminus \Delta)/\Sigma_n$$

is actually a covering map. As is well-known, covering maps induce injective homomorphisms at the π_1 level, so this is another way to think of the inclusion $P_n \subset B_n$.

It was observed in [14] that $\mathbb{C}^n \setminus \Delta$ has trivial homotopy groups in dimension greater than one. That is, it is an Eilenberg-Maclane space, also known as a $K(P_n, 1)$. Therefore its cohomology groups coincide with the group cohomology of P_n. By covering theory, the quotient space $(\mathbb{C}^n \setminus \Delta)/\Sigma_n$ also has trivial higher homotopy, so it is a $K(B_n, 1)$. Since these spaces have real dimension $2n$, this view of braid groups gives us the following observation.

Theorem 2.9 *The groups B_n and P_n have finite cohomological dimension.*

If a group contains an element of finite order, standard homological algebra implies that the cohomological dimension of the group must be infinite. Thus there are no braids of finite order.

Corollary 2.10 *The braid groups are torsion-free.*

Finally, we note that the space $(\mathbb{C}^n \setminus \Delta)/\Sigma_n$ can be identified with the space of all complex polynomials of degree n which are monic and have n distinct roots

$$p(z) = (z - r_1) \cdots (z - r_n).$$

This is one way in which the braid groups play a role in classical algebraic geometry, as fundamental groups of such spaces of polynomials.

Definition 7: B_n is the fundamental group of the space of all monic polynomials of degree n with complex coefficients and only simple roots.

3   Knot theory, braids and the Jones polynomial

Knots and Reidemeister moves

One often pictures a knot or link by drawing a projection onto the plane, with only double points, and indicating which string goes under by putting a small gap in it, as in Figure 1. If one deforms the plane by a planar isotopy, an equivalent projection results. One can also make local changes to a knot diagram. The first two vignettes in Figure 2 are sometimes called Reidemeister moves of type 2 and 3 (respectively). One can remember the numbering, as type 2 involves two strands and type 3 involves three. There is also a type 1 Reidemeister move.
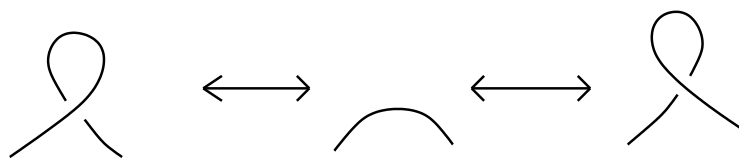
Reidemeister move of type 1

Markov moves

It has already been mentioned that different braids can, upon forming the closure, give rise to the same knot or link. You can easily convince yourself that if α and β are n-string braids, then the closure of $\alpha^{-1}\beta\alpha$ is equivalent to the closure of β. Because of the trivial strings added in forming the closure, α and its inverse can annihilate each other! Therefore we see that conjugate braids close to the same link. Another example of this phenomenon is to consider the n-braid β as an element of B_{n+1} by adding a trivial string at the top, and compare the original closure $\hat{\beta}$ with the closure of $\beta \times \sigma_n$, the latter braid taken in B_{n+1}. If you sketch this, you can easily convince yourself that these result in the same link. The first move (conjugation) and the second move just described constitute the two Markov moves. They are the key to understanding the connection between braid theory and knot theory. The following theorem is due to J. W. Alexander (first part) and A. A. Markov. A full proof appeared first, to my knowledge, in [6].

Theorem 3.1 *Every knot or link is the closure of some braid. Two braids close to equivalent knots or links if and only if they are related by a finite sequence of the two Markov moves.*

Exercise 3.2 *If you want to learn a nice, elementary proof of the above theorem, read the paper [24].*

Exercise 3.3 *Consider the 3-braid $\beta = \sigma_1\sigma_2$ and identify the knot or link, simplifying, if possible by using Reidemeister moves: $\hat{\beta}$, $\hat{\beta^2}$, $\hat{\beta^3}$. For which integers k is the closure of β^k a knot?*
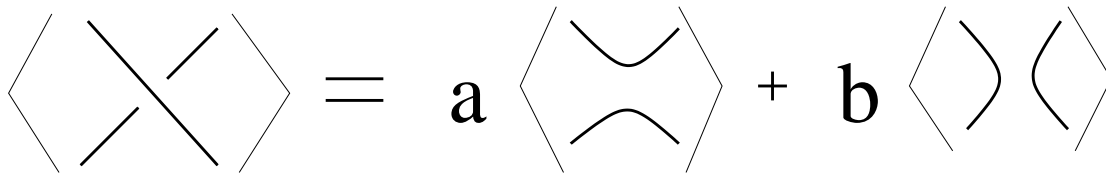
Kauffman's bracket and Jones' polynomial

The original construction of the Jones polynomial involved a family of representations
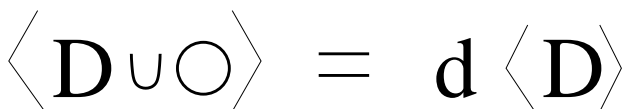
$$B_n \to \mathcal{A}$$

of the braid groups into an algebra $\mathcal{A}$ involving a parameter t. This algebra has a linear trace function into the ring of (Laurent) polynomials. The composite of the representation and the trace, together with some correction terms to account for the Markov moves, defined the original Jones polynomial.

We will present this in a sort of reverse order, as there is a very simple derivation of the Jones polynomial discovered a few years later by L. Kauffman. From this, we can define an algebra, called the Temperley-Lieb algebra, and reconstruct what is essentially Jones' representation.

Consider a planar diagram D of a link L, which has only simple transverse crossings. Kauffman's bracket $\langle D \rangle$ is (at first) a polynomial in three commuting variables, a, b and d defined by the equations:

Equations defining Kauffman's bracket polynomial.

Some explanation is in order. The vignettes in the brackets in the first equation stand for complete link diagrams, which differ only near the crossing in question. Those on the right side represent diagrams with fewer crossings. In the second equation, one can introduce a closed curve which has no intersections with the remainder of the diagram D, resulting in a diagram whose bracket polynomial is d times the bracket of D.

Exercise 3.4 *Verify that the bracket polynomial is well-defined, if we decree that the bracket of a single curve with no crossings is equal to 1.*

Exercise 3.5 *Show that the bracket will be invariant under the type 2 Reidemeister move, if we have the following relations among the variables:*

$$a^2 + b^2 + abd = 0 \quad \text{and} \quad ab = 1.$$

Thus we make the substitutions $b = a^{-1}$ and $d = -a^2 - a^{-2}$ and now consider the bracket to be a Laurent polynomial in the single variable a.

Exercise 3.6 *Show that the invariance under the type 2 Reidemeister moves implies the bracket is invariant under the type 3 move, too.*

Exercise 3.7 *Calculate the bracket of the two trefoil knot diagrams with 3 crossings. Show that they are the same, except for reversal of the sign of the exponents.*

Exercise 3.8 *Investigate the effect of Reidemeister move 1 on the bracket of a diagram, and show that it changes the bracket by a factor of $-a^{\pm 3}$, the sign of the exponent depending on the sense of the curl removed.*

Because of this, one can define a polynomial invariant under all three Reidemeister moves, by counting the number of positive minus the number of negative crossings, and modifying the bracket polynomial by an appropriate factor. A positive crossing corresponds to a (positive) braid generator, if both strings are oriented from left to right. This gives, up to change of variable, the Jones polynomial of the knot. Specifically, we define the writhe of an *oriented* diagram $\vec{D}$ for an oriented knot (or link) $\vec{K}$ to be

$$w(D) = \sum_c \epsilon_c$$

where the sum is over all crossings and $\epsilon_c = 1$ if the crossing c is positive, and -1 if negative. Then we define

$$f_{\vec{K}}(a) = (-a^3)^{-w(\vec{D})} \langle D \rangle.$$

This is an invariant of the oriented link $\vec{K}$. If K happens to be a knot, it is independent of the orientation, as reorienting both strands of a crossing does not change its sign. It is related to the Jones polynomial $V_K(t)$ by a simple change of variables:

$$V_{\vec{K}}(t) = f_{\vec{K}}(t^{-1/4}).$$

Exercise 3.9 *Show that all exponents of a in $\langle D \rangle$ are divisible by 4 if D is a diagram of a knot (or link with an odd number of components). In other cases, they are congruent to 2 mod 4. Thus the Jones polynomial is truly a (Laurent) polynomial in t for knots and odd component links, but a polynomial in $\sqrt{t}$ in the other cases.*

Representations

This is a very big subject, which I will just touch upon. By a representation of a group we will mean a homomorphism of the group into a group of matrices, or more generally into some other group, or ring or algebra. Often, but not always, we want the target to be finite-dimensional. We've already encountered the Artin representation $B_n \to Aut(F_n)$, which is faithful. Here the target group is far from being "finite-dimensional."

Another very important representation is the one defined by Jones [16] which gave rise to his famous knot polynomial, and the subsequent revolution in knot theory. The version I will discuss is more thoroughly described in [17]; it is based on the Kauffman bracket, an elementary combinatorial approach to the Jones polynomial. First we need to describe the Temperley-Lieb Algebras $\mathcal{T}_n$, in their geometric form. The elements of $\mathcal{T}_n$ are something like braids: we consider strings in a box, visualized as a square in the plane, endpoints being exactly n specified points on each of the left and right sides. The strings are not required to be monotone, or even to run across from one side to the other. There also may be closed components. Really what we are looking at are "tangle" diagrams. Two tangle diagrams are considered equal if there is a planar isotopy, fixed on the boundary of the square, taking one to the other.
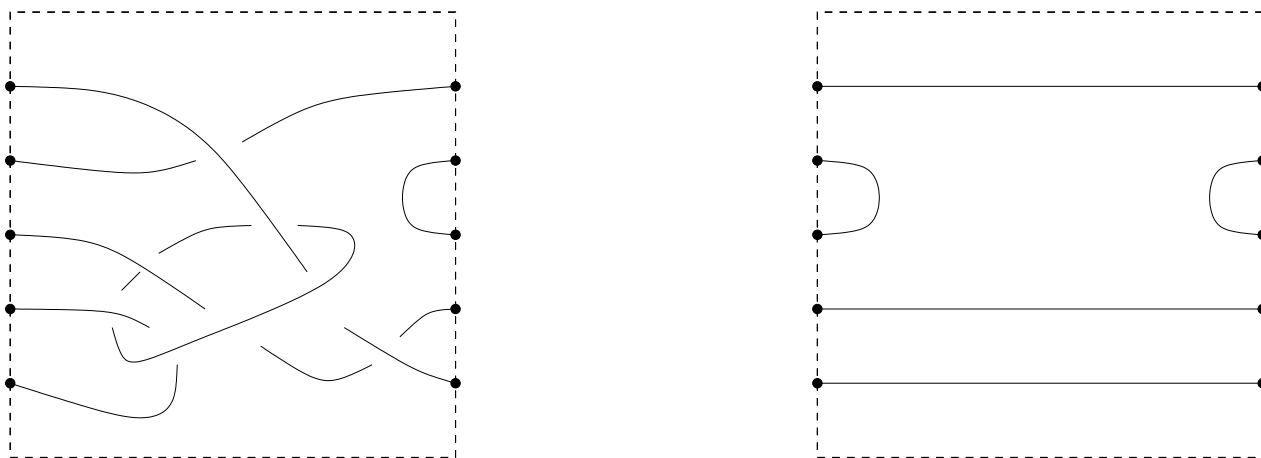
A typical element in $\mathcal{T}_5$ and the generator e_3.

Now we let A be a fixed complex number (regarded as a parameter), and formally define $\mathcal{T}_n$ to be the complex vector space with basis the set of all tangles, as described above, but modulo the following relations, which correspond to similar relations used to define Kauffman's bracket version of the Jones polynomial (we have promoted the variable a to upper case).

$$\times \;=\; A \,\asymp\; +\; A^{-1} \,)($$

$$K \cup \bigcirc \;=\; d\,K, \qquad d = -A^{2} - A^{-2}$$

Relations in $\mathcal{T}_n$.

The first relation means that we can replace a tangle with a crossing by a linear combination of two tangles with that crossing removed in two ways. As usual, the pictures mean that the tangles are identical outside the part pictured. The second relation means that we can remove any closed curve in the diagram, if it does not have any crossings with the rest of the tangle, at the cost of multiplying the tangle by the scalar $-A^2 - A^{-2}$. Using the relations, we see that any element of $\mathcal{T}_n$ can be expressed as a linear combination of tangles which have no crossings and no closed curves – that is, disjoint planar arcs connecting the $2n$ points of the boundary. This gives a finite generating set, which (for generic values of A) can be shown to be a basis for $\mathcal{T}_n$ as a vector space. But there is also a multiplication of $\mathcal{T}_n$, a concatenation of tangles, in exactly the same way braids are multiplied. This enables us to consider $\mathcal{T}_n$ to be generated *as an algebra* by the elements $e_1, \ldots, e_{n-1}$. In e_i all the strings go straight across, except those at level i and $i+1$ which are connected by short caps; the generator e_3 of $\mathcal{T}_5$ is illustrated in Figure 2. The identity of this algebra is simply the diagram consisting of n horizontal lines (just like the identity braid). $\mathcal{T}_n$ can be described abstractly as the associative algebra with the generators $e_1, \ldots, e_{n-1}$, subject to the relations:

$$e_i e_j = e_j e_i \text{ if } |i - j| > 1 \qquad e_i e_{i \pm 1} e_i = e_i \qquad e_i^2 = (-A^2 - A^{-2}) e_i$$

It is an enjoyable exercise to verify these relations from the pictures. Now the Jones representation $J : B_n \to \mathcal{T}_n$ can be described simply by considering a braid diagram as an element of the algebra. In terms of generators, this is just

$$J(\sigma_i) = A + A^{-1} e_i$$

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The Burau representation

One of the classical representations of the braid groups is the Burau representation, which can be described as follows. Consider the definition of B_n as the mapping class group of the punctured disk D_n (Definition 5). As already noted, the fundamental group of D_n is a free group, with generator x_i represented by a loop, based at a point on the boundary of the disk, which goes once around the ith puncture. Consider the subgroup of $\pi_1(D_n)$ consisting of all words in the x_i whose exponent sum is zero. This is a normal subgroup, and so defines a regular covering space $\bar{D}_n \to D_n$. The group of covering translations is infinite cyclic. Therefore, the homology $H_1(\bar{D}_n)$ can be considered as a module over the polynomial ring $Z[t, t^{-1}]$, where t represents the generator of the covering translation group. A braid β can be represented as (an isotopy class of) a homeomorphism $\beta : D_n \to D_n$ fixing the basepoint. This lifts to a homeomorphism $\bar{\beta} : \bar{D}_n \to \bar{D}_n$, which is unique if we insist that it fix some particular lift of the basepoint. The induced homomorphism on homology, $\bar{\beta}_* : H_1(\bar{D}_n) \to H_1(\bar{D}_n)$ is a linear map of these finite-dimensional modules, and so can be represented by a matrix with entries in $\mathbb{Z}[t, t^{-1}]$. The mapping

$$\beta \to \bar{\beta}_*$$

is the Burau representation of B_n.

Exercise 3.10 *Show that, with appropriate choice of basis, the Burau representation sends σ_j to the matrix*

$$I_{j-1} \oplus \begin{bmatrix} 1-t & 1 \\ 1 & 0 \end{bmatrix} \oplus I_{n-j-1},$$

where I_k denotes the $k \times k$ identity matrix.

It has been known for many years that this representation is faithful for $n \le 3$, and it is only within the last decade that it was found to be unfaithful for any n at all. John Moody showed in 1993 [23] that it is unfaithful for $n \ge 9$. This has since been improved by Long and Paton [20] and very recently by Bigelow to $n \ge 5$. The case $n = 4$ remains open, as far as I am aware.

Linearity of the braid groups.

It has long been questioned, whether the braid groups are *linear*, meaning that there is a faithful representation $B_n \to GL(V)$ for some finite dimensional vector space V. A candidate had been the so-called Burau representation, but as already mentioned it has been known for several years ([23], [20]) that Burau is unfaithful, in general. Just recently, Bachmuth [3] published an argument that the Gassner representation (of pure braid groups) is faithful, from which it would follow that the full braid groups

are also linear. Unfortunately, his arguments had serious gaps, which were pointed out in the review [7].

The question was finally settled recently by Daan Krammer and Stephen Bigelow, using equivalent representations, but different methods. They use a representation defined very much like the Burau representation. But instead of a covering of the punctured disk D_n, they use a covering of the *configuration space* of pairs of points of D_n, upon which B_n also acts. This action induces a linear representation in the homology of an appropriate covering, and provides just enough extra information to give a faithful representation!

Theorem 3.11 (Krammer [19], Bigelow [5]) *The braid groups are linear.*

In fact, Bigelow has announced that the BMW representation (Birman, Murakami, Wenzl) [8] is also faithful. Another open question is whether the Jones representation $J : B_n \rightarrow T_n$, discussed earlier, is faithful.

4 Ordering braid groups

This is, to me, one of the most exciting of the recent developments in braid theory. Call a group G *right orderable* if its elements can be given a strict total ordering $<$ which is right-invariant:

$$\forall x, y, z \in G, \quad x < y \Rightarrow xz < yz.$$

Theorem 4.1 (Dehornoy[10]) B_n *is right-orderable.*

Interestingly, I know of three quite different proofs. The first is Dehornoy's, the second is one that was discovered jointly by myself and four other topologists. We were trying to understand difficult technical aspects of Dehornoy's argument, then came up with quite a different way of looking at exactly the same ordering, but using the view of B_n as the mapping class group of the punctured disk D_n. Yet a third way is due to Thurston, using the fact that the universal cover of D_n embeds in the hyperbolic plane. Here are further details.

Dehornoy's approach: It is routine to verify that a group G is right-orderable if and only if there exists a subset Π (positive cone) of G satisfying:

(1) $\Pi \cdot \Pi = \Pi$

(2) The identity element does not belong to Π, and for every $g \neq 1$ in G exactly one of $g \in \Pi$ or $g^{-1} \in \Pi$ holds.

One defines the ordering by $g < h$ iff $hg^{-1} \in \Pi$.

Exercise 4.2 *Verify that the transitivity law holds, and that the ordering is right-invariant. Show that the ordering defined by this rule is also left-invariant if and only if $g\Pi g^{-1} = \Pi$ for every $g \in G$; that is, Π is "normal."*

Dehornoy's idea is to call a braid i-positive if it is expressible as a word in $\sigma_j, j \geq i$ in such a manner that all the exponents of σ_i are positive. Then define the set $\Pi \subset B_n$ to be all braids which are i-positive for some $i = 1, \ldots, n - 1$. To prove (1) above is quite easy, but (2) requires an extremely tricky argument.

Here is the point of view advocated in [13]. Consider B_n as acting on the complex plane, as described above. Our idea is to consider the image of the real axis $\beta(\mathbb{R})$, under a mapping class $\beta \in B_n$. Of course there are choices here, but there is a unique "canonical form" in which (roughly speaking) $\mathbb{R} \cap \beta(\mathbb{R})$ has the fewest number of components. Now declare a braid β to be positive if (going from left to right) the first departure of the canonical curve $\beta(\mathbb{R})$ from $\mathbb{R}$ itself is into the *upper* half of the complex plane. Amazingly, this simple idea works, and gives exactly the same ordering as Dehornoy's combinatorial definition.

Finally, Thurston's idea for ordering B_n again uses the mapping class point of view, but a different way at looking at ordering a group. This approach, which has the advantage of defining infinitely many right-orderings of B_n is described by H. Short and B. Wiest in [28]. The Dehornoy ordering (which is discrete) occurs as one of these right-orderings – others constructed in this way are order-dense. A group G acts on a set X (on the right) if the mapping $x \to xg$ satisfies: $x(gh) = (xg)h$ and $x1 = x$. An action is *effective* if the only element of G which acts as the identity is the identity $1 \in G$. The following is a useful criterion for right-orderability:

Lemma 4.3 *If the group G acts effectively on $\mathbb{R}$ by order-preserving homeomorphisms, then G is right-orderable.*

By way of a proof, consider a well-ordering of the real numbers. Define, for g and $h \in G$,

$$g < h \Leftrightarrow xg < xh \quad \text{at the first } x \in \mathbb{R} \text{ such that } xg \neq xh.$$

It is routine to verify that this defines a right-invariant strict total ordering of G. (By the way, we could have used any ordered set in place of $\mathbb{R}$.) For those wishing to avoid the axiom of choice (well-ordering $\mathbb{R}$) we could have just used an ordering of the rational numbers.

The universal cover $\tilde{D}_n$ of D_n can be embedded in the hyperbolic plane $\mathbb{H}^2$ in such a way that the covering translations are isometries. This gives a hyperbolic structure on D_n. It also gives a beautiful tiling of $\mathbb{H}^2$, illustrated in Figure 4 for the case $n = 2$.
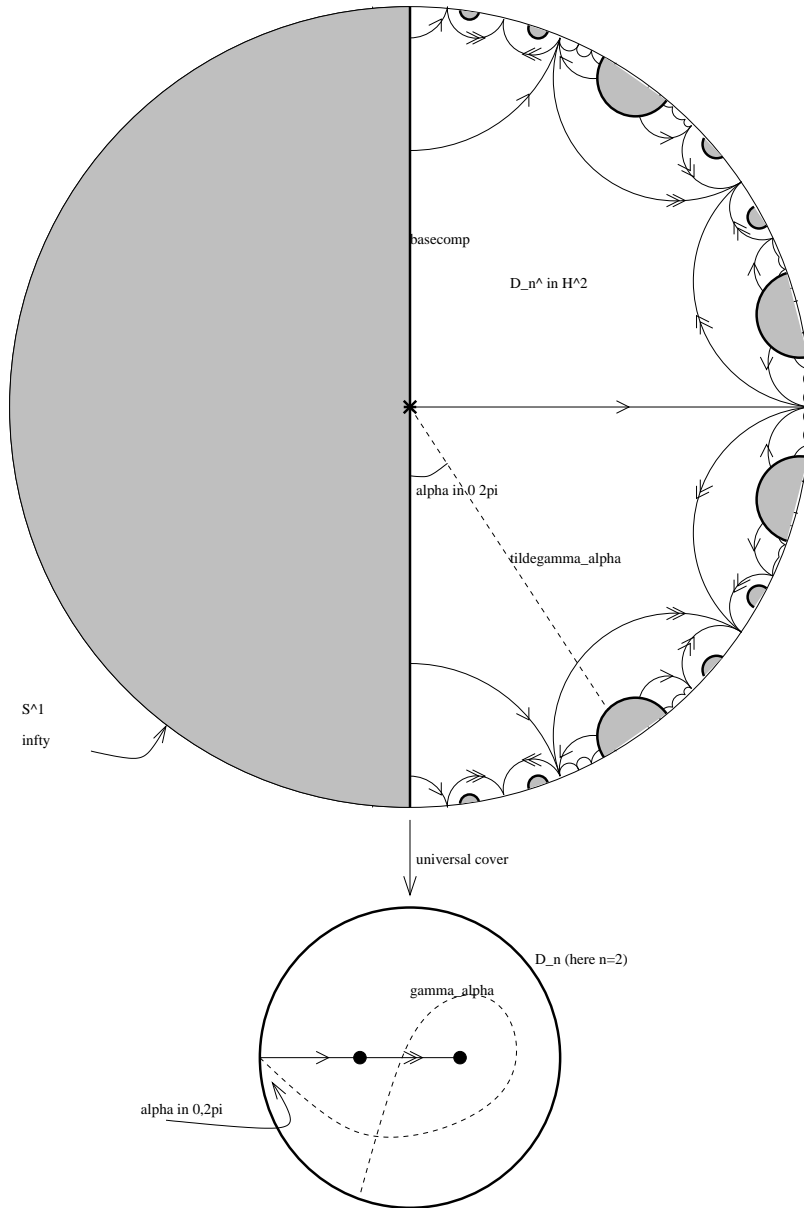
The universal cover of a twice-punctured disk, with a lifted geodesic. (Courtesy of H. Short and B. Wiest [28]

Choose a basepoint $* \in \partial D$ and a specific lift $\tilde{*} \in \mathbb{H}^2$. Now a braid is represented by a homeomorphism of D_n, which fixes $*$. This homeomorphism lifts to a homeomorphism of $\tilde{D}_n$, unique if we specify that it fixes $\tilde{*}$. In turn, this homeomorphism extends to a homeomorphism of the boundary of $\tilde{D}_n$ But in fact, this homeomorphism fixes the interval of $\partial \tilde{D}_n$ containing $\tilde{*}$, and if we identify the complement of this interval with the real line $\mathbb{R}$, a braid defines a homeomorphism of $\mathbb{R}$. This defines an action of B_n upon $\mathbb{R}$ by order-preserving homeomorphisms, and hence a right-invariant ordering of B_n.

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Two-sided invariance? Any right-invariant ordering of a group can be converted to a left-invariant ordering, by comparing inverses of elements, but that ordering is in general different from the given one. We will say that a group G with strict total ordering $<$ is fully-ordered, or bi-ordered, if

$$x < y \Rightarrow xz < yz \text{ and } zx < zy, \forall x, y, z \in G.$$

There are groups which are right-orderable but not bi-orderable – in fact the braid groups!

Proposition 4.4 (N. Smythe) *For $n > 2$ the braid group B_n cannot be bi-ordered.*

The reason for this is that there exists a nontrivial element which is conjugate to its inverse: take $x = \sigma_1\sigma_2^{-1}$ and $y = \sigma_1\sigma_2\sigma_1$ and note that $yxy^{-1} = x^{-1}$. In a bi-ordered group, if $1 < x$ then $1 < yxy^{-1} = x^{-1}$, contradicting the other conclusion $x^{-1} < 1$. If $x < 1$ a similar contradiction arises.

Exercise 4.5 *Show that in a bi-ordered group $g < h$ and $g' < h'$ imply $gg' < hh'$. Conclude that if $g^n = h^n$ for some $n \neq 0$, then $g = h$. That is, roots are unique. Use this to give an alternative proof that B_n is not bi-orderable if $n \geq 3$.*

Theorem 4.6 *The pure braid groups P_n can be bi-ordered.*

This theorem was first noticed by J. Zhu, and the argument appears in [26], based on the result of Falk and Randall [15] that the pure braid groups are "residually torsion-free nilpotent." Later, in joint work with Djun Kim, we discovered a really natural, and I think beautiful, way to define a bi-invariant ordering of P_n. We've already done half the work, by discussing Artin combing. Now we need to discuss ordering of free groups.

Bi-ordering free groups

Lemma 4.7 *For each $n \geq 1$, the free group F_n has a bi-invariant ordering $<$ with the further property that it is invariant under any automorphism $\phi : F_n \to F_n$ which induces the identity upon abelianization: $\phi_{ab} = id : \mathbb{Z}^n \to \mathbb{Z}^n$.*

The construction depends on the Magnus expansion of free groups into rings of formal power series. Let F be a free group with free generators $x_1, \ldots, x_n$. Let $Z[[X_1, \ldots, X_n]]$ denote the ring of formal power series in the *non-commuting* variables $X_1, \ldots, X_n$. Each term in a formal power series has a well-defined (total) degree, and

we use $O(d)$ to denote terms of degree $\geq d$. The subset $\{1 + O(1)\}$ is actually a multiplicative subgroup of $Z[[X_1, \ldots, X_n]]$. Moreover, there is an embedding

$$\mu : F \to Z[[X_1, \ldots, X_n]]$$

defined by

$$\mu(x_i) = 1 + X_i$$
$$\mu(x_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots$$

$$(2)$$

There is a very nice proof that μ is injective in [21], as well as discussion of some if its properties. One such property is that commutators have zero linear terms. For example (dropping the μ)

$$[x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1}$$
$$= (1 + X_1)(1 + X_2)(1 - X_1 + X_1^2 - \cdots)(1 - X_2 + X_2^2 - \cdots)$$
$$= 1 + X_1 X_2 - X_2 X_1 + O(3).$$

$$(3)$$

Now there is a fairly obvious ordering of $Z[[X_1, \ldots, X_n]]$. Write a power series in ascending degree, and within each degree list the monomials lexicographically according to subscripts). Given two series, order them according to the coefficient of the first term (when written in the standard form just described) at which they differ. Thus, for example, 1 and $[x_1, x_2]$ first differ at the $X_1 X_2$ term, and we see that $1 < [x_1, x_2]$. It is not difficult to verify that this ordering, restricted to the group $\{1 + O(1)\}$ is invariant under both left- and right-multiplication.

Exercise 4.8 *Write x_1, $x_2 x_1 x_2^{-1}$ and $x_2^{-1} x_1 x_2$ in increasing order, according to the ordering just described.*

Now we define an ordering of P_n. If α and β are pure braids, compare their Artin coordinates (as described earlier)

$$(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \text{ and } (\beta_1, \beta_2, \ldots, \beta_{n-1})$$

lexicographically, using within each F_k the Magnus ordering described above. This all needs choices of conventions, for example, for generators of the free groups, described in detail in [18]. The crucial fact is that the action associated with the semidirect product, by automorphisms φ, has the property mentioned (Added in proof: these results appear in [19] and [5]). in Lemma 4.5. We recall the definition of a positive braid according to Garside: a braid is Garside-positive if it can be expressed as a word in the standard generators σ_i with only positive exponents.

Theorem 4.9 (Kim-Rolfsen) *P_n has a bi-ordering with the property that Garside-positive pure braids are greater than the identity, and the set of all Garside-positive pure braids is well-ordered by the ordering.*

Algebraic consequences: The orderability of the braid groups has implications beyond what we already knew – e. g. that they are torsion-free. In the theory of representations of a group G, it is important to understand the group algebra $\mathbb{C}G$ and the group ring $\mathbb{Z}G$. These rings also play a role in the theory of Vassiliev invariants. A basic property of a ring would be whether it has (nontrivial) zero divisors.

Exercise 4.10 *If R is a ring without zero divisors, and G is a right-orderable group, then the group ring RG has no zero divisors. Moreover, the only units of RG are the monomials rg, $g \in G, r$ a unit of R.*

Proposition 4.11 (Malcev, Neumann) *If G has a bi-invariant ordering, then its group ring $\mathbb{Z}G$ embeds in a division algebra, that is, an extension in which all nonzero elements have inverses.*

These results give us new information about the group rings of the braid groups.

Theorem 4.12 *$\mathbb{Z}B_n$ has no zero divisors. Moreover, $\mathbb{Z}P_n$ embeds in a division algebra.*

A proof of the theorem of Malcev and Neumann [22] can be found in [25].

Exercise 4.13 *Which subgroups of $Aut(F_n)$ are right-orderable?*

Of course, $Aut(F_n)$ itself is not right-orderable, because it has elements of finite order, e.g. permuting the generators.

A final note regarding orderings: As we've seen, the methods we've used for ordering B_n and P_n are quite different. One might hope there could be compatible orderings: a bi-ordering of P_n which extends to a right-invariant ordering of B_n. But, in recent work with Akbar Rhemtulla, we showed this is hopeless!

Theorem 4.14 (Rhemtulla, Rolfsen). *For $n \geq 5$, there is no right-invariant ordering of B_n, which, upon restriction to P_n, is also left-invariant.*

References

[1] E. Artin, Theorie der Zöpfe, *Abh. Math. Sem. Hamburg. Univ.* 4(1926), 47 – 72.

[2] E. Artin, Theory of braids, *Annals of Math. (2)* 48(1947), 101 – 126.

[3] S. Bachmuth, Braid groups are linear groups, *Adv. Math.* 121(1996), 50 – 61.

[4] G. Baumslag, Automorphism groups of residually finite groups, *J. London Math. Soc.* 38(1963), 117–118.

[5] Bigelow, Stephen J. Braid groups are linear. J. Amer. Math. Soc. 14 (2001), no. 2, 471–486 (electronic).

[6] J. Birman, *Braids, links and mapping class groups*, Annals of Math. Studies 82, Princeton Univ. Press, 1974.

[7] J. Birman, review of [Ba], *Math. Reviews* 98h:20061.

[8] J. Birman and H. Wenzl, Braids, link polynomials and a new algebra, *Trans. Amer. Math. Soc.* 313(1989), 249 – 273.

[9] G. Burde and H. Zieschang, *Knots*, de Gruyter, 1985.

[10] P. Dehornoy, Braid groups and left distributive operations, *Trans Amer. Math. Soc.* 345(1994), 115 – 150.

[11] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, *Word processing in groups,* Jones and Bartlett, 1992.

[12] R. Fenn, An elementary introduction to the theory of braids, notes by B. Gemein, 1999, available at the author's website.

[13] R. Fenn, M. Green, D. Rolfsen, C. Rourke, B. Wiest, Ordering the braid groups, *Pacific J. Math.* 191(1999), 49–74.

[14] E. Fadell and L. Neuwirth, Configuration spaces, *Math. Scand.* 10(1962), 111 – 118.

[15] M. Falk and R. Randell, Pure braid groups and products of free groups, *Braids (Santa Cruz 1986), Contemp. Math. 78* 217–228.

[16] V. F. R. Jones, A polynomial invariant for knots via Von Neumann algebras, *Bull. Amer. Math. Soc.* 12(1985), 103–111.

[17] L. Kauffman, *Knots and Physics*, World Scientific, 1993.

[18] D. Kim and D. Rolfsen, An ordering for groups of pure braids and hyperplane arrangements, to appear, Can. Math. Bulletin.

[19] D. Krammer, The braid group B_4 is linear. Invent. Math. 142 (2000), no. 3, 451–486.

[20] D. Long and M. Paton, The Burau representation is not faithful for $n \geq 6$, *Topology* 32(1993), 439 – 447.

[21] W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Wiley 1966.

[22] A. I. Mal'cev, *On the embedding of group algebras in division algebras,* Dokl. Akad. Nauk SSSR 60 (1948) 1944-1501.

[23] J. Moody, The faithfulness question for the Burau representation, *Proc. Amer. Math. Soc.* 119(1993), 439 – 447.

[24] H. R. Morton, Threading knot diagrams. *Math. Proc. Cambridge Philos. Soc.* 99 (1986), no. 2, 247–260.

[25] R. Mura and A. Rhemtulla, *Orderable groups*, Lecture Notes in Pure and Applied Mathematics, vol. 27, Marcel Dekker, New York, 1977.

[26] D. Rolfsen, J. Zhu, Braids, orderings and zero divisors, *J. Knot Theory and its Ramifications* 7(1998), 837-841.

[27] D. S. Passman, *The algebraic structure of group rings*, Pure and Applied Mathematics, Wiley Interscience, 1977.

[28] H. Short and B. Wiest, Orderings of mapping class groups after Thurston, to appear, *l'Enseignement Math.*