Introductory Notes in Topology

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May 29, 2007

Contents

1 Definition of Categories and Functors. Some Constructions.

Definition 1.1. A **category** $\mathcal{C}$ consists of a class of objects, denoted by $\mathrm{Ob}(\mathcal{C})$, such that for each pair A, B of objects in $\mathrm{Ob}(\mathcal{C})$, there is a set (possibly empty), denoted by $\mathrm{Mor}(A, B)$, called the set of **morphisms** from A to B. An morphism $f \in \mathrm{Mor}(A, B)$ is written as $f : A \to B$. The following properties are also satisfied.

1. The sets of morphisms are pairwise disjoint, namely, if either $A \neq X$ or $B \neq Y$ for objects $A, X, B, Y \in \mathrm{Ob}(\mathcal{C})$, then $\mathrm{Mor}(A, B) \cap \mathrm{Mor}(X, Y) = \emptyset$.
2. For all objects $C \in \mathrm{Ob}(\mathcal{C})$, there is a morphism in $\mathrm{Mor}(C, C)$ called the **identity morphism on** C, which is denoted by id_C.
3. For all morphisms $\alpha : A \to B$ and $\beta : B \to C$, we can associate a **unique** morphism $\beta \circ \alpha : A \to C$. The morphism $\beta \circ \alpha$ is called the **composite** of α and β.
4. If $\alpha : A \to B$, $\beta : B \to C$ and $\gamma : C \to D$ are morphisms, then **associativity** holds:
$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$
5. For all morphisms $\alpha : A \to B$ and $\beta : B \to C$, we have $\mathrm{id}_B \circ \alpha = \alpha$ and $\beta \circ \mathrm{id}_B = \beta$.

If $\alpha : A \to B$, it is customary to say that the **domain** of α is A, and the **codomain** of α in B; each morphism has a unique domain and codomain since all the sets $\mathrm{Mor}(A, B)$ are pairwise disjoint. In a category $\mathcal{C}$, a morphism $\alpha : A \to B$ is known as an **isomorphism** if in $\mathcal{C}$ there is a morphism $\beta : B \to A$ such that $\beta \circ \alpha = \mathrm{id}_A$ and $\alpha \circ \beta = \mathrm{id}_B$ (in which case $\beta : B \to A$ is also an isomorphism). If there is an isomorphism $\alpha : A \to B$, then we say that the objects A, B are **isomorphic**.

The identity morphism is always unique for any object C because if id_C and id'_C are two identity morphisms on C, then by Definition 1.1(5), we have $\mathrm{id}_C \circ \mathrm{id}'_C = \mathrm{id}'_C$ since id_C is identity, and also $\mathrm{id}_C \circ \mathrm{id}'_C = \mathrm{id}_C$ since id'_C is identity.

Lemma 1.2. *Let $\mathcal{C}$ be a category.*

1. *For all objects $C \in \mathrm{Ob}(\mathcal{C})$, the identity morphism $\mathrm{id}_C : C \to C$ is an isomorphism.*
2. *The composite of two isomorphisms is an isomorphism.*
3. *If $\alpha : A \to B$ is an isomorphism, then there is one and only one morphism $\beta : B \to A$ in the category $\mathcal{C}$ such that $\beta \circ \alpha = \mathrm{id}_A$ and $\alpha \circ \beta = \mathrm{id}_B$.*

Examples 1.3.

1. Let Set denote the class of all sets. For every $A, B \in$ Set, let $\mathrm{Mor}(A, B)$ be the set of all functions from A to B. Then Set is easily seen to be a category, in which the composition of morphisms is the same as the composition of functions. This requires a small technical proviso: functions with different codomains must be distinguished even if their actions on a common domain are identical; a function $f : A \to B$ must be distinguished from the function $f : A \to C$ in which C is a strict subset of B that contains $\mathrm{Im}(f)$. In the category Set, the identity morphism on a set is the usual identity function, and a morphism f in Set is an isomorphism if and only if f is a bijective function.
2. Let Grp denote the category whose objects are all groups, and for any groups G, H, $\mathrm{Mor}(G, H)$ is the set of group homomorphisms from G to H (respecting the same technical proviso as in Example 1). A morphism ϕ in Grp is an isomorphism if and only if ϕ is bijective as a function. The category Ab of abelian groups is similarly defined.
3. A particular multiplicative group G can also be made into a category. Let the category have only one object, G itself, and let $\mathrm{Mor}(G, G)$ be the set of elements of G; composition of morphisms a, b is simply the product ab given by the binary operation in G. The group identity 1_G is the identity morphism on G. Every morphim is an isomorphism since every element of G has a unique inverse.

4. Let $\mathcal{C}$ be a category, $\mathrm{Mor}(\mathcal{C})$ be the class of all morphisms of $\mathcal{C}$, and for any pair of morphisms $f : A \to B$ and $g : C \to D$ of $\mathcal{C}$, define $\mathrm{Mor}(f, g)$ to be the set of all ordered pairs (α, β), where $\alpha : A \to C$ and $\beta : B \to D$ are morphisms of $\mathcal{C}$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow[g]{} & D \end{array}$$

   Then the class $\mathrm{Mor}(\mathcal{C})$ together with the sets $\mathrm{Mor}(f, g)$ is a category under this definition.

5. If $\mathcal{C}$ is a category, the category $\mathcal{C}^{\mathbb{Z}}$ is the category whose objects consists of infinite sequences $\{(C_n)\}_{n\in\mathbb{Z}}$ of objects in $\mathcal{C}$ and whose morphisms are collections $\{f_n : C_n \to D_n\}_{n\in\mathbb{Z}}$ of morphisms of $\mathcal{C}$ indexed by $\mathbb{Z}$. We call $\mathcal{C}^{\mathbb{Z}}$ the category of $\mathcal{C}$-objects **graded by** $\mathbb{Z}$.

Definition 1.4. Let $\mathcal{C}$ be a catgeory. The **opposite category** $\mathcal{C}^{\mathrm{op}}$ is defined to be the category whose objects are the objects of $\mathcal{C}$, namely, the class $\mathrm{Ob}(\mathcal{C})$, and in which the following properties hold.

1. For any pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, there is a bijective correspondence between the set of morphisms $\mathrm{Mor}(A, B)$ in $\mathcal{C}$ and the set of morphisms $\mathrm{Mor}(B, A)$ in $\mathcal{C}^{\mathrm{op}}$, such that for all morphisms $\alpha : A \to B$ in the category $\mathcal{C}$, we have a unique morphism $\alpha^{\mathrm{op}} : B \to A$ in $\mathcal{C}^{\mathrm{op}}$.

2. For all objects C, $(\mathrm{id}_C)^{\mathrm{op}} = \mathrm{id}_C$.

3. For all morphisms $\alpha : A \to B$ and $\beta : B \to C$ in $\mathcal{C}$, we have $(\beta \circ \alpha)^{\mathrm{op}} = \alpha^{\mathrm{op}} \circ \beta^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$.

In other words, the opposite category $\mathcal{C}^{\mathrm{op}}$ is constructed from $\mathcal{C}$ by reversing the direction of the morphisms and compositions.

Example 1.5. SET^{op} is the category of sets and mappings, but with mappings written on the right, namely, if $f : S \to T$ is a function and $x \in S$, then the image of x under f is written as xf. Composition of mappings is done from left to right, so that if $g : T \to U$ is another mapping, then $f \circ g : S \to U$ is the mapping the sends x to $(xf)g = x(f \circ g)$.

Metatheorem 1.6. *A theorem which applies to all categories remain true if all the morphisms and compositions mentioned in the theorem have their directions reversed.*

Definitions and results that are obtained from each other by reversing the directions of the morphisms and compositions are said to be **dual** of each other. A definition or result is said to be **self-dual** if it remains unchanged when the direction of morphisms and compositions are reversed. For example, the concept of an isomorphism is a self-dual concept.

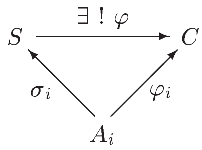
Definition 1.7. Let $\mathcal{C}$ be a category and $\{A_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ indexed by a nonempty set I. Suppose that there is an object $P \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{\pi_i : P \to A_i\}_{i\in I}$ such that for any object $C \in \mathrm{Ob}(\mathcal{C})$ and any family of morphisms $\{\varphi_i : C \to A_i\}_{i\in I}$, there is a **unique** morphism $\varphi : C \to P$ such that $\pi_i \circ \varphi = \varphi_i$ for all $i \in I$.

$$\begin{array}{ccc} C & \xrightarrow{\exists\,!\,\varphi} & P \\ & {\scriptstyle\varphi_i}\searrow \quad \swarrow{\scriptstyle\pi_i} & \\ & A_i & \end{array}$$

Then we say that $\{P, \{\pi_i\}_{i\in I}\}$ is a **product** in the category $\mathcal{C}$. Casually, we may say that P is a product of $\{A_i\}_{i\in I}$ and denote it by $\prod_{i\in I} A_i$.

Dualizing Definition 1.7 yields the following.

Definition 1.8. Let $\mathcal{C}$ be a category and $\{A_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ indexed by a nonempty set I. Suppose that there is an object $S \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{\sigma_i : A_i \to S\}_{i\in I}$ such that for any object $C \in \mathrm{Ob}(\mathcal{C})$ and any family of morphisms $\{\varphi_i : A_i \to C\}_{i\in I}$, there is a **unique** morphism $\varphi : S \to C$ such that $\varphi \circ \sigma_i = \varphi_i$ for all $i \in I$.



Then we say that $\{S, \{\sigma_i\}_{i\in I}\}$ is a **coproduct** or **sum** in the category $\mathcal{C}$. Casually, we may say that S is a coproduct of $\{A_i\}_{i\in I}$ and denote it by $\coprod_{i\in I} A_i$.

Theorem 1.9. *Let $\mathcal{C}$ be a category and $\{A_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ indexed by a nonempty set I. Suppose that $\{P, \{\pi_i\}_{i\in I}\}$ and $\{Q, \{\pi'_i\}_{i\in I}\}$ are products of $\{A_i\}_{i\in I}$ in the category $\mathcal{C}$. Then P and Q are isomorphic.*

Similarly, dualizing Theorem 1.9 yields the following result.

Theorem 1.10. *Let $\mathcal{C}$ be a category and $\{A_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ indexed by a nonempty set I. Suppose that $\{S, \{\sigma_i\}_{i\in I}\}$ and $\{T, \{\sigma'_i\}_{i\in I}\}$ are coproducts of $\{A_i\}_{i\in I}$ in the category $\mathcal{C}$. Then S and T are isomorphic.*

Definition 1.11. Let $\mathcal{C}, \mathcal{D}$ be categories. A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a function F that maps objects in $\mathcal{C}$ to objects in $\mathcal{D}$, and morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$, such that the following are satisfied:

Func1: $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any object $X \in \mathcal{C}$;

Func2: If α, β are morphisms in $\mathcal{C}$ for which the composite $\beta \circ \alpha$ is well-defined, then the composite $F(\beta) \circ F(\alpha)$ is well-defined in $\mathcal{D}$, and $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$.

A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a function F that maps objects in $\mathcal{C}$ to objects in $\mathcal{D}$, and morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$ such that (1) above is satisfied, and in place of (2), we have:

Func2': If α, β are morphisms in $\mathcal{C}$ for which the composite $\beta \circ \alpha$ is well-defined, then the composite $F(\alpha) \circ F(\beta)$ is well-defined in $\mathcal{D}$, and $F(\beta \circ \alpha) = F(\alpha) \circ F(\beta)$.

Examples 1.12.

1. For any category $\mathcal{C}$, there is a covariant functor $\mathcal{C} \to \mathcal{C}$ that sends every object and morphism to itself. This is known as the identity functor on $\mathcal{C}$ and is denoted by $\mathrm{Id}_{\mathcal{C}}$.

2. If $\mathcal{D}$ is a subcategory of $\mathcal{C}$ (a subclass of the class of objects and morphisms of $\mathcal{C}$ that forms category), there is a covariant functor $\mathcal{D} \to \mathcal{C}$ known as the inclusion functor defined in the obvious way.

3. If $\mathcal{C}$ is a category, there is a contravariant functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ that sends every object $X \in \mathcal{C}$ to X^{op} and every morphism f in $\mathcal{C}$ to f^{op}. This is known as the opposite functor on $\mathcal{C}$.

4. There is a covariant functor $\textsc{Grp} \to \textsc{Ab}$ that sends every group G to the quotient group G/G', where G' is the commutator subgroup of G; G' is the normal subgroup of G generated by all the elements of the form $xyx^{-1}y^{-1}$, where $x, y \in G$.

5. Let $\mathcal{C}$ be a category and let an object $A \in \mathcal{C}$ be fixed. There is a covariant functor

$$\mathrm{Mor}(A, -) : \mathcal{C} \to \textsc{Set}$$

that sends each object $X \in \mathcal{C}$ to the morphism set $\mathrm{Mor}(A, X)$ and each morphism $f : X \to Y$ in $\mathcal{C}$ to the set function $\mathrm{Mor}(A, f) : \mathrm{Mor}(A, X) \to \mathrm{Mor}(A, Y)$ defined by $\mathrm{Mor}(A, f) : \alpha \mapsto f \circ \alpha$.

6. Let $\mathcal{C}$ be a category and let an object $A \in \mathcal{C}$ be fixed. There is a contravariant functor

$$\mathrm{Mor}(-, A) : \mathcal{C} \to \textsc{Set}$$

that sends each object $X \in \mathcal{C}$ to the morphism set $\mathrm{Mor}(X, A)$ and each morphism $f : Y \to X$ in $\mathcal{C}$ to the set function $\mathrm{Mor}(f, A) : \mathrm{Mor}(X, A) \to \mathrm{Mor}(Y, A)$ defined by $\mathrm{Mor}(f, A) : \alpha \mapsto \alpha \circ f$.

The composite of two functors is again a functor.

Definition 1.13. Let $F, G : \mathcal{C} \to \mathcal{D}$ be covariant functors. A **natural transformation** from F to G is a collection $\eta = \{\eta_X : F(X) \to G(X)\}_{X \in \mathcal{C}}$ of morphisms in $\mathcal{D}$ indexed by the objects of $\mathcal{C}$ such that for all morphisms $f : X \to Y$ in $\mathcal{C}$, we have the following commutative diagram:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y) \end{array}$$

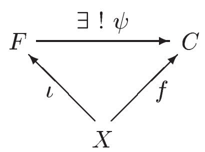
If $F, G : \mathcal{C} \to \mathcal{D}$ are contravariant functors, a natural transformation from F to G is a collection $\eta = \{\eta_X : F(X) \to G(X)\}_{X \in \mathcal{C}}$ of morphisms in $\mathcal{D}$ indexed by the objects of $\mathcal{C}$ such that for all morphisms $f : Y \to X$ in $\mathcal{C}$, we have same diagram as above.
If η_X is an isomorphism for all objects $X \in \mathcal{C}$, we call η a **natural isomorphism**.

The composite of two natural transformations is again a natural transformation.

Definition 1.14. A **concrete category** is a category $\mathcal{C}$ with a functor $\upsilon : \mathcal{C} \to \textsc{Set}$.

The category of groups, equipped with the function that assigns to each group its underlying set in the usual sense, is a concrete category. Similarly, the categories of abelian groups and of partially-ordered sets, with the obvious underlying sets, are concrete categories. Since in the great majority of examples of concrete categories, the functor υ assigns to an object its underlying set in the usual sense (such as in examples above), we shall denote both the object and its underlying set by the same symbol and leave out any explicit reference to the functor υ.

Definition 1.15. Let F be an object in a concrete category $\mathcal{C}$, X be a nonempty set, and $\iota : X \to F$ be a function of sets. We say that F, or more exactly (F, ι), is **free on** X if for any object $C \in \mathrm{Ob}(\mathcal{C})$ and any set function $f : X \to C$, there exists a **unique** morphism $\psi : F \to C$ in the category $\mathcal{C}$ such that $\psi \circ \iota = f$ as functions of sets.



In the concrete category $\mathcal{C}$, an object that is free on some set is called a **free object**.

Lemma 1.16. *In a concrete category $\mathcal{C}$ that contains objects which are sets with more than one element, if (F, ι) is a free on a set X, then the set function $\iota : X \to F$ is injective.*

Proof. Suppose that $\iota(x_1) = \iota(x_2)$ with $x_1 \neq x_2$. Choose an object C which is set containing at least two distinct elements, say c_1 and c_2. Let $f : X \to C$ be any set function such that $f(x_1) = c_1$ and $f(x_2) = c_2$. Since F is free on X, we have a unique morphism $\psi : F \to C$ in the category $\mathcal{C}$ such that $\psi \circ \iota = f$ as functions of sets. From the fact that $\psi(\iota(x_1)) = \psi(\iota(x_2))$, we deduce that $f(x_1) = f(x_2)$, whence $c_1 = c_2$, which contradicts our earlier assumptions. □

Theorem 1.17. *If $\mathcal{C}$ is a concrete category in which (F_1, ι_1) is free on X_1 and (F_2, ι_2) free on X_2 with $|X_1| = |X_2|$, then F_1 and F_2 are isomorphic.*

2 Group Actions on Sets

Definition 2.1. Let X be a nonempty set and G be a group. Let $\alpha : G \times X \to X$ be a function and write an image $\alpha(g, x)$ simply as gx. We call α a **group action** if $1_G x = x$ and $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$. If there is a group action $\alpha : G \times X \to X$, we say that X is a **G-set**, and that G **acts on** the set X.

Examples 2.2. Let G be a group.

1. G can be made to act on itself. The map $(g, x) \mapsto gx$ for $g, x \in G$ is easily seen to be a group action. We say that G acts on G by **left multiplication**.
2. Similarly, the map $(g, x) \mapsto xg^{-1}$ for $g, x \in G$ is also a group action. We say that G acts on G by **right multiplication**.
3. The map $(g, x) \mapsto gxg^{-1}$ for $g, x \in G$ is yet another group action of G on itself. We say that G acts on G by **conjugation**.
4. The group of permutations on a set X, $\mathrm{Sym}(X)$, acts on X in the natural manner via the map $(\sigma, x) \mapsto \sigma(x)$, for $\sigma \in S_X$ and $x \in X$. We say that $\mathrm{Sym}(X)$ acts on X **canonically**.

Proposition 2.3. *If X is a G-set with group action $\alpha : G \times X \to X$, then there is a group homomorphism $\psi : G \to \mathrm{Sym}(X)$ given by $\psi(g) : x \mapsto gx = \alpha(g, x)$. Conversely, if we can find a group homomorphism $\phi : G \to \mathrm{Sym}(X)$, then the map $(g, x) \mapsto \phi(g)(x)$ is a group action that makes X into a G-set. These processes are inverse to one another, so that we have a bijective correspondence between the G-set structures that can be defined on X and the group homomorphisms of G into* $\mathrm{Sym}(X)$.

If $\psi : G \to \mathrm{Sym}(X)$ is a group homomorphism of G into $\mathrm{Sym}(X)$, then we say that G **acts on** X **via** ψ, and the group action we are referring to will be that which arises from the abovementioned correspondence, namely $(g, x) \mapsto \psi(g)(x)$.

Definition 2.4. Let X be a G-set and $x \in X$. The **orbit** of x in G, denoted by $\mathcal{O}_G(x)$ or by Gx, is defined by

$$\mathcal{O}_G(x) = \{gx \ : \ g \in G\}.$$

Proposition 2.5. *If X is a G-set, then the orbits of X form a partition of X.*

In particular, this would mean that if G acts on X, then the relation $\sim$ on X defined by

$$x \sim x' \ \Leftrightarrow \ gx = x' \text{ for some } g \in G$$

is an equivalence relation on X, and the equivalence classes of X under $\sim$ are simply the orbits $\mathcal{O}_G(x)$.

If X is a G-set, the collection of orbits $\mathcal{O}_G(x)$ is denoted by X/G. Thus X/G is the collection of equivalence classes of X under the relation $\sim$ defined above.

Definition 2.6. Let X be a G-set and $x \in X$. The **stabilizer** or **isotropy group** of x in G, denoted by G_x, is defined by

$$G_x = \{g \in G \ : \ gx = x\}.$$

Example 2.7. Let G be a group. Consider G acting on itself by conjugation. The orbit of $x \in G$ is the conjugacy class

$$\mathrm{Cl}_G(x) = \{gxg^{-1} \ : \ g \in G\}$$

and the stabilizer of $x \in G$ is the centralizer

$$\mathrm{C}_G(x) = \{g \in G \ : \ gxg^{-1} = x\}.$$

Proposition 2.8. *Let X be a G-set and let $g \in G$, $x \in X$.*

1. *G_x is a subgroup of G.*
2. *$G_{gx} = gG_xg^{-1}$.*
3. *There is a bijective correspondence between the elements of $\mathcal{O}_G(x)$ and the left cosets of G_x in G.*

3 Basic Definitions in Topology

Definition 3.1. A **topology** on a set X is a collection $\Im$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\Im$.
2. If $\{U_i\}_{i\in I}$ are a collection of sets in $\Im$, then $\bigcup_{i\in I} U_i$ is in $\Im$.
3. If $\{U_i\}_{i=1}^n$ is a finite collection of sets in $\Im$, then $\bigcap_{i=1}^n U_i$ is in $\Im$.

A set X for which a topology $\Im$ has been specified is known as a **topological space**, denoted by $(X, \Im)$.

We often leave out mention of $\Im$ is there is no confusion. If X is a topological space with topology $\Im$ and $U \in \Im$, we say that U is an **open set** of X, and that $X \setminus U$ is a **closed set** of X, with respect to the topology $\Im$.

Examples 3.2.

1. The **discrete** topology on a set X is the topology that consists of all subsets of X.
2. The **indiscrete** or **trivial** topology on X is the topology that consists only of $\emptyset$ and X.
3. The **finite complement** topology on X is the collection U of subsets of X such that $X \setminus U$ is either finite or the whole of X.

Definition 3.3. Suppose that $\Im$ and $\Im'$ are two topologies on a set X. If $\Im \subseteq \Im'$, we say that $\Im$ is **coarser** than $\Im'$, and that $\Im'$ is **finer** than $\Im$.

Definition 3.4. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X, whose members are called **basic sets**, such that:

1. $\emptyset \in \mathcal{B}$.
2. $\bigcup_{B\in\mathcal{B}} B = X$.
3. If $C_1, C_2, ..., C_n$ is a finite collection of elements of $\mathcal{B}$, then there is a collection $\{B_i\}_{i\in I}$ of elements of $\mathcal{B}$ such that
$$\bigcup_{i\in I} B_i = \bigcap_{j=1}^{n} C_j.$$

Let $\mathcal{B}$ be a basis for a topology on X. Let $\Im$ be the collection of all arbitrary unions of elements of $\mathcal{B}$. Then $(X, \Im)$ is a topological space.

Definition 3.5. The topology $\Im$ constructed from a basis $\mathcal{B}$ by taking arbitrary union of basic sets is known as the topology **generated** by $\mathcal{B}$, and the set $\mathcal{B}$ is known as a **basis for** $\Im$.

Example 3.6. Let M be a nonempty set. Suppose there is a function $\mathrm{d}_M : M \times M \to \mathbb{R}$ such that:

1. $\mathrm{d}_M(x, y) \geq 0$ for all $x, y \in M$, equality holding if and only if $x = y$;
2. $\mathrm{d}_M(x, y) = \mathrm{d}_M(y, x)$ for all $x, y \in M$;
3. $\mathrm{d}_M(x, z) \leq \mathrm{d}_M(x, y) + \mathrm{d}_M(y, z)$ for all $x, y, z \in M$.

Then we call (M, d_M) a **metric space**. The collection of sets of the form $B_x^\epsilon = \{y \in M \ : \ \mathrm{d}_M(x, y) < \epsilon\}$ for $x \in M$ and $\epsilon > 0$ forms a basis for a topology on M known as the topology **induced by the metric**.

Definition 3.7. If X is a set, a **sub-basis** for a topology on X is a collection $\mathcal{S}$ of subsets of X whose members are called **sub-basic sets** such that:

1. $\emptyset \in \mathcal{S}$.
2. $\bigcup_{S\in\mathcal{S}} S = X$.

Obviously, a sub-basis for a topology on X can be extended to a basis by collecting all the intersections of finite collections of sub-basic sets.

Definition 3.8. The topology $\Im$ constructed from a sub-basis $\mathcal{S}$ by first extending $\mathcal{S}$ to a basis $\mathcal{B}$ by collecting all the intersections of finite collections of sub-basic sets, and then generating $\Im$ from $\mathcal{B}$, is known as the topology **generated** by $\mathcal{S}$, and the set $\mathcal{S}$ is known as a **sub-basis for** $\Im$.

Definition 3.9. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open set V in Y, the set $f^{-1}(V)$ is open in X.

Example 3.10. In classical metric space theory, a function $f : M \to N$ between metric spaces is said to be continuous if given any fixed point $x \in M$ and any $\epsilon > 0$, there exists some $\delta > 0$ such that $\mathrm{d}_N(f(x), f(y)) < \epsilon$ whenever $\mathrm{d}_M(x, y) < \delta$. A function between metric spaces is continuous in this definition if and only if it is continuous in the sense of Definition 3.9.

Definition 3.11. A bijective function $f : X \to Y$ between topological spaces is said to be a **homeomorphism** if both f and f^{-1} are continuous functions. If X is homeomorphic to Y we denote it by $X \cong Y$.

Definition 3.12. A function $f : X \to Y$ between topological spaces is said to be an **open map** if whenever U is an open set in X, $f(U)$ is an open set in Y.

In other words, an open map sends open sets to open sets. In particular, a homeomorphism is a bijective continuous open map.

From now on, the word "map" is always assumed to mean "continuous function".

Definition 3.13. Let $(X, \Im)$ be a topological space. If Y is a subset of X, the collection

$$\Im_Y = \{U \cap Y \ : \ U \in \Im\}$$

is a topology on Y known as the **subspace** or **induced** topology. We call Y a **subspace** of X.

Example 3.14. The open sets of $[0, 1]$ as a subspace of $\mathbb{R}$ are those of the form (a, b), where $0 \leq a, b \leq 1$, or of the form $[0, a)$ or $(a, 1]$ where $0 \leq a \leq 1$, in addition to the set $[0, 1]$ itself.

Definition 3.15. Let $\{X_i\}_{i\in I}$ be a collection of topological spaces. The **box topology** on $\prod_{i\in I} X_i$ is the topology having as basis all the elements of the form $\prod_{i\in I} U_i$, where U_i is an open set in X_i for each $i \in I$.

For each $j \in I$, let $\pi_j : \prod_{i\in I} X_i \to X_j$ be the map that sends each element $(x_i)_{i\in I}$, where $x_i \in X_i$ for all i, to the element $x_j \in X_j$. We call π_j the **projection map** at coordinate j.

Definition 3.16. Let $\{X_i\}_{i\in I}$ be a collection of topological spaces. For each $i \in I$, let $\mathcal{S}_i$ denote the collection

$$\mathcal{S}_i = \{\pi_i^{-1}(U_i) \ : \ U_i \text{ is open in } X_i\},$$

and let

$$\mathcal{S} = \bigcup_{i\in I} \mathcal{S}_i.$$

Then the collection $\mathcal{S}$ forms a sub-basis for a topology on $\prod_{i\in I} X_i$ known as the **product topology**.

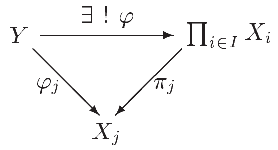
The box and product topologies are generalizations for the topology on $X \times Y$. For the case where I is a finite set, the box and product topologies on $\prod_{i\in I} X_i$ coincide. If I is an infinite set, then the box topology is in general finer than the product topology.

The product topology on $\prod_{i\in I} X_i$ has as basis all sets of the form $\prod_{i\in I} U_i$, where U_i is open in X_i for all $i \in I$ and $U_i = X_i$ except for at most finitely many i.

If $\prod_{i\in I} X_i$ is given the product topology, the canonical projections $\pi_j : \prod_{i\in I} X_i \to X_j$ are continuous for each $j \in I$.

Let $\Delta : X \to \prod_{i\in I} X$ be defined by $\Delta : x \mapsto (x_i)_{i\in I}$, where $x_i = x$ for all $i \in I$. We call Δ the **diagonal map**, and Δ is continuous if $\prod_{i\in I} X$ is given the product topology.

Proposition 3.17. *Let $\{X_i\}_{i\in I}$ be a collection of topological spaces indexed by a nonempty set I, and let $\prod_{i\in I} X_i$ be given the product topology. Then for any topological space Y and any collection of continuous maps $\{\varphi_i : Y \to X_i\}_{i\in I}$, there exists a unique continuous map $\varphi : Y \to \prod_{i\in I} X_i$ such that $\pi_j \circ \varphi = \varphi_j$ for each $j \in I$, where the π_j's are the canonical projections. Hence, $\prod_{i\in I} X_i$ with the product topology is a product of $\{X_i\}_{i\in I}$ in the category* Top *of topological spaces and continuous maps.*



Definition 3.18. Let X be a topological space, and let A be a subset of X.

1. The **closure** of A in X, denoted by $\mathrm{Cl}_X(A)$ (or $\overline{A}$ if there is no ambiguity), is the intersection of all closed sets of X containing A.
2. The **interior** of A in X, denoted by $\mathrm{Int}_X(A)$ (or A° if there is no ambiguity), is the union of all open sets of X that are contained within A.
3. The **boundary** of A in X, denoted by $\mathrm{Bd}_X(A)$ (or ∂A if there is no ambiguity), is defined by

$$\mathrm{Bd}_X(A) = \mathrm{Cl}_X(A) \cap \mathrm{Cl}_X(X \setminus A).$$

Obviously, $\overline{A}$ is a closed set and A° is an open set, implying that $A = \mathrm{Cl}_X(A)$ if and only if A is closed in X, and $A = \mathrm{Int}_X(A)$ if and only if A is open in X. Furthermore,

$$A^\circ \subseteq A \subseteq \overline{A}.$$

Example 3.19. Let $A = [0, 1) \subseteq \mathbb{R}$. Then $\mathrm{Cl}_{\mathbb{R}}(A) = [0, 1]$, $\mathrm{Int}_{\mathbb{R}}(A) = (0, 1)$, and $\mathrm{Bd}_{\mathbb{R}}(A) = \{0, 1\}$.

Definition 3.20. Let A be a subset of a topological space X. We say that $x \in X$ is a **limit point** or an **accumulation point** of A if for every open set U of X containing x, there is some $y \in A$ with $y \neq x$ and $y \in U$. We denote the set of limit points of A in X by $\mathrm{L}_X(A)$ (or by A' if there is no ambiguity).

Note that elements of A are not necessarily limit points of A. A subspace is closed if and only if it contains all its limit points.

Example 3.21. Let $A = [0, 1) \cup \{2\}$. Then $\mathrm{L}_{\mathbb{R}}(A) = [0, 1]$.

Definition 3.22. Let X and Y be topological spaces, and let $p : X \to Y$ be a surjective function. We say that p is a **quotient map** if every subset V of Y is open in Y if and only if $p^{-1}(V)$ is open in X.

A quotient map between topological spaces is necessarily continuous.

Let $p : X \to A$ is a surjective function, and let $\Im$ consist of those subsets V of A for which $p^{-1}(V)$ is open in X. Then it is straightforward to check that $\Im$ is a topology on A. This is called the **quotient topology** on A induced by $p : X \to A$, and is the unique topology $\Im$ on A relative to which p is a quotient map.

Definition 3.23. Let R be an equivalence relation on a space X. Let $p : X \to X/R$ be the canonical surjection that maps each element of X to its equivalence class. Under the quotient topology induced on X/R by p, the space X/R is called a **quotient space** or **identification space** of X.

If A is a subspace of X, the space X/A is defined to be the quotient space of X obtained by identifying A to a single point.

4 Compact, Locally Compact, Hausdorff, Connected Spaces

Let X be a subspace of a topological space Y. A **covering** of X in Y is a collection $\{U_i\}_{i\in I}$ of subsets of Y such that $X \subseteq \bigcup_{i\in I} U_i$. We call a covering $\{U_i\}_{i\in I}$ of X in Y an **open covering** in Y if every set U_i is open in Y.

If every U_i is a subset of X and $X = \bigcup_{i\in I} U_i$, we simply say that $\{U_i\}_{i\in I}$ is a covering of X.

Definition 4.1. A topological space X is said to be **compact** if given any open covering $\{U_i\}$ of X, there exists a finite collection $U_1, U_2, ..., U_n$ of sets drawn from $\{U_i\}$ such that $X = \bigcup_{j=1}^{n} U_j$. We say that the open covering $\{U_i\}$ has a **finite subcover**.

Proposition 4.2. *The notion of compactness of X is independent of the space that contains X. More precisely, X is compact if and only if for all $Y \supseteq X$ in which X is given the subspace topology of Y, any open covering of X in Y admits a finite subcover.*

Proposition 4.3. *The continuous image of a compact space is compact.*

Proposition 4.4. *A closed subspace of a compact space is compact.*

Proof. Let X be a compact topological space and let S be a closed subspace of X. Let $\{U_i\}_{i\in I}$ be an open cover of S in X. Since $X = (\bigcup_{i\in I} U_i) \cup (X \setminus S)$ and $(X \setminus S)$ is open in X, it follows that there is a finite collection $U_1, U_2, ..., U_n$ of sets drawn from $\{U_i\}_{i\in I}$ such that $X = (\bigcup_{j=1}^{n} U_j) \cup (X \setminus S)$. But then this would mean that $\{U_1, U_2, ..., U_n\}$ is a finite subcover of S in X. □

Definition 4.5. Let x be a point in a topological space X. We say that U is a **neighbourhood** of x if there is some open set V of X such that $V \subseteq U$ and $x \in V$.

Definition 4.6. A space X is said to be **locally compact** if every point in X has a compact neighbourhood.

Clearly, any compact space is locally compact.

Example 4.7. $\mathbb{R}^n$ is locally compact but not compact. The Heine-Borel Theorem states that any closed bounded subset of $\mathbb{R}^n$ is compact.

Definition 4.8. A topological space X is said to be **Hausdorff** if given any distinct points $u, v \in X$, there are disjoint open sets U, V of X such that $u \in U$ and $v \in V$.

Proposition 4.9. *Every compact subspace of a Hausdroff space is closed.*

Proposition 4.10. *Let X be a locally compact Hausdorff space. Given a point $x \in X$ and a neighborhood U of x, there is an open set V such that $x \in V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact.*

Let $S^0 = \{0, 1\}$ denote the topological space consisting of two distinct points and given the discrete topology.

Definition 4.11. A topological space X is said to be **disconnected** if there is a continuous surjective map $X \to S^0$. Such a map is termed a **disconnection** of X.

If $f : X \to S^0$ is a disconnection of X, then $X = f^{-1}(0) \sqcup f^{-1}(1)$ describes a partition of X into two disjoint subsets, each of which are both open and closed.

Definition 4.12. A topological space X is said to be **connected** if it is no disconnection.

Proposition 4.13. *The continuous image of a connected space is connected.*

Definition 4.14. A topological space X is said to be **path-connected** for any points $u, v \in X$, there is a continuous map $f : [0, 1] \to X$ such that $f(0) = u$ and $f(1) = v$.

Proposition 4.15. *The continuous image of a path-connected space is path-connected.*

Proposition 4.16. *A path-connected space is connected.*

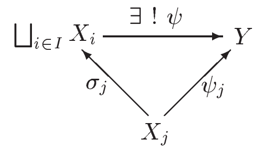
Example 4.17. The subspace $Y = \{(x, y) \in \mathbb{R}^2 \;:\; y = \sin(1/x),\; x > 0\} \cup \{(0, 0)\}$ of $\mathbb{R}^2$ is connected but not path-connected.

5 Some Important Topological Constructions and Examples

Definition 5.1. Given a collection $\{X_i\}_{i\in I}$ of topological spaces, the **disjoint union** $\bigsqcup_{i\in I} X_i$ is the union of the spaces X_i regarded as pairwise disjoint sets, with topology given by open sets of the form $\bigcup_{i\in I} U_i$, where U_i is open in X_i.

The canonical inclusions $\sigma_j : X_j \to \bigsqcup_{i\in I} X_i$ are continuous for all $j \in I$.

Proposition 5.2. *Let $\{X_i\}_{i\in I}$ be collection of topological spaces. If $\{\psi_i : X_i \to Y\}_{i\in I}$ is a collection of continuous maps, then there is a unique continuous map $\psi : \bigsqcup_{i\in I} X_i \to Y$ such that $\psi \circ \sigma_j = \psi_j$ for all $j \in I$, where $\sigma_j : X_j \to \bigsqcup_{i\in I} X_i$ is the canonical inclusion. Hence, the disjoint union $\bigsqcup_{i\in I} X_i$ is a coproduct of $\{X_i\}_{i\in I}$ in the category* Top *of topological spaces.*



Definition 5.3. A **pair** of spaces (X, Y) is a space X together with a subspace Y. Given pairs (X, Y) and (A, B) of spaces, a **map of pairs** $f : (X, Y) \to (A, B)$ is a continuous map $f : X \to A$ that satisfies $f(Y) \subseteq B$. We call f a **homeomorphism of pairs** if f is a homeomorphism and the inverse map f^{-1} is a map of pairs $(A, B) \to (X, Y)$.

Let (X, Y) and (A, B) be pairs of spaces, and let $f : Y \to B$ be a continuous map. Define an equivalence relation $\sim$ on the disjoint union $X \sqcup A$ by identifying y with $f(y)$ for all $y \in Y$. The space $(X \cup A)/\sim$ is denoted by $A \cup_f X$ and is referred to as an **adjunction space**.

If (X, Y) is a pair of spaces in which $Y = \{*\}$ is a singleton set, we denote it simply by $(X, *)$ and say that X is a **pointed space** with **basepoint** $*$. A map of pairs between pointed spaces is called a **pointed map**. Thus a pointed map sends the basepoint of one space to that of the other. We often denote $(X, *)$ simply by X if there is no danger of confusion.

Definition 5.4. If X, Y are pointed spaces with base points x_0, y_0 respectively, the **wedge** $X \vee Y$ is the adjunction space $Y \cup_f X$ where $f : \{x_0\} \to \{y_0\}$. In other words, $X \vee Y$ is obtained from the disjoint union of X and Y by identifying the base points of X, Y.

For a space Z, define the **fold map** $\bigtriangledown : Z \vee Z \to Z$ by $\bigtriangledown : (z, *) \mapsto z$ and $\bigtriangledown : (*, z) \mapsto z$. The fold map is continuous.

Definition 5.5. If X, Y are pointed spaces, the **smash** $X \wedge Y$ is defined to be $(X \times Y)/(X \vee Y)$, where we identify $X \vee Y$ as a subspace of $X \times Y$.

Examples 5.6.

1. For a space X with subspace A, the quotient space X/A is often regarded as a pointed space with basepoint being the equivalence class containing the elements of A.

2. For all nonnegative integers n, the n-sphere is denoted by S^n and is the subspace of $\mathbb{R}^{n+1}$ consisting of all points of distance 1 from the origin. S^n is often regarded as a pointed space with basepoint being the north pole. We have $S^n \cong D^n/\partial D^n$ for all $n \geq 1$, where $D^n = [0, 1]^n \subseteq \mathbb{R}^n$. We often identify S^{n-1} as the boundary of D^n, and write $S^n \cong D^n/S^{n-1}$ for $n \geq 1$.

3. The **real projective space** $\mathbb{R}P^n$ $(n \geq 0)$ is defined to be the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation $\sim$ defined by $x \sim y \;\Leftrightarrow\; x = ry$ for some real number r. $\mathbb{R}P^n$ is homeomorphic to the quotient space of S^n under the equivalence relation R defined by $xRy \;\Leftrightarrow\; x = \pm y$.

Let X and Y be topological spaces. The **mapping space** $\mathrm{Map}(X,Y)$ is the space of all continuous maps from X to Y under the **compact-open topology** defined as follows:

Let K be a compact set in X and U be an open set in Y. Let $W_{K,U} = \{f \in \mathrm{Map}(X,Y) \ : \ f(K) \subseteq U\}$. Then all the elements $W_{K,U}$ as K varies over the compact sets in X and U varies over the open sets in Y form a sub-basis for the compact-open topology on $\mathrm{Map}(X,Y)$. If X and Y are pointed spaces, then the pointed mapping space $\mathrm{Map}_*(X,Y)$, also commonly denoted as Y^X, is the subspace of $\mathrm{Map}(X,Y)$ consisting of all pointed maps $f : X \to Y$.

Definition 5.7. Let $f : A \to X$ and $g : Y \to B$ be continuous maps. Then the function $g^f : \mathrm{Map}(X,Y) \to \mathrm{Map}(A,B)$ is defined by $g^f(\lambda) = g \circ \lambda \circ f$. If f and g are pointed continuous maps, then they induce a function $g^f : \mathrm{Map}_*(X,Y) \to \mathrm{Map}_*(A,B)$ also defined as above.

Proposition 5.8. *Let $f : A \to X$ and $g : Y \to B$ be continuous maps (or pointed continuous maps). Then the function $g^f : \mathrm{Map}(X,Y) \to \mathrm{Map}(A,B)$ (resp. $g^f : Y^X \to B^A$) is a continuous map.*

Remark 5.9. If X and Y are pointed spaces, then $\mathrm{Map}(X,Y)$ and $\mathrm{Map}_*(X,Y)$ are also pointed spaces with the base point being the constant map $X \to \{y_0\}$, where y_0 is the base point of Y. If $g : Y \to B$ is a pointed continuous map, then for any continous map $f : A \to X$, the induced function g^f is also a pointed continuous map between the mapping spaces.

Let $\textsc{Top}_*$ denote the category of pointed topological sapces. Then both $\mathrm{Map}(X,-)$ and $\mathrm{Map}_*(X,-)$ are covariant functors on $\textsc{Top}$ and $\textsc{Top}_*$ respectively for any space X (respectively pointed space X), and both $\mathrm{Map}(-,X)$ and $\mathrm{Map}_*(-,X)$ are contravariant functors on $\textsc{Top}$ and $\textsc{Top}_*$ respectively for any space X (respectively pointed space X).

Proposition 5.10. *Let X, Y and Z be topological spaces.*

1. *If Z is a subspace of Y, then $\mathrm{Map}(X,Z)$ is a subspace of $\mathrm{Map}(X,Y)$.*
2. *If X, Y and Z are pointed spaces with Z being a subspace of Y (sharing the same base point as Y), then $\mathrm{Map}_*(X,Z)$ is a subspace of $\mathrm{Map}_*(X,Y)$.*

Definition 5.11. Let X and Y be topological spaces. The **evaluation map** $e : \mathrm{Map}(X,Y) \times X \to Y$ is defined by $e : (f,x) \mapsto f(x)$. If X and Y are pointed spaces, the restriction of e to $\mathrm{Map}_*(X,Y)$ gives the evaluation map $e : \mathrm{Map}_*(X,Y) \times X \to Y$, with the property that if f is the constant pointed map or x is the base point of X, then $e(f,x) = y_0$, where y_0 is the base point of Y.

Remark 5.12. It follows from the definition that for pointed spaces X and Y,

$$e(\mathrm{Map}_*(X,Y) \vee X) = \{y_0\},$$

and so e induces the evaluation map $\overline{e} : \mathrm{Map}_*(X,Y) \wedge X \to Y$.

Proposition 5.13. *Let X and Y be pointed spaces. If X is locally compact Hausdorff, then the evaluation maps $e : \mathrm{Map}(X,Y) \times X \to Y$ and $\overline{e} : \mathrm{Map}_*(X,Y) \wedge X \to Y$ are continuous.*

Proposition 5.14. *Let X, Y and Z be pointed spaces with X and Y Hasudorff. Then*

1. $\mathrm{Map}(X \sqcup Y, Z) \cong \mathrm{Map}(X,Z) \times \mathrm{Map}(Y,Z)$.
2. $\mathrm{Map}_*(X \vee Y, Z) \cong \mathrm{Map}_*(X,Z) \times \mathrm{Map}_*(Y,Z)$.

Proposition 5.15. *Let X, Y and Z be pointed spaces with X Hausdorff. Then*

1. $\mathrm{Map}(X, Y \times Z) \cong \mathrm{Map}(X,Y) \times \mathrm{Map}(X,Z)$.
2. $\mathrm{Map}_*(X, Y \times Z) \cong \mathrm{Map}_*(X,Y) \times \mathrm{Map}_*(X,Z)$.

Let X, Y and Z be topological spaces, and let $\lambda : X \times Y \to Z$ be a continuous map. For a given $x \in X$, we define a function $\overline{\lambda}_x : Y \to Z$ by $\overline{\lambda}_x(y) = \lambda(x,y)$. Then the function $\overline{\lambda}_x$ is continuous, and the function $\alpha(\lambda) : X \to \mathrm{Map}(Y,Z)$ defined by $\alpha(\lambda)(x) = \overline{\lambda}_x$ is also continuous.

Definition 5.16. Let X, Y and Z be topological spaces. The **association map** is the function

$$\alpha : \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

defined by $[\alpha(\lambda)(x)](y) = \lambda(x, y)$ for $x \in X$, $y \in Y$ and $\lambda : X \times Y \to Z$.

We consider the pointed case. Let X, Y and Z be pointed spaces and let $p : X \times Y \to X \wedge Y$ be the quotient map. Then we have the continuous map $(\mathrm{id}_Z)^p : \mathrm{Map}_*(X \wedge Y \to Z) \to \mathrm{Map}_*(X \times Y, Z)$. Clearly α maps the image of $(\mathrm{id}_Z)^p$ into the subspace $\mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z))$ of $\mathrm{Map}(X, \mathrm{Map}(Y, Z))$. Thus α induces the **reduced association map** $\overline{\alpha} : \mathrm{Map}_*(X \wedge Y, Z) \to \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z))$ with $[\overline{\alpha}(\lambda)(x)](y) = \lambda(x \wedge y)$ for $x \in X$, $y \in Y$ and $\lambda : X \wedge Y \to Z$. In fact, $\overline{\alpha}$ is the composite

$$Z^{X \wedge Y} \xrightarrow{(\mathrm{id}_Z)^p} Z^{X \times Y} \xrightarrow{\alpha|_{Z^{X \times Y}}} (Z^Y)^X$$

Proposition 5.17. *If X is Hausdorff, the association map $\alpha : \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ is continuous, and therefore the reduced association map $\overline{\alpha} : \mathrm{Map}_*(X \wedge Y, Z) \to \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z))$ is also continuous.*

Proposition 5.18. *Let X, Y and Z be topological spaces. Then α and $\overline{\alpha}$ are injective. If Y is locally compact Hausdorff, then α and $\overline{\alpha}$ are bijective. If X and Y are locally compact Hausdorff, then α is a homeomorphism. If X and Y are compact Hausdorff, then $\overline{\alpha}$ is a homeomorphism.*

Examples 5.19.

1. For a pointed space Y, define $\Omega(Y) = \mathrm{Map}_*(S^1, Y)$. In general, for all nonnegative integers n, define $\Omega^n(Y) = \mathrm{Map}_*(S^n, Y)$. $\Omega^n(Y)$ is termed the n-fold **loop space** of Y. For pointed spaces X, Y, we have $\Omega^n(X \times Y) \cong \Omega^n(X) \times \Omega^n(Y)$ and for nonnegative integers n, m, we have $\Omega^{n+m}(X) \cong \Omega^n(\Omega^m(X))$.

2. For a pointed space X, define $\Sigma(X) = S^1 \wedge X$. In general, for all nonnegative integers n, define $\Sigma^n(X) = S^n \wedge X$. $\Sigma^n(X)$ is termed the n-fold **suspension** of X. For nonnegative integers n, m, we have $\Sigma^{n+m}(X) \cong \Sigma^n(\Sigma^m(X))$. For any pointed spaces X, Y, we have

$$\mathrm{Map}_*(\Sigma^n(X), Y) \cong \mathrm{Map}_*(X, \Omega^n(Y)).$$

Both the loop Ω and suspension Σ are functors on the category $\textsc{Top}_*$ of pointed topological spaces.

6 Introduction to Homotopy

Definition 6.1. Let $f, g : X \to Y$ be two maps. We say that f is **homotopic** to g if we can find a map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. The map F is called a **homotopy from f to g**. We write $f \simeq g$, or $F : f \simeq g$. A map $f : X \to Y$ is called **null-homotopic** if f is homotopic to a constant map from X to Y.

Definition 6.2. Let A be a subspace of X and suppose $f, g : X \to Y$ are maps. We say that f is homotopic to g **relative to** A if we can find a map $F : X \times I \to Y$ such that F is a homotopy from f to g and $F(a, t) = f(a)$ for all $a \in A$, $t \in I$. We write $f \simeq g \operatorname{rel} A$ or $F : f \simeq g \operatorname{rel} A$. We say that f is **null-homotopic** relative to A if f is homotopic to a constant map from X to Y relative to A.

If $f \simeq g \operatorname{rel} A$, then we have $g(a) = f(a)$ for all $a \in A$. Hence the maps f and g agree on the subspace A. If in addition, g is a constant map, then $f|_A$ is also a constant map. Homotopy relative to a subspace A is an equivalence relation on the set of maps from X to Y.

The notation $[X, Y]_A$ will be used to denote the set of equivalence classes of maps from X to Y under the relation of homotopy relative to the subspace A of X. Given a map $f : X \to Y$, we use $[f]_A$ to denote the equivalence class in $[X, Y]_A$ to which f belongs. For unpointed spaces X and Y, we can write $[X, Y]_\emptyset$ simply as $[X, Y]$, and $[f]_\emptyset$ simply as $[f]$.

Proposition 6.3. *Let A be a subspace of X and B be a subspace of Y. Let $f_0, f_1 : X \to Y$ be homotopic relative to A and $g_0, g_1 : Y \to Z$ be homotopic relative to B. Suppose further that $f_0(A) = f_1(A) \subseteq B$. Then $g_0 \circ f_0 \simeq g_1 \circ f_1 \operatorname{rel} A$.*

Definition 6.4. Let X and Y be pointed spaces and let $f, g : X \to Y$ be pointed maps. f is called **pointed homotopic** to g if $f \simeq g \operatorname{rel} x_0$, where x_0 is the base point of X. We can simply write $f \simeq g$ if there is no ambiguity.

For pointed spaces X and Y, the notation $[X, Y]$ is used to denote the set of equivalence classes of pointed maps from X to Y under the relation of pointed homotopy, that is, homotopy relative to the base point x_0 of X. For any pointed map $f : X \to Y$, $[f]$ denotes the equivalence class in $[X, Y]$ to which f belongs.

Let TopH and TopH$_*$ respectively denote the category of topological spaces (respectively pointed spaces) in which the morphism sets are the equivalence classes of maps (respectively pointed maps) under homotopy. Then for any pointed space X, $[X, -]$ is a covariant functor on TopH$_*$ and $[-, X]$ is a contravariant functor on TopH$_*$. For a map $f : X \to Y$, we often denote $[X, f]$ simply as f_*^X or f_*, and $[f, X]$ simply as f_X^* or f^*. Thus $f_* : [\alpha] \mapsto [f \circ \alpha]$ for all $\alpha : Z \to X$ and $f^* : [\beta] \mapsto [\beta \circ f]$ for all $\beta : Y \to Z$.

Proposition 6.5. *Let X, Y and Z be pointed spaces.*

1. *There is a bijection $\theta : [X \vee Y, Z] \to [X, Z] \times [Y, Z]$ given by $\theta : [\lambda] \mapsto ([\lambda \circ i_X], [\lambda \circ i_Y])$, where $i_X : X \to X \vee Y$ and $i_Y : Y \to X \vee Y$ are inclusion maps.*

2. *There is a bijection $\gamma : [X, Y \times Z] \to [X, Y] \times [X, Z]$ given by $\gamma : [\lambda] \mapsto ([p_Y \circ \lambda], [p_Z \circ \lambda])$, where $p_Y : Y \times Z \to Y$ and $p_Z : Y \times Z \to Z$ are projection maps.*

Proposition 6.6. *Let X, Y, Z be pointed spaces. If Y is locally compact Hausdorff, then the reduced association map $\overline{\alpha} : \operatorname{Map}_*(X \wedge Y, Z) \to \operatorname{Map}_*(X, \operatorname{Map}_*(Y, Z))$ induces a bijection $\overline{\alpha}_* : [X \wedge Y, Z] \to [X, \operatorname{Map}_*(Y, Z)]$.*

7 Homotopy Equivalences and Contractible Spaces

Definition 7.1. Let X and Y be topological spaces. A map $f : X \to Y$ is called a **homotopy equivalence** if there is a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. The map g is called a **homotopy inverse** of f. A space X is **homotopy equivalent** to Y if we can find a homotopy equivalence $f : X \to Y$. In this case, we say that X has the **same homotopy type** as Y, and we write $X \simeq Y$, or $X \simeq_f Y$. Definitions remain the same if X and Y are pointed spaces and the word "homotopy" is replaced by "pointed homotopy".

Definition 7.2. A space X is **contractible** if the identity map $\text{id}_X : X \to X$ is homotopic to a constant map on X, namely, id_X is null-homotopic.

Proposition 7.3. *Any two maps from any arbitrary space to a contractible space, or from a contractible space to any arbitrary space, are homotopic, and in particular, are null-homotopic.*

Corollary 7.4. *If Y is a contractible space, then any two maps on Y are homotopic. In particular, any two constant maps on Y are homotopic, so the identity map id_Y is homotopic to any constant map on Y.*

Proposition 7.5. *A space X is contractible if and only if X is homotopy equivalent to the one-point space.*

Corollary 7.6. *Any two contractible spaces have the same homotopy type. If X and Y are contractible spaces, then any map $f : X \to Y$ is a homotopy equivalence.*

8 The Fundamental Group

Definition 8.1. Let X be a topological space and let $\lambda : I \to X$ and $\mu : I \to X$ be two paths in X with $\lambda(1) = \mu(0)$. The product $\lambda * \mu : I \to X$ is defined by

$$(\lambda * \mu)(t) = \begin{cases} \lambda(2t) & 0 \leq t \leq 1/2 \\ \mu(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

Clearly the product $\lambda * \mu$ is also a path in X.

Let $\lambda, \mu : I \to X$ be two paths in X. λ and μ are briefly said to be **homotopic**, denoted by $\lambda \simeq \mu$, if λ is homotopic to μ relative to $\partial I = \{0, 1\}$. Note that if $\lambda \simeq \mu$ then $\lambda(0) = \mu(0)$ and $\lambda(1) = \mu(1)$.

Lemma 8.2. *Let $\lambda_0, \lambda_1, \mu_0, \mu_1$ be paths in a topological space X with $\lambda_0(1) = \mu_0(0)$ and $\lambda_1(1) = \mu_1(0)$. If $\lambda_0 \simeq \lambda_1$ and $\mu_0 \simeq \mu_1$, then $\lambda_0 * \mu_0 \simeq \lambda_1 * \mu_1$.*

Lemma 8.3. *Suppose that $\lambda_0, \lambda_1, \lambda_2$ are paths in X with $\lambda_0(1) = \lambda_1(0)$ and $\lambda_1(1) = \lambda_2(0)$. Then $(\lambda_0 * \lambda_1) * \lambda_2 \simeq \lambda_0 * (\lambda_1 * \lambda_2)$.*

For each $x \in X$, we define a constant path $\epsilon_x : I \to X$ by $\epsilon_x(t) = x$.

Lemma 8.4. *Let λ be a path in X with $\lambda(0) = x$ and $\lambda(1) = y$. Then $\epsilon_x * \lambda \simeq \lambda \simeq \lambda * \epsilon_y$.*

Given a path λ in X, the inverse of λ, denoted by λ^{-1}, is defined by $\lambda^{-1} : I \to X$, $\lambda^{-1}(t) = \lambda(1 - t)$. Clearly the inverse of a path is itself a path.

Lemma 8.5. *Let λ be a path in X with $\lambda(0) = x$ and $\lambda(1) = y$. Then $\lambda * \lambda^{-1} \simeq \epsilon_x$ and $\lambda^{-1} * \lambda \simeq \epsilon_y$.*

Let X be a pointed space with base point x_0. By Lemmas 8.2-8.5, the set of homotopy classes of paths in X having start and end points x_0 is a group with identity $[\epsilon_{x_0}]$ under the well-defined multiplication given by

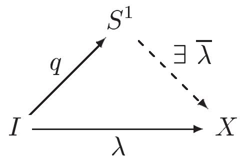
$$[\lambda] \cdot [\mu] = [\lambda * \mu].$$

Definition 8.6. Let X be a pointed space with base point x_0. The n^{th}**-homotopy group** $\pi_n(X, x_0)$ is defined by

$$\pi_n(X, x_0) = [S^n, X]$$

for $n \geq 0$. If there is no danger of confusion, we may omit mentioning the base point and simply write $\pi_n(X)$.

Since $S^1 \cong I/\partial I$, a path $\lambda : I \to X$ with start and end points x_0 factors uniquely into:



where $q : I \to S^1$ is the canonical quotient. We have just shown that $\pi_1(X, x_0) = [S^1, X]$ is a group. This is known as the **fundamental group** of X with base point x_0.

Corollary 8.7. *A map $f : X \to Y$ induces a group homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ given by $f_* : [\lambda] \mapsto [f \circ \lambda]$, where we regard X as having base point x_0 and Y as having base point $f(x)$ when we pass to the homotopy group. If $f, g : X \to Y$ and $f \simeq g \operatorname{rel} x$, then $f_* = g_*$. If X and Y are pointed spaces which are pointed homotopy equivalent, we have $\pi_1(X) \cong \pi_1(Y)$ as groups.*

It is known that $\pi_n(X)$ is an abelian group for all $n \geq 2$.

Proposition 8.8. *Let $x, y \in X$. If there is a path from x to y in X, then the groups $\pi_n(X, x) \cong \pi_n(X, y)$ as groups.*

By Proposition 6.5(2), we have $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$ as sets for any $n \geq 0$.

Definition 8.9. A pointed space X is said to be **n-connected** if $\pi_m(X) = 0$ for all $0 \leq m \leq n$. A pointed space X is said to be **simply connected** if it is 1-connected.

If a space X is simply connected, then for any $x, y \in X$, any two paths from x to y are homotopic.

Proposition 8.10. *A contractible space X is n-connected for all $n \geq 0$.*

9 The Fundamental Group of S^1

We define a map $\mathbf{e} : \mathbb{R} \to S^1 \subseteq \mathbb{C}$ by $\mathbf{e}(t) = \exp(2\,\mathrm{i}\,\pi t)$. We observe that $\mathbf{e}$ is continuous and that $\mathbf{e}|_{(-1/2,1/2)}$ is a homeomorphism from $(-1/2, 1/2)$ onto $S^1 \setminus \{\exp(\mathrm{i}\,\pi)\}$. Let

$$\log : S^1 \setminus \{\exp(\mathrm{i}\,\pi)\} \to (-1/2, 1/2)$$

be the inverse of $\mathbf{e}|_{(-1/2,1/2)}$.

Definition 9.1. A subset $X \subseteq \mathbb{R}^n$ is said to be **starlike from a point** x_0 if whenever $x \in X$, the closed segment $[x_0, x]$ from x_0 to x lies in X.

Lemma 9.2. *Let $X \subseteq \mathbb{R}^n$ be compact and starlike from a point $x_0 \in X$. Then given any map $f : X \to S^1$ and any $t_0 \in \mathbb{R}$ such that $\mathbf{e}(t_0) = f(x_0)$, there exists a map $\overline{f} : X \to \mathbb{R}$ such that $\overline{f}(x_0) = t_0$ and $\mathbf{e} \circ \overline{f} = f$.*

Lemma 9.3. *Let X be a connected subspace of $\mathbb{R}^n$ and let $f, g : X \to \mathbb{R}$ be maps such that $\mathbf{e} \circ f = \mathbf{e} \circ g$ and $f(x_0) = g(x_0)$ for some $x_0 \in X$. Then $f = g$.*

Let $\alpha : I \to S^1$ be a closed path at $1 \in S^1$. Since I is compact, connected and starlike from $0 \in I$ and $\alpha(0) = \alpha(1) = 1$, it follows from Lemmas 9.2-9.3 that there exists a unique lifting $\overline{\alpha} : I \to \mathbb{R}$ such that $\overline{\alpha}(0) = 0$ and $\mathbf{e} \circ \overline{\alpha} = \alpha$. Since $(\mathbf{e} \circ \overline{\alpha})(1) = \alpha(1) = 1$, it follows that $\overline{\alpha}(1)$ is an integer. We define the degree of α by

$$\deg(\alpha) = \overline{\alpha}(1).$$

Lemma 9.4. *Let $\alpha, \beta : I \to S^1$ be homotopic closed paths at $1 \in S^1$. Then $\deg(\alpha) = \deg(\beta)$.*

It follows that there is a well-defined function $\deg : \pi_1(S^1, 1) \to \mathbb{Z}$ defined by

$$\deg([\alpha]) = \overline{\alpha}(1).$$

Theorem 9.5. *The function $\deg : \pi_1(S^1, 1) \to \mathbb{Z}$ is a group isomorphism.*

10 Free Groups and Free Products of Groups With Amalagamation

For a nonempty set X, denote by X^{-1} a set disjoint from X with the property that there is a bijection $X \to X^{-1}$ in which we associate every element $x \in X$ with a corresponding element (called its **inverse**) in X^{-1} that we label as x^{-1}. It is convenient to use the same notation $y \mapsto y^{-1}$ for the inverse bijection $X^{-1} \to X$; in particular we have $(x^{-1})^{-1} = x$ for all $x \in X \cup X^{-1}$. Hence we may denote the inverse of an element $x \in X \cup X^{-1}$ unambiguously as x'.

A **word** in X is a finite product $x_1 x_2 ... x_n$, where $x_i \in X \cup X^{-1}$; in the case that $n = 0$, the word w is the **empty word** which is written simply as 1, where we may regard 1 as an element disjoint from $X \cup X^{-1}$. The **product** of two words is defined by juxtaposition, namely, if $w = x_1 x_2 ... x_n$ and $v = y_1 y_2 ... y_m$, then

$$wv = x_1 x_2 ... x_n y_1 y_2 ... y_m,$$

with the convention that $w1 = w = 1w$ for all words w. A word w in X is said to be **reduced** if no pair of elements x and x' are adjacent. By convention, the empty word 1 is reduced.

For a nonempty set X, we let $\mathbb{X}$ denote the set of all words in X. Define a relation $\simeq$ on $\mathbb{X}$ by the following. Two words w and v in X are said to be **equivalent** (written $w \simeq v$) if it is possible to pass from w to v by a finite sequence of operations of the following type:

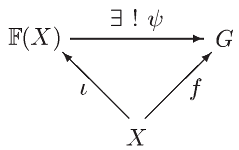
1. insertion of xx', where $x \in X \cup X^{-1}$, as a block of two consecutive elements;
2. deletion of such xx', with the additional rule that when they are the only elements left, they must be replaced by the element 1.

The reader will find it a straightforward exercise to check that $\simeq$ is an equivalence relation on $\mathbb{X}$. For a word w in X, we denote by $[w]$ the equivalence class containing w under $\simeq$.

Proposition 10.1. *Every word is equivalent to a reduced word and every equivalence class $[w]$ of words in X contains a unique reduced word. Let $\mathbb{F}(X) = \mathbb{X}/\simeq$ be the set of all equivalence classes of words in X. We can define a binary operation on $\mathbb{F}(X)$ by setting $[w][v] = [wv]$. Then $\mathbb{F}(X)$ is a group under this operation.*

Definition 10.2. The group $\mathbb{F}(X)$ is known as the **free group on the set** X.

Theorem 10.3. *Let X be a nonempty set, $\mathbb{F}(X)$ be the free group on X, and $\iota : X \to \mathbb{F}(X)$ be the canonical inclusion $x \mapsto [x]$. The given any group G and set function $f : X \to G$, there exists a unique group homomorphism $\psi : \mathbb{F}(X) \to G$ such that $\psi \circ \iota = f$. Hence, $(\mathbb{F}(X), \iota)$ is free in the category* GRP *of groups, and so this* **universal property** *determines the group $\mathbb{F}(X)$ uniquely up to isomorphism with respect to the given set X.*



Definition 10.4. Let X be a subset of a group G that does not contain the identity 1_G. We say that G **is free on** X if every nonidentity element $g \in G$ has a unique expression of the form $g = x_1^{n_1} x_2^{n_2} ... x_k^{n_k}$, where $x_i \in X$, $n_j = \mathbb{Z} \setminus \{0\}$, and $x_i \neq x_{i+1}$ for each i. We call such an expression of $g \in G$ a **normal form of** g **with respect to** X. Thus G is free on X if and only if every nonidentity $g \in G$ has a unique normal form with respect to X.

Proposition 10.5. *Suppose that X is a subset of a group G that does not contain the identity 1_G. Then there is a group homomorphism $\psi : \mathbb{F}(X) \to G$ such that ψ is an isomorphism if and only if G is free on X.*

Corollary 10.6. *Every group is a homomorphic image of a free group.*

Proposition 10.7. *Suppose that X_1 and X_2 are nonempty sets for which $|X_1| = |X_2|$. Then we have the isomorphism $\mathbb{F}(X_1) \cong \mathbb{F}(X_2)$.*

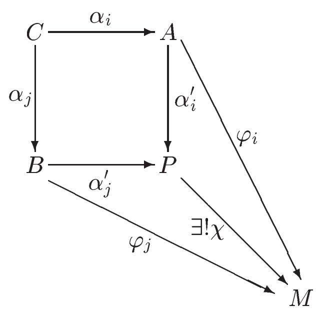
Definition 10.8. Let $\mathcal{C}$ be a category and fix an object C in $\mathcal{C}$. Let $\Phi = \{\alpha_i : C \to A_i\}_{i\in I}$ be a family of morphisms. A **pushout** for Φ consists of an object P together with a family of morphisms $\{\alpha'_i : A_i \to P\}_{i\in I}$ that make each of the following squares commute

$$\begin{array}{ccc} C & \xrightarrow{\alpha_i} & A_i \\ {\scriptstyle\alpha_j}\big\downarrow & & \big\downarrow{\scriptstyle\alpha'_i} \\ A_j & \xrightarrow[\alpha'_j]{} & P \end{array}$$

and which satisfy the property that whenever there is another object M and a family of morphisms $\{\varphi_i : A_i \to M\}_{i\in I}$ that make each of the following squares (with the same morphisms α_i) commute

$$\begin{array}{ccc} C & \xrightarrow{\alpha_i} & A_i \\ {\scriptstyle\alpha_j}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_i} \\ A_j & \xrightarrow[\varphi_j]{} & M \end{array}$$

then there exists a **unique** morphism $\chi : P \to M$ such that $\chi \circ \alpha'_i = \varphi_i$ for all $i \in I$.



Theorem 10.9. *If in a category $\mathcal{C}$, $\{P, \{\alpha'_i\}_{i\in I}\}$ and $\{P', \{\alpha''_i\}_{i\in I}\}$ are pushouts for the family of morphisms $\Phi = \{\alpha_i : C \to A_i\}_{i\in I}$, then P and P' are isomorphic.*

Let $\{G_i\}_{i\in I}$ be a family of groups such that all their identity elements are identified and there is a group H such that $G_i \cap G_j = H$ whenever $i \neq j$. Label the common identity element of all the groups as 1 (which is equal to 1_H). Let $X = \bigcup_{i\in I} G_i$, and let $\mathbb{X}$ be the set of all nonempty words in X. We say that a word is **reduced** if it is either the identity 1, or none of its symbols is the identity and no two adjacent symbols belong to the same group.

1. Define a relation $\sim$ on $\mathbb{X}$ by the following rule: Two words w, v are equivalent (written $w \sim v$) if it is possible to pass from w to v by a finite sequence of operations of the following type:

   (a) insertion of 1;

   (b) deletion of 1 except when it is the only symbol left;

   (c) replacing an element $g \in G_i$ with a pair x_1x_2, where $x_1, x_2 \in G_i$;

   (d) deleting a pair x_1x_2, where $x_1, x_2 \in G_i$, and replacing it with g, where $g = x_1x_2$.

   Then $\sim$ is an equivalence relation on $\mathbb{X}$, and each equivalence class of words under $\sim$ contains a (**not** necessarily unique) reduced word that is either an element of H, or a reduced word of the form $g_{\lambda_1}g_{\lambda_2}...g_{\lambda_k}$, where $g_{\lambda_i} \in G_{\lambda_i} \setminus H$ and $\lambda_i \neq \lambda_{i+1}$ for each i whenever $k \geq 2$. Denote the equivalence class containing the word w as $[w]$.

2. Similar to the construction of free products, there exists a group whose elements consist of the equivalence classes of $\mathbb{X}$ under $\sim$ in which the group operation is given by $[w][v] = [wv]$ — we term this the **free product amalgamating** H of the groups $\{G_i\}_{i\in I}$, and denote it by $\coprod_{i\in I}{}_H G_i$.
3. The inclusion functions $\sigma_i : g \mapsto [g]$ are group monomorphisms from G_i into $\coprod_{i\in I}{}_H G_i$ for each i, and the group $\coprod_{i\in I}{}_H G_i$ together with the inclusion maps σ_i forms a pushout for the family of canonical inclusions $\{\iota_i : H \to G_i\}_{i\in I}$ in the category GRP of groups.

Examples 10.10.

1. If K is the trivial group, $G \coprod_* K = G$.
2. If $f : H \to G$ is a group homomorphism and K is the trivial group, $G \coprod_H K$ is the quotient group of G by the normal subgroup generated by $f(H)$.
3. $\mathbb{Z} \coprod_* \mathbb{Z}$ is the free group generated by two elements, ie, $\mathbb{F}(x_1, x_2)$. In general, the n**-fold free product** of $\mathbb{Z}$ is a free group of rank n.
4. $\mathbb{Z}/m \coprod_* \mathbb{Z}/n$ is the quotient group of $\mathbb{F}(x_1, x_2)$ by the relations $x_1^m = x_2^n = 1$.

11 The Seifert-Van Kampen Theorem

Theorem 11.1. *Let X be a topological space. Suppose $X = U_1 \cup U_2$ with U_1, U_2 open, $U_1 \cap U_2$ nonempty and path-connected. Let $x_0 \in U_1 \cap U_2$ be the base point of X. Then*

$$\pi_1(X, x_0) = \pi_1(U_1, x_0) \coprod_{\pi_1(U_1 \cap U_2, x_0)} \pi_1(U_2, x_0).$$

Examples 11.2.

1. Suppose $X = U \cup V$ with U, V open in X and simply connected, and $U \cap V$ is nonempty and path-connected. Then X is simply connected.
2. S^n is simply connected for $n \geq 2$. In particular, $\pi_1(S^n)$ is trivial for $n \geq 2$.
3. Suppose that $x_0 \in X$ and $y_0 \in Y$ are base points of X, Y respectively such that each base point is contained in a contractible neighbourhood. Then $\pi_1(X \vee Y) = \pi_1(X) \coprod_* \pi_1(Y)$.
4. $\pi_1(S^1 \vee S^1) = \mathbb{F}(x_1, x_2)$. In general $\pi_1(\vee^n S^1) = \mathbb{F}(x_1, x_2, ..., x_n)$.
5. $\pi_1(\mathbb{R}P^1) = \mathbb{Z}$ and $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$ for $n \geq 2$.

Lemma 11.3. *Let $\phi : \mathbb{F}(x_1, x_2, ..., x_m) \to \mathbb{F}(y_1, y_2, ..., y_n)$ be any group homomorphism. Then there is a continuous map*

$$f : \bigvee^m S^1 \to \bigvee^n S^1$$

such that $\phi = f_* : \pi_1(\bigvee^m S^1) \to \pi_1(\bigvee^n S^1)$.

Theorem 11.4. *For all groups G, there is a space $X = X(G)$ such that $\pi_1(X) = G$. If $\phi : G \to H$ is a group homomorphism, then there is a natural continuous map $f : X(G) \to X(H)$ such that*

$$\phi = f_* : \pi_1(X(G)) \to \pi_1(X(H)).$$

12 Deformations, Cofibrations, Fibrations

Definition 12.1. A subspace A of X is said to be a **retract** of X if the inclusion $i : A \hookrightarrow X$ has a left inverse, that is, there is a map $r : X \to A$ such that $r \circ i = \mathrm{id}_A$.

A subspace A of X is said to be a **weak retract** of X if the inclusion $i : A \hookrightarrow X$ has a left homotopy inverse, that is, there is a map $r : X \to A$ such that $r \circ i \simeq \mathrm{id}_A$.

Definition 12.2. Given a subspace X' of X, a **deformation** of X' in X is a homotopy $D : X' \times I \to X$ satisfying $D(x', 0) = x'$ for all $x' \in X'$, that is, D is a homotopy from the inclusion $i : X' \hookrightarrow X$ to some other map $X' \to X$. If $D(X' \times \{1\}) \subseteq A \subseteq X$, then D is said to be a **deformation of** X' **into** A, and the subspace X' is said to be **deformable into** A **in** X.

A space X is said to be deformable into its subspace A if X is deformable into A in X. In particular, a space X is contractible if and only if X is deformable into one of its points in X.

Proposition 12.3. *A space X is deformable into a subspace A if and only if the inclusion $i : A \hookrightarrow X$ has a right homotopy inverse, that is, there is a map $h : X \to A$ such that $h = i \circ h \simeq \mathrm{id}_X$.*

Definition 12.4. A subspace A of X is said to be a **weak deformation retract** of X if the inclusion $i : A \hookrightarrow X$ is a homotopy equivalence.

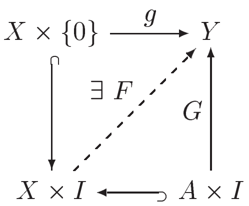
Hence, a subspace A of X is a weak deformation retract of X if and only if it is A is weak retract of X and X is deformable into A in X.

Definition 12.5. A subspace A of X is called a **deformation retract** of X if there exists a map $h : X \to A$ such that $h \circ i = \mathrm{id}_A$ and $i \circ h \simeq \mathrm{id}_X$, where $i : A \hookrightarrow X$ is the inclusion.

Hence, a subspace A of X is a deformation retract of X if and only if it is A is retract of X and X is deformable into A in X.

Definition 12.6. A subspace A of X is a **strong deformation retract** of X if there is a retraction $r : X \to A$ such that $i \circ r \simeq \mathrm{id}_X \operatorname{rel} A$, where $i : A \hookrightarrow X$ is the inclusion.

Definition 12.7. Let (X, A) be a pair of spaces, and let Y be any space. The pair (X, A) is said to have the **homotopy extension property** with respect to Y if for all maps $g : X \to Y$ and all maps $G : A \times I \to Y$ satisfying $G(a, 0) = g(a)$ for all $a \in A$, there exists a map $F : X \times I \to Y$ such that $F(x, 0) = g(x)$ for all $x \in X$ and $F(a, t) = G(a, t)$ for all $a \in A$ and $t \in [0, 1]$.



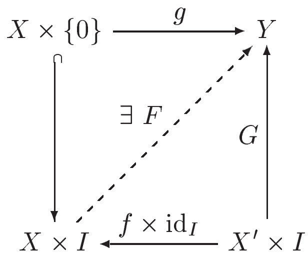
Proposition 12.8. *Suppose that (X, A) has the homotopy extension property with respect to Y, and $f_0, f_1 : A \to Y$ are homotopic. If f_0 has an extension to X, then so does f_1, and their respective extensions are homotopic as well.*

Proposition 12.9. *Let (X, A) be a pair of spaces. If A is contractible and the pair (X, A) satisfies the homotopy extension property with respect to X, then the canonical quotient $X \to X/A$ is a homotopy equivalence.*

Proposition 12.10. *If (X, A) has the homotopy extension property with respect to A, then A is a weak retract (resp. weak deformation retract) of X if and only if A is a retract (resp. deformation retract) of X.*

The notion of a cofibration is a generalization of the notion of having a homotopy extension property.

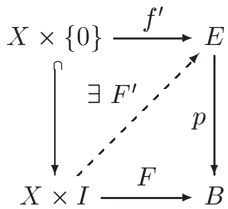
Definition 12.11. A map $f : X' \to X$ between two spaces is said to be a **cofibration** if for all spaces Y and all maps $g : X \to Y$, $G : X' \times I \to Y$ satisfying $G(x', 0) = g(f(x'))$ for all $x' \in X'$, there exists a map $F : X \times I \to Y$ such that $F(x, 0) = g(x)$ for all $x \in X$ and $F(f(x'), t) = G(x', t)$ for all $x' \in X'$ and $t \in [0, 1]$.



Proposition 12.12. *Let $i : A \hookrightarrow X$ be the inclusion. The following are equivalent:*

1. *i is a cofibration.*
2. *(X, A) has the homotopy extension property with respect to any space Y.*
3. *$(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$.*

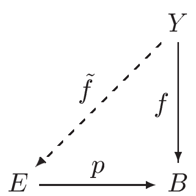
Definition 12.13. A map $p : E \to B$ is said to have the **homotopy lifting property** with respect to a space X if for all maps $f' : X \to E$ and $F : X \times I \to B$ satisfying $F(x, 0) = p(f'(x))$ for all $x \in X$, there exists a map $F' : X \times I \to E$ such that $F'(x, 0) = f'(x)$ for all $x \in X$ and $p \circ F' = F$.



Proposition 12.14. *If $p : E \to B$ has the homotopy lifting property with respect to a space X, and $f_0, f_1 : X \to B$ are homotopic, then f_0 can be lifted to E (namely, there exists $F_0 : X \to E$ such that $p \circ F_0 = f_0$) if and only if f_1 can be lifted to E.*

Definition 12.15. A map $p : E \to B$ is called a **fibration** if p has the homotopy lifting property with respect to every space X. In this case we say that E is the **total space** and B is the **base space** of the fibration. For $b \in B$, the set $p^{-1}(b)$ is called the **fibre over** b.

If $p : E \to B$ and $f : Y \to B$ are maps, a **lifting** of f is a map $\tilde{f} : Y \to E$ such that $p \circ \tilde{f} = f$.



Proposition 12.16. *If $p : E \to B$ is a fibration, any path ω in B satisfying $\omega(0) \in p(E)$ can be lifted to a path in E.*

Definition 12.17. A map $p : E \to B$ is said to have **unique path-lifting** if given paths λ, λ' in E such that $p \circ \lambda = p \circ \lambda'$ and $\lambda(0) = \lambda'(0)$, then $\lambda = \lambda'$.

Lemma 12.18. *If a map $p : E \to B$ has unique path-lifting, then it has the* **unique lifting property** *for path-connected spaces, in the sense that for any maps $f, g : Y \to E$ where Y is path-connected, $p \circ f = p \circ g$, and $f(y_0) = g(y_0)$ for some $y_0 \in Y$, then $f = g$.*

Proposition 12.19. *A fibration $p : E \to B$ has unique path lifting if and only if every fibre $p^{-1}(b)$ has no nonconstant paths.*

Proposition 12.20. *The composite of two fibrations is a fibration.*

Lemma 12.21. *Let $p : E \to B$ be a fibration. If A is any path-connected component of E, then $p(A)$ is a path-connected component of B, and $p|_A : A \to p(A)$ is a fibration.*

Definition 12.22. A space is said to be **locally path-connected** if for any $x \in X$ and any neighbourhood U of x, there exists a path-connected open neighbourhood V of x such that $V \subseteq U$.

Remark 12.23. In Maunder [3] (Chapter 6, Exercise 23), the definition of locally path-connected is as follows: A space X is said to be locally path-connected if, for each $x \in X$ and any neighborhood U of x, there is an open neighborhood V of x such that $x \in V \subseteq U$ and any two points in V can be connected by a path in U. Thus our definition of locally path-connected, which is the definition given by Hatcher [1] and Massey [2], is stronger than Maunder's definition.

Proposition 12.24. *Let $p : E \to B$ be a map. If E is locally path-connected, then p is a fibration if and only if for each path-connected component A of E, $p(A)$ is a path-connected component of B, and $p|_A : A \to p(A)$ is a fibration.*

Theorem 12.25. *Let $p : E \to B$ be a fibration with unique path-lifting. If λ, λ' are paths in E such that $\lambda(0) = \lambda'(0)$ and $p \circ \lambda \simeq p \circ \lambda' \operatorname{rel} \partial I$, then $\lambda \simeq \lambda' \operatorname{rel} \partial I$.*

Corollary 12.26. *Let $p : (E, e_0) \to (B, b_0)$ be a fibration with unique path-lifting. Then*

$$p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$$

is a group monomorphism.

Theorem 12.27. *Let $p : E \to B$ be a fibration with unique path-lifting. If B is path-connected, then any two fibres are homeomorphic.*

If B is path-connected and $p : E \to B$ is a fibration with unique path-lifting, the **number of sheets** of p, or the **multiplicity** of p, is defined to be the cardinality of the fibre $p^{-1}(b)$, which is independent of the choice of $b \in B$ by Theorem 12.27.

Theorem 12.28. *Let $p : E \to B$ be a fibration with unique path-lifting and suppose that E, B are path-connected. Then the multiplicity of p is the index of $p_*(\pi_1(E, e_0))$ in $\pi_1(B, p(e_0))$.*

Theorem 12.29. *Let $p : (E, e_0) \to (B, b_0)$ be a fibration with unique path-lifting. Let Y be a connected, locally path-connected space. Then a map $f : (Y, y_0) \to (B, b_0)$ has a lifting $(Y, y_0) \to (E, e_0)$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0))$.*

13 Covering Spaces

Definition 13.1. A map $p : \tilde{X} \to X$ is called a **covering projection** if:

1. p is surjective;
2. For all $x \in X$, there exists an open neighbourhood U of x, called an **elementary neighbourhood** of x, such that
$$p^{-1}(U) = \bigsqcup_{\alpha \in \Lambda} U_\alpha,$$
is a topological disjoint union of open sets U_α called **sheets** and p maps each U_α homeomorphically onto U.

We call $\tilde{X}$ the **covering space** and call X the **base space** of the covering projection p.

Examples 13.2.

1. Any homeomorphism $p : X \to X$ is a one-sheeted covering projection.
2. Let F be a space endowed with the discrete topology, and let $\tilde{X} = X \times F$. Then the coordinate projection $p : \tilde{X} \to X$ is a covering projection.
3. The canonical quotient $p : S^n \to \mathbb{R}P^n$ is a two-sheeted covering projection.
4. The map $p : S^1 \to S^1$ given by $p(z) = z^n$ is an n-sheeted covering projection.
5. The exponential map $\mathbf{e} : \mathbb{R} \to S^1$ is a covering projection with $\aleph_0$-many sheets.

Proposition 13.3. *A covering projection exhibits its base space as a quotient of its covering space.*

Lemma 13.4. *A covering projection $p : \tilde{X} \to X$ has the unique lifting property for connected spaces, namely, if $f, g : Y \to \tilde{X}$ are liftings of the same map $p \circ f = p \circ g : Y \to X$, Y is connected and $f(y_0) = g(y_0)$ for some $y_0 \in Y$, then $f = g$.*

Theorem 13.5. *A covering projection is a fibration.*

Proposition 13.6. *Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering projection.*

1. *Every path $\lambda : (I, 0) \to (X, x_0)$ has a unique lifting $\tilde{\lambda} : (I, 0) \to (\tilde{X}, \tilde{x_0})$. In particular, p has unique path-lifting (Definition 12.17).*
2. *Every map $F : (I \times I, (0,0)) \to (X, x_0)$ has a unique lifting $\tilde{F} : (I \times I, (0,0)) \to (\tilde{X}, \tilde{x_0})$.*

Since a covering projection $p : \tilde{X} \to X$ is a fibration with unique path-lifting, it follows from Theorem 12.25 that if λ, λ' are paths in $\tilde{X}$ such that $\lambda(0) = \lambda'(0)$ and $p \circ \lambda \simeq p \circ \lambda' \operatorname{rel} \partial I$, then $\lambda \simeq \lambda' \operatorname{rel} \partial I$. By Corollary 12.26, $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ is a group monomorphism.

Definition 13.7. A group G is termed a **topological group** if G is a topological space, and the group multiplication $G \times G \to G$, $(g, h) \mapsto gh$, and the inverse function $G \to G$, $g \mapsto g^{-1}$, are both continuous maps.

Let G be a group and let Y be a G-space. For an element $g \in G$ and a subset S of Y, let

$$gS = \{gs \ : \ s \in S\}.$$

Definition 13.8. Let G be a discrete group (a topological group endowed with the discrete topology). Let Y be a G-space. We call the G-action on Y **properly dicontinuous** if for every $y \in Y$, there exists a neighbourhood W_y of y such that for all $g_1, g_2 \in G$,

$$g_1 \neq g_2 \ \Rightarrow \ g_1 W_y \cap g_2 W_y = \emptyset.$$

This is equivalent to saying that for all $g \in G$,

$$g \neq 1_G \ \Rightarrow \ gW_y \cap W_y = \emptyset.$$

Proposition 13.9. *Let X be a G-space. The G-action on X is properly discontinuous if and only if the canonical quotient $p : X \to X/G$ is a covering projection.*

Definition 13.10. A group G is said to act freely on a space X if $gx \neq x$ for all $x \in X$ and $1_G \neq g \in G$. This is equivalent to saying that $g_1 x \neq g_2 x$ for all $x \in X$ and $g_1 \neq g_2 \in G$.

Proposition 13.11. *Let X be a G-space. Suppose that G is a finite group and X is Hausdorff. Then the G-action on X is properly discontinuous if and only if G acts freely on X.*

Examples 13.12.

1. Let an action of $\mathbb{Z}$ on $\mathbb{R}$ be defined by $(n, x) \mapsto n + x$. This action is properly discontinuous, and so the canonical quotient $\mathbf{e} : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ is a covering projection.

2. Let G be a Hausdorff topological group and let H be a finite subgroup of G. Let G/H be the set of left cosets of H in G endowed with the quotient topology. H acts on G by left multiplication and this action is free. Hence the canonical quotient $G \to G/H$ is a covering projection.

Fix a covering projection $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$. Let a loop $\alpha : (I, 0, 1) \to (X, x_0, x_0)$ be given. Suppose that $\tilde{\alpha} : (I, 0) \to (\tilde{X}, \tilde{x_0})$ is its unique lifting. Then

$$p \circ \tilde{\alpha}(1) = \alpha(1) = \alpha(0) = x_0,$$

and so $\tilde{\alpha}(1) \in p^{-1}(x_0)$. There is a well-defined function $\psi : \pi_1(X, x_0) \to p^{-1}(x_0)$ given by $\psi : [\alpha] \to \tilde{\alpha}(1)$.

Proposition 13.13. *If $\tilde{X}$ is path-connected, then ψ is surjective, and if $\tilde{X}$ is simply connected, then ψ is bijective.*

Suppose now that $\tilde{X}$ is a G-space and that the quotient $p : \tilde{X} \to \tilde{X}/G$ is a covering projection. Let $[\tilde{x_0}] \in \tilde{X}/G$ denote the orbit of $\tilde{x_0}$ under the G-action. Since the G-action is properly discontinuous, we can identify $p^{-1}([\tilde{x_0}])$ with G by the correspondence $g\tilde{x_0} \leftrightarrow g$.

Theorem 13.14. *If $\tilde{X}$ is path-connected, then the function $\psi : \pi_1(\tilde{X}/G, [\tilde{x_0}]) \to p^{-1}(x_0) = G$ is a group epimorphism with kernel $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$. In particular, if $\tilde{X}$ is simply connected, then $\pi_1(\tilde{X}/G, [\tilde{x_0}]) \cong G$.*

Examples 13.15.

1. Since S^n is simply connected for all $n \geq 2$ and there is a properly discontinuous $\mathbb{Z}_2$-action on S^n given by $(n, x) \mapsto nx$, where $n = \pm 1$, so $\pi_1(\mathbb{R}P^n) = \pi_1(S^n/\mathbb{Z}_2) \cong \mathbb{Z}_2$ for all $n \geq 2$.
2. Since $\mathbb{R}$ is contractible and there is a properly discontinuous $\mathbb{Z}$-action on $\mathbb{R}$ given by $(n, x) \mapsto n+x$, so we have $\pi_1(S^1) = \pi_1(\mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}$. In this example, ψ as defined above is simply the degree map $\deg : \pi_1(S^1) \to \mathbb{Z}$ defined in Section 9.

Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering. Suppose that (Y, y_0) is simply connected and locally path-connected. Since p is a fibration with unique path-lifting, so by Theorem 12.29 and Lemma 13.4, every map $f : (Y, y_0) \to (X, x_0)$ admits a unique lifting $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$.

Corollary 13.16. *Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering. Then for all $n \geq 1$ and all maps*

$$f : (Y, y_0) \to (X, x_0)$$

with Y simply connected and locally path-connected, there exists some $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ such that $p_* \circ \tilde{f}_* = f_* : \pi_n(Y, y_0) \to \pi_n(X, x_0)$.

Example 13.17. Since S^n is always locally path-connected, and is simply connected for all $n \geq 2$, so all pointed maps $f : S^n \to S^1$ admit a unique lifting $\tilde{f} : S^n \to \mathbb{R}$. Since $\mathbb{R}$ is contractible, so $\pi_n(S^1) = 0$ for all $n \geq 2$.

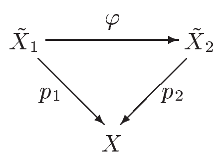
Corollary 13.18. *For all $n \geq 2$ and any covering projection $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$, the map*

$$p_* : \pi_n(\tilde{X}, \tilde{x_0}) \to \pi_n(X, x_0)$$

is a group isomorphism.

Example 13.19. $\pi_m(S^n) \cong \pi_m(\mathbb{R}P^n)$ for all $m \geq 2$ and $n \geq 1$. In particular, $\pi_m(\mathbb{R}P^1) = \pi_m(S^1) = 0$ for all $m \geq 2$.

Definition 13.20. Let $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ be covering projections. A **homomorphism** of $(\tilde{X}_1, p_1)$ to $(\tilde{X}_2, p_2)$ is a continuous map $\varphi : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ \varphi = p_1$.



Fix a space X. The class of all covering projections with base space X and their homomorphisms form a category. Two covering projections with the same base space X are called **isomorphic** if they are isomorphic objects in this category.

Theorem 13.21. *Two covering projections $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ with the same base space X and where $\tilde{X}_1$, $\tilde{X}_2$ are path-connected and locally path-connected are isomorphic if and only if for any points $\tilde{x}_1 \in \tilde{X}_1$ and $\tilde{x}_2 \in \tilde{X}_2$ such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x_0 \in X$, the subgroups $(p_1)_*(\pi_1(\tilde{X}_1, \tilde{x}_1))$ and $(p_2)_*(\pi_2(\tilde{X}_2, \tilde{x}_2))$ of $\pi_1(X, x_0)$ are conjugate.*

Lemma 13.22. *Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be covering projections with the same base space X where $\tilde{X}_1$, $\tilde{X}_2$ are path-connected and locally path-connected. Let $\varphi : (\tilde{X}_1, p_1) \to (\tilde{X}_2, p_2)$ be a homomorphism. Then $\varphi : \tilde{X}_1 \to \tilde{X}_2$ is a covering projection.*

Definition 13.23. A covering projection $p : \tilde{X} \to X$ is termed **universal** if X is path-connected and $\tilde{X}$ is simply connected.

Let $p : \tilde{X} \to X$ be a covering projection, with $\tilde{X}$ simply connected and locally path-connected. If $(\tilde{X}', p')$ is any other covering space of X, then there exists a homomorphism φ of $(\tilde{X}, p)$ onto $(\tilde{X}', p')$, and by Lemma 13.22, $\varphi : \tilde{X} \to \tilde{X}'$ is a covering projection. In other words, $\tilde{X}$ can serve as a covering space of any covering space of X.

Definition 13.24. A space X is called **semi-locally simply connected** if for each point $x \in X$, there exists a neighborhood U of x such that $i_* : \pi_1(U, x) \to \pi_1(X, x)$ is the trivial group homomorphism, where $i : U \hookrightarrow X$ is the inclusion map. is trivial.

Theorem 13.25. *Let X be path-connected, locally path-connected, and semi-locally simply connected. Then there is a universal covering $\tilde{X}$ of X.*

Corollary 13.26. *Let X be path-connected, locally path-connected, and semi-locally simply connected. Then for all subgroups H of $\pi_1(X, x_0)$, there exists a covering projection $p_H : \tilde{X}_H \to X$ such that $(p_H)_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$ for a suitably chosen base point $\tilde{x}_0 \in \tilde{X}_H$.*

14 Barratt-Puppe Exact Sequences

Let S, T be pointed sets with base points s_0, t_0 respectively. Let $f : S \to T$ be a pointed function. Set $\mathrm{Ker}(f) = f^{-1}(t_0)$. A sequence of pointed sets

$$\dots \longrightarrow S_{n+1} \xrightarrow{f_{n+1}} S_n \xrightarrow{f_n} S_{n-1} \longrightarrow \dots$$

is set to be **exact** if each function f_n is pointed and $\mathrm{Ker}(f_n) = \mathrm{Im}(f_{n+1})$ for all n.

For a pointed space (X, x_0), the **reduced cone** is defined by $C(X) = (X \times I)/((\{x_0\} \times I) \cup (X \times \{1\}))$. We identify X as a subspace of $C(X)$ via the correspondence $x \leftrightarrow (x, 0)$.

Let $f : (X, x_0) \to (Y, y_0)$ be a pointed map. The **reduced mapping cone** is the adjunction space $Y \cup_f C(X) = (C(X) \sqcup Y)/((x, 0) \sim f(x) \ : \ x \in X)$ and is denoted by C_f.

Let (Y, y_0) be a pointed space. The **path space** $P(Y)$ is defined to be the space of all paths λ in Y satisfying $\lambda(0) = y_0$ under the compact-open topology. Let $f : (X, x_0) \to (Y, y_0)$ be a pointed map. The **mapping path space** P_f is defined by

$$P_f = \{(x, \lambda) \in X \times P(Y) \ : \ f(x) = \lambda(1)\}.$$

Theorem 14.1 (Barratt-Puppe). *Let $f : (X, x_0) \to (Y, y_0)$ be a pointed map. Let $j : Y \to C_f$ be the canonical inclusion, $p : P_f \to X$ be the canonical quotient, and let Z be any pointed space. Then there are long exact sequences*

$$\dots \longrightarrow [\Sigma(C_f), Z] \xrightarrow{(\Sigma j)^*} [\Sigma(Y), Z] \xrightarrow{(\Sigma f)^*} [\Sigma(X), Z] \longrightarrow [C_f, Z] \xrightarrow{j^*} [Y, Z] \xrightarrow{f^*} [X, Z]$$

and

$$\dots \longrightarrow [Z, \Omega(P_f)] \xrightarrow{(\Sigma p)_*} [Z, \Omega(X)] \xrightarrow{(\Omega f)_*} [Z, \Omega(Y)] \longrightarrow [Z, P_f] \xrightarrow{p_*} [Z, X] \xrightarrow{f_*} [Z, Y].$$

15 CW-Complexes

Definition 15.1. Let X^0 be a discrete space whose points are called 0-cells. Inductively form X^n from X^{n-1} by attaching n**-cells** as follows: Let X^n be the adjunction space

$$X^n = X^{n-1} \bigcup_{\varphi_\alpha \,:\, \alpha\in\Lambda_n} \left(\bigsqcup_{\alpha\in\Lambda_n} D^n_\alpha \right) = \left(\left(\bigsqcup_{\alpha\in\Lambda_n} D^n_\alpha \right) \bigsqcup X^{n-1} \right) / (s \sim \varphi_\alpha(s) \ : \ s \in S^{n-1}_\alpha, \alpha \in \Lambda_n),$$

where $\{D^n_\alpha\}_{\alpha\in\Lambda_n}$ is a collection of discs and $\{\varphi_\alpha : S^{n-1}_\alpha \to X^{n-1}\}_{\alpha\in\Lambda_n}$ is a collection of maps for each n, where we canonically identify $S^{n-1}_\alpha = \partial(D^n_\alpha)$ for each $\alpha \in \Lambda_n$. Thus as a set, X^n is the disjoint union of X^{n-1} and $\bigsqcup_{\alpha\in\Lambda_n} e^n_\alpha$, where each e^n_α is an open disc called an n-cell. Let $X = \bigcup_n X^n$ be given the weak topology: a subset A of X is closed if and only if $A \cap X^n$ is closed in X^n for all $n \geq 0$. We call X a **CW-complex**.

In the special case where the cells being attached have a maximum finite dimension, we call X a finite-dimensional CW-complex.

Examples 15.2.

1. S^n has the structure of a CW-complex with just two cells e^0 and e^n. The n-cell e^n is attached via the constant map $S^{n-1} \to e^0$.

2. $\mathbb{R}P^n$ has the cell structure $e^0 \sqcup e^1 \sqcup ... \sqcup e^n$.

An alternative description of CW-complexes is as follows: A CW-complex is a Hausdorff space X together with an indexing set Λ_n for all $n \geq 0$ and **characteristic maps** $\phi^n_\alpha : D^n \to X$ for all $\alpha \in \Lambda_n$, such that the following properties are satisfied, where $e^n = (D^n)^\circ$ for all $n \geq 1$:

1. $X = \bigcup \phi^n_\alpha(e^n)$, the union being taken over all $n \geq 0$ and $\alpha \in \Lambda_n$ (set $e^0 = D^0 = \{*\}$).

2. $\phi^n_\alpha(e^n) \cap \phi^m_\beta(e^m) = \emptyset$ unless $n = m$ and $\alpha = \beta$.

3. $\phi^n_\alpha|_{e^n}$ is injective for all $n \geq 0$ and $\alpha \in \Lambda_n$.

4. Let $X^n = \bigcup_{m=0}^n \phi^m_\alpha(e^m)$, where the union is taken over all $\alpha \in \Lambda_m$ for $0 \leq m \leq n$. Then $\phi^n_\alpha(S^{n-1}) \subseteq X^{n-1}$ for each $n \geq 1$ and $\alpha \in \Lambda_n$.

5. A subset K of X is closed in X if and only if $(\phi^n_\alpha)^{-1}(K)$ is closed in D^n for each $n \geq 0$ and $\alpha \in \Lambda_n$.

6. For each $n \geq 0$ and $\alpha \in \Lambda_n$, $\phi^n_\alpha(D^n)$ is contained in the union of a finite number of sets of the form $\phi^m_\beta(e^m)$.

It is known that the above definition is equivalent to Definition 15.1. Condition 6 is known as **closure-finiteness**, which is equivalent to saying that the closure of each cell (Definition 15.1) is contained in the union of a finite number of cells.

Definition 15.3. A **subcomplex** of a CW-complex X is a closed subspace A of X that is a union of cells of X.

Proposition 15.4. *A wedge of CW-complexes is again a CW-complex.*

For a CW-complex X, let the **skeleton** $\mathrm{sk}_n(X)$ be the subspace of X consisting of the cells up to dimension n.

Proposition 15.5. *For each $n \geq 1$, the inclusion $\mathrm{sk}_{n-1}(X) \hookrightarrow \mathrm{sk}_n(X)$ is a cofibration and*

$$\mathrm{sk}_n(X)/\,\mathrm{sk}_{n-1}(X) \cong \bigvee_{\alpha\in\Lambda_n} S^n,$$

a wedge of n-spheres.

16 Homology

Two maps of pairs $f, g : (X, A) \to (Y, B)$ are said to be **homotopic**, written $f \simeq g$, if there is a homotopy $F : X \times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, and $F(a, t) \in B$ for all $x \in X$, $a \in A$, and $t \in [0, 1]$. A special example is when $f \simeq g \operatorname{rel} A$ for $f, g : X \to Y$ and $A \subseteq X$.

Let G, H be groups and let $f : G \to H$ be a group homomorphism. The kernel of f, denoted $\operatorname{Ker}(f)$, is defined by

$$\operatorname{Ker}(f) = f^{-1}(1_H) = \{x \in G \ : \ f(x) = 1_H\}.$$

A sequence of groups and group homomorphisms

$$\dots \longrightarrow G_{n+1} \xrightarrow{f_{n+1}} G_n \xrightarrow{f_n} G_{n-1} \longrightarrow \dots$$

is set to be **exact** if $\operatorname{Ker}(f_n) = \operatorname{Im}(f_{n+1})$ for all n. A **short exact sequence** of groups is an exact sequence of the form

$$1 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 1.$$

If H is an abelian group, we use additive notation, writing the identity of H as 0.

An **unreduced homology theory** h_* consists of the following items:

1. A sequence $\{h_n(X, A)\}_{n \in \mathbb{Z}}$ of abelian groups for any pair of spaces (X, A). The abelian group $h_n(X, A)$ is called the n^{th} **relative homology group of X modulo A**, and is simply written as $h_n(X)$ if $A = \emptyset$.
2. A sequence of group homomorphisms $\{f_n : h_n(X, A) \to h_n(Y, B)\}_{n \in \mathbb{Z}}$ corresponding to any map of pairs $f : (X, A) \to (Y, B)$.
3. Group homomorphisms $\partial_n(X, A) : h_n(X, A) \to h_{n-1}(A)$ for all $n \in \mathbb{Z}$ and pair of spaces (X, A). These are called the **boundary operators**.

The above items satisfying the following **Eilenberg-Steenrod** axioms:

1. If $f = \mathrm{id}_{(X,A)}$, then $f_n = \mathrm{id}_{h_n(X,A)}$ for all $n \in \mathbb{Z}$.
2. If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$, then $(g \circ f)_n = g_n \circ f_n : h_n(X, A) \to h_n(Z, C)$.

   Hence, h_n is a covariant functor from the category of pairs of topological spaces to the category $\mathrm{A}\textsc{b}$ of abelian groups for all $n \in \mathbb{Z}$. More explicitly, h_* is a covariant functor from the category of pairs of topological spaces to the category $\mathrm{A}\textsc{b}^{\mathbb{Z}}$ of graded abelian groups.

   For brevity, we denote by f_* the collection of all group homomorphism f_n for any map of pairs f. If a statement holds for all f_n, we simply say that it holds for f_*. We may view f_* as the image of f in $\mathrm{A}\textsc{b}^{\mathbb{Z}}$ under the functor h_*.
3. For all map of pairs $f : (X, A) \to (Y, B)$, we have $\partial \circ f_* = (f|_A)_* \circ \partial$. In particular, the diagram

$$\begin{array}{ccc} h_q(X, A) & \xrightarrow{f_q} & h_q(Y, B) \\ \downarrow{\scriptstyle \partial_q(X,A)} & & \downarrow{\scriptstyle \partial_q(Y,B)} \\ h_{q-1}(A) & \xrightarrow{(f|_A)_{q-1}} & h_{q-1}(B) \end{array}$$

   commutes for all $q \in \mathbb{Z}$.

   Hence, there is a natural transformation $\partial_n : h_n \to h_{n-1} \circ R$ for all $n \in \mathbb{Z}$, where R is the functor sending (X, A) to $(A, \emptyset)$.

4. (Exactness) For any pair of spaces (X, A), there is a long exact sequence of abelian groups

$$\dots \longrightarrow h_{n+1}(X,A) \xrightarrow{\partial_{n+1}} h_n(A) \xrightarrow{i_n} h_n(X) \xrightarrow{j_n} h_n(X,A) \xrightarrow{\partial_n} h_{n-1}(A) \longrightarrow \dots$$

   where $i : (A, \emptyset) \to (X, \emptyset)$ and $j : (X, \emptyset) \to (X, A)$ are the inclusion maps.

5. (Homotopy) If $f \simeq g : (X, A) \to (Y, B)$, then $f_* = g_* : h_*(X, A) \to h_*(Y, B)$.

6. (Excision) If U is an open subset of X, and $\overline{U} \subseteq A^\circ$, then the inclusion $j : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism $j_* : h_*(X \setminus U, A \setminus U) \xrightarrow{\cong} h_*(X, A)$. This is equivalent to saying that for all subspaces X_1, X_2 of X such that X_1 is closed and $X = (X_1)^\circ \cup (X_2)^\circ$, the inclusion $i : (X_1, X_1 \cap X_2) \to (X, X_2)$ induces an isomorphism $i_* : h_*(X_1, X_1 \cap X_2) \xrightarrow{\cong} h_*(X, X_2)$.

   An **ordinary homology theory** is a homology theory h_* that satisfies an addition axiom:

7. (Dimension Axiom) Let P be a one-point space. Then $h_q(P) = 0$ for all $q \neq 0$.

   In this case, $h_*(X, A)$ is called the **homology** of (X, A) with **coefficients** in $G = h_0(P)$, and we denote $h_*(X, A)$ more precisely by $H_*(X, A; G)$.

   We write $H_*(X, A)$ for the integral homology $H_*(X, A; \mathbb{Z})$.

Proposition 16.1. *If A is a weak deformation retract of X, then $h_*(X, A) = 0$. In particular, $h_*(X, X) = 0$.*

Lemma 16.2. *Suppose that the inclusion $i : A \hookrightarrow X$ is a cofibration. Then the reduced mapping cone C_i is homotopy equivalent to the quotient space X/A.*

Corollary 16.3. *Suppose that the inclusion $i : A \hookrightarrow X$ is a cofibration. Then there is an exact sequence of groups:*

$$\pi_1(A) \xrightarrow{i_*} \pi_1(X) \longrightarrow \pi_1(X/A) \longrightarrow 1.$$

Theorem 16.4. *Let (X, A) be a pair of spaces such that the inclusion $i : A \hookrightarrow X$ is a cofibration. Then the quotient map $p : (X, A) \to (X/A, *)$ induces an isomorphism $p_* : h_*(X, A) \xrightarrow{\cong} h_*(X/A, *)$.*

Given a homology theory h_*, the **reduced homology** $\bar{h}_*$ is defined as follows: For a pointed space X with base point x_0,

$$\bar{h}_*(X) = h_*(X, x_0).$$

In general, we have $\bar{h}_*(X) = \mathrm{Ker}(\epsilon_* : h_*(X) \to h_*(P))$ and $h_*(X) = \bar{h}_*(X) \oplus h_*(P)$, where P is the one-point space and $\epsilon : X \to P$ is the constant projection.

Corollary 16.5. *Let (X, A) be a pair of spaces such that the inclusion $i : A \hookrightarrow X$ is a cofibration. Then there is a long exact sequence*

$$\dots \longrightarrow \bar{h}_{n+1}(X/A) \xrightarrow{\partial_{n+1}} \bar{h}_n(A) \xrightarrow{i_n} \bar{h}_n(X) \xrightarrow{p_n} \bar{h}_n(X/A) \xrightarrow{\partial_n} \bar{h}_{n-1}(A) \longrightarrow \dots$$

where $p : X \to X/A$ is the canonical quotient.

Theorem 16.6. *Let X be a pointed space. Then there is a natural isomorphism*

$$\sigma_n : \bar{h}_n(X) \to \bar{h}_{n+1}(\Sigma(X))$$

for each $n \in \mathbb{Z}$.

Theorem 16.7 (Mayer-Vietoris). *Let $X = A^\circ \cup B^\circ$ where A, B are subspaces of X. Then there is a long exact sequence*

$$\dots \xrightarrow{\Delta_{n+1}} h_n(A \cap B) \xrightarrow{\alpha_n} h_n(A) \oplus h_n(B) \xrightarrow{\beta_n} h_n(X) \xrightarrow{\Delta_n} h_{n-1}(A \cap B) \dots$$

where $\alpha_n = \begin{pmatrix} (i_1)_n \\ (i_2)_n \end{pmatrix}$, $i_1 : A \cap B \to A$ and $i_2 : A \cap B \to B$ are the inclusions, $\beta_n = (j_1)_n - (j_2)_n$, $j_1 : A \to X$ and $j_2 : B \to X$ are the inclusions.

Proposition 16.8. *If* $i_X : X \to X \sqcup Y$ *and* $i_Y : Y \to X \sqcup Y$ *are the inclusions, then*

$$((i_X)_*, (i_Y)_*) : h_*(X) \oplus h_*(Y) \to h_*(X \sqcup Y)$$

is an isomorphism.

Proposition 16.9. *For all* $n \geq 0$,

$$\bar{H}_i(S^n) = \begin{cases} G = H_0(P) & i = n; \\ 0 & i \neq n. \end{cases}$$

Corollary 16.10. *For all* n*, the sphere* S^{n-1} *is not a weak retract of the disc* D^n.

Any pointed map $f : S^n \to S^n$ induces a group homomorphism $f_* : H_n(S^n) \to H_n(S^n)$. For all $n \geq 1$, we have $f_* : \mathbb{Z} \to \mathbb{Z}$. It follows that there is a unique integer $\deg(f)$ such that $f_*(\alpha) = \deg(f)\alpha$ for all $\alpha \in H_n(S^n)$ which depends only on the homotopy class of f.

Proposition 16.11. *For ordinary homology over* $\mathbb{Z}$*, we have the following for pointed maps*

$$f, g : S^n \to S^n :$$

1. $\deg(\mathrm{id}_{S_n}) = 1$.
2. *If* f *is not surjective, then* $\deg(f) = 0$.
3. $f \simeq g$ *if and only if* $\deg(f) = \deg(g)$.
4. $\deg(g \circ f) = \deg(g)\deg(f)$.
5. *If* f *is a reflection fixing points in a subspace* S^{n-1}*, then* $\deg(f) = -1$.
6. *The antipodal map* $x \mapsto -x$ *has degree* $(-1)^{n+1}$.
7. *If* f *has no fixed points, then* $\deg(f) = (-1)^{n+1}$.

Corollary 16.12. *If* n *is even, then* $\mathbb{Z}/2\mathbb{Z}$ *is the only nontrivial group that can act freely on* S^n.

17 Cohomology

An **unreduced cohomology theory** h^* consists of the following items:

1. A sequence $\{h^n(X, A)\}_{n \in \mathbb{Z}}$ of abelian groups for any pair of spaces (X, A). The abelian group $h^n(X, A)$ is called the n^{th} **relative cohomology group of** X **modulo** A, and is simply written as $h^n(X)$ if $A = \emptyset$.
2. A sequence of group homomorphisms $\{f^n : h^n(Y, B) \to h^n(X, A)\}_{n \in \mathbb{Z}}$ corresponding to any map of pairs $f : (X, A) \to (Y, B)$.
3. Group homomorphisms $\delta^n(X, A) : h^n(A) \to h^{n+1}(X, A)$ for all $n \in \mathbb{Z}$ and pair of spaces (X, A). These are called the **boundary operators**.

The above items satisfying the following **Eilenberg-Steenrod** axioms:

1. If $f = \mathrm{id}_{(X,A)}$, then $f^n = \mathrm{id}_{h^n(X,A)}$ for all $n \in \mathbb{Z}$.
2. If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$, then $(g \circ f)^n = f^n \circ g^n : h^n(Z, C) \to h^n(X, A)$.

   Hence, h^n is a contravariant functor from the category of pairs of topological spaces to the category $\mathrm{A{\small B}}$ of abelian groups for all $n \in \mathbb{Z}$. More explicitly, h^* is a contravariant functor from the category of pairs of topological spaces to the category $\mathrm{A{\small B}}^{\mathbb{Z}}$ of graded abelian groups.

   For brevity, we denote by f^* the collection of all group homomorphism f^n for any map of pairs f. If a statement holds for all f^n, we simply say that it holds for f^*. We may view f^* as the image of f in $\mathrm{A{\small B}}^{\mathbb{Z}}$ under the functor h^*.

3. For all map of pairs $f : (X, A) \to (Y, B)$, we have $f^* \circ \delta = \delta \circ (f|_A)^*$. In particular, the diagram

$$\begin{array}{ccc} h^{q-1}(B) & \xrightarrow{(f|_A)^{q-1}} & h^{q-1}(A) \\ \delta^{q-1}(Y,B) \downarrow & & \downarrow \delta^{q-1}(X,A) \\ h^q(Y,B) & \xrightarrow{f^q} & h^q(X,A) \end{array}$$

   commutes for all $q \in \mathbb{Z}$.

   Hence, there is a natural transformation $\delta^{n-1} : h^{n-1} \circ R \to h^n$ for all $n \in \mathbb{Z}$, where R is the functor sending (X, A) to $(A, \emptyset)$.

4. (Exactness) For any pair of spaces (X, A), there is a long exact sequence of abelian groups

$$\dots \longrightarrow h^{n-1}(A) \xrightarrow{\delta^{n-1}} h^n(X,A) \xrightarrow{j^n} h^n(X) \xrightarrow{i^n} h^n(A) \xrightarrow{\delta^n} h^{n+1}(X,A) \longrightarrow \dots$$

   where $i : (A, \emptyset) \to (X, \emptyset)$ and $j : (X, \emptyset) \to (X, A)$ are the inclusion maps.

5. (Homotopy) If $f \simeq g : (X, A) \to (Y, B)$, then $f^* = g^* : h^*(Y, B) \to h^*(X, A)$.

6. (Excision) If U is an open subset of X, and $\overline{U} \subseteq A^\circ$, then the inclusion $j : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism $j^* : h^*(X, A) \xrightarrow{\cong} h^*(X \setminus U, A \setminus U)$. This is equivalent to saying that for all subspaces X_1, X_2 of X such that X_1 is closed and $X = (X_1)^\circ \cup (X_2)^\circ$, the inclusion $i : (X_1, X_1 \cap X_2) \to (X, X_2)$ induces an isomorphism $i^* : h^*(X, X_2) \xrightarrow{\cong} h^*(X_1, X_1 \cap X_2)$.

   An **ordinary cohomology theory** is a cohomology theory h^* that satisfies an addition axiom:

7. (Dimension Axiom) Let P be a one-point space. Then $h^q(P) = 0$ for all $q \neq 0$.

   In this case, $h^*(X, A)$ is called the **cohomology** of (X, A) with **coefficients** in $G = h^0(P)$, and we denote $h^*(X, A)$ more precisely by $H^*(X, A; G)$.

   We write $H^*(X, A)$ for the integral cohomology $H^*(X, A; \mathbb{Z})$.

18 Exercises

Exercises 18.1.

1. Let $\mathcal{C}$ be a category, $\text{Mor}(\mathcal{C})$ be the class of all morphisms of $\mathcal{C}$, and for any pair of morphisms $f : A \to B$ and $g : C \to D$ of $\mathcal{C}$, define $\text{Mor}(f, g)$ to be the set of all ordered pairs (α, β), where $\alpha : A \to C$ and $\beta : B \to D$ are morphisms of $\mathcal{C}$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow[g]{} & D \end{array}$$

   Show that the class $\text{Mor}(\mathcal{C})$ together with the sets $\text{Mor}(f, g)$ is a category under this definition.

2. Describe how a nonempty class A can be made into a category in which the only morphisms are identity morphisms.

3. A **pointed set** is a pair (S, x), where S is a set and $x \in S$. A morphism of pointed sets $(S, x) \to (T, y)$ is a set function $f : S \to T$ such that $f(x) = y$. Show that the pointed sets form a category, which we denote by SET_0.

4. Let $\{A_i\}_{i \in I}$ be a family of sets. Let S be the set of all ordered pairs (a, i), where $a \in A_i$ and $i \in I$. For each $i \in I$, define a function $\sigma_i : A_i \to S$ by $\sigma_i(a) = (a, i)$. Show that $\{S, \{\sigma_i\}_{i \in I}\}$ is a coproduct of $\{A_i\}_{i \in I}$ in the category SET of sets.

5. Show that every family $\{A_i\}_{i \in I}$ of pointed sets (Exercise 3) has a product and coproduct in the category SET_0 of pointed sets.

6. An object I in a category $\mathcal{C}$ is said to be **initial** if for any object C, there exists precisely one morphism $I \to C$ in $\mathcal{C}$. An object T is said to be **terminal** if for any object C, there exists precisely one morphism $C \to T$ in $\mathcal{C}$.

   (a) Show that any two initial (respectively terminal) objects in a category are isomorphic.

   (b) Show that the trivial group $\{1\}$ is both initial and terminal in the category GRP of groups.

7. Let X be a topological space. Prove the following:

   (a) $\emptyset$ and X are closed sets.

   (b) Finite unions of closed sets are closed.

   (c) Arbitrary intersections of closed sets are closed.

8. Let $\Re$ be a collection of subsets of X such that $\emptyset, X \in \Re$, $\Re$ is closed under finite unions, and $\Re$ is closed under arbitrary intersections. Show that $\Im = \{X \setminus C \ : \ C \in \Re\}$ is a topology on X.

9. Let $\mathcal{B}$ be a basis for a topology on X. Let $\Im$ be the collection of all arbitrary unions of elements of $\mathcal{B}$. Show that $(X, \Im)$ is a topological space.

10. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\Im$ and $\Im'$ respectively. Show that $\Im \subseteq \Im'$ if and only if for every finite collection $C_1, C_2, ..., C_n$ of basis elements of $\mathcal{B}$, there is a collection $\{B'_i\}_{i \in I}$ of basis elements in $\mathcal{B}'$ such that

$$\bigcup_{i \in I} B'_i = \bigcap_{j=1}^{n} C_j.$$

11. Let X be a topological space. Suppose that $\mathcal{B}$ is a collection of open sets of X such that for any open set U of X, there is a collection $\{B_i\}_{i \in I}$ of elements of $\mathcal{B}$ such that

$$\bigcup_{i \in I} B_i = U.$$

   Show that $\mathcal{B}$ is a basis for the topology on X.

12. Let X and Y be topological spaces, and let $x \in X$. A function $f : X \to Y$ is said to be **continuous at** x if whenever V is an open set in Y containing $f(x)$, there is an open set U in X such that $x \in U$ and $U \subseteq f^{-1}(V)$. Show that a function $f : X \to Y$ is continuous if and only if f is continuous at every $x \in X$.

13. Let X and Y be topological spaces, and let $f : X \to Y$ be a function. Suppose that $\mathcal{B}$ is a basis for the topology on Y, and $\mathcal{S}$ is a sub-basis for the topology on Y such that the basis $\mathcal{B}$ is the natural extension of $\mathcal{S}$ obtained by taking intersections of all finite collections of sub-basic sets of $\mathcal{S}$. Show that the following are equivalent:

    (a) f is continuous.

    (b) For every basic set $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X.

    (c) For every sub-basic set $S \in \mathcal{S}$, the set $f^{-1}(S)$ is open in X.

14. Let $\mathcal{B}$ be a basis for the topology on X and Y be a subset of X. Show that the collection

$$\mathcal{B}_Y = \{B \cap Y \ : \ B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y. Let $\mathcal{S}$ be a sub-basis for the topology on X and Y be a subset of X. Show that the collection

$$\mathcal{S}_Y = \{S \cap Y \ : \ S \in \mathcal{S}\}$$

is a sub-basis for the subspace topology on Y.

15. Let Y be a subspace of X and U be a subset of Y. If U is open in Y and Y is open in X, show that U is open in X.

16. Let Y be a subspace of X and A be a subset of Y. Prove that the subspace topology on A relative to Y is the same as the subspace topology on A relative to X.

17. Let Y be a subspace of X. Prove that a subset A of Y is closed in Y if and only if $A = B \cap Y$ for some closed set B of X.

18. Let Y be a subspace of X and U be a subset of Y. If U is closed in Y and Y is closed in X, show that U is closed in X.

19. If U is open in X and A is closed in X, show that $U \setminus A$ is open in X and $A \setminus U$ is closed in X. Hence, an open set complement out a closed set remains open, and a closed set complement out an open set remains closed.

20. Let X, Y, Z be topological spaces. Prove the following:

    (a) (Constant function) If $f : X \to Y$ maps all of X onto a single point $y_0 \in Y$, then f is continuous.

    (b) (Inclusion) If A is a subspace of X, then the inclusion function $\iota : A \to X$ given by $\iota(a) = a$ for all $a \in A$ is continuous. Furthermore, the subspace topology on A is the coarsest topology on A that makes the inclusion map continuous.

    (c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is also continuous.

    (d) (Domain restriction) If $f : X \to Y$ is continuous and A is a subspace of X, then the function $f|_A : A \to Y$ is continuous.

    (e) (Codomain restriction or expansion) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing $f(X)$, then the function $g : X \to Z$, $g(x) = f(x)\ \forall\, x \in X$, obtained by restricting the codomain of f, is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$, $h(x) = f(x)\ \forall\, x \in X$, obtained by expanding the codomain of f, is continuous.

21. Let X, Y, Z be topological spaces. Prove the following:

    (a) (Inclusion) If A is an **open subspace** of X, then the inclusion function $\iota : A \to X$ given by $\iota(a) = a$ for all $a \in A$ is an open map.

    (b) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are open maps, then $g \circ f : X \to Z$ is also an open map.

    (c) (Domain restriction) If $f : X \to Y$ is an open map and A is an **open subspace** of X, then the function $f|_A : A \to Y$ is an open map.

(d) (Codomain restriction or expansion) Let $f : X \to Y$ be an open map. If Z is a subspace of Y containing $f(X)$, then the function $g : X \to Z$, $g(x) = f(x)\ \forall\ x \in X$, obtained by restricting the codomain of f, is an open map. If Z is a space having Y as a subspace, then the function $h : X \to Z$, $h(x) = f(x)\ \forall\ x \in X$, obtained by expanding the codomain of f, is an open map.

22. Let X and Y be topological spaces, and let $\{U_i\}_{i\in I}$ be a collection of open sets in X such that $X = \bigcup_{i\in I} U_i$. Prove that a function $f : X \to Y$ is continuous if and only if every $f|_{U_i} : U_i \to Y$ is continuous. Suppose that $C_1, C_2, ..., C_n$ is a finite collection of closed sets in X such that $X = \bigcup_{j=1}^{n} C_j$. Show that a function $f : X \to Y$ is continuous if and only if every $f|_{C_j} : C_j \to Y$ is continuous.

23. Let $X = \bigcup_{i\in I} U_i$, where each U_i is an open set in X. For each $i \in I$, let $f_i : U_i \to Y$ be a function such that $f_j(x) = f_k(x)$ for all $x \in \bigcap_{i\in I} U_i$ and $j, k \in I$. Define a function $h : X \to Y$ by $h(x) = f_i(x)$ for $x \in U_i$. Show that h is continuous if and only if each f_i is continuous.

24. Let $X = \bigcup_{i=1}^{n} C_i$, where $C_1, C_2, ..., C_n$ is a finite collection of closed sets in X. For each $i = 1, 2, ..., n$, let $f_i : C_i \to Y$ be a function such that $f_j(x) = f_k(x)$ for all $x \in \bigcap_{i=1}^{n} C_i$ and $1 \le j, k \le n$. Define a function $h : X \to Y$ by $h(x) = f_i(x)$ for $x \in C_i$. Show that h is continuous if and only if each f_i is continuous.

25. If A is closed in X and B is closed in Y, prove that $A \times B$ is closed in $X \times Y$. More generally, if C_i is closed in X_i for each $i \in I$, show that $\prod_{i\in I} C_i$ is closed in $\prod_{i\in I} X_i$ in both the box and product topologies.

26. Let $\mathcal{B}_i$ be a basis for the topology on each space X_i. Prove that the collection

$$\mathcal{D} = \left\{ \prod_{i\in I} B_i \ : \ B_i \in \mathcal{B}_i \text{ for all } i \in I \right\}$$

forms a basis for the box topology on $\prod_{i\in I} X_i$, and the collection

$$\mathcal{D}' = \left\{ \prod_{i\in I} B_i \ : \ B_i \in \mathcal{B}_i \text{ for finitely many } i \in I,\ B_j = X_j \text{ for all other } j \in I \right\}$$

forms a basis for the product topology on $\prod_{i\in I} X_i$.

27. Let A_i be a subspace of X_i for each $i \in I$. Show that the box (respectively product) topology on $\prod_{i\in I} A_i$ is the same as the topology that $\prod_{i\in I} A_i$ inherits as a subspace of $\prod_{i\in I} X_i$ endowed with the box (respectively product) topology.

28. Let X_i be a topological space for each $i \in I$. Let $k \in I$ be fixed and for each $i \in I$, $i \ne k$, let $b_i \in X_i$ be given. Define the canonical inclusion map $\sigma_{\{b_i\}_{i\in I, i\ne k}} : X_k \to \prod_{i\in I} X_i$ by

$$\sigma_{\{b_i\}_{i\in I, i\ne k}} : x \mapsto (y_i)_{i\in I},$$

where $y_k = x$ and $y_i = b_i$ for all $i \ne k$. Let $\prod_{i\in I} X_i$ be endowed with either the box or product topology.

(a) Show that $\sigma_{\{b_i\}_{i\in I, i\ne k}}$ are continuous open maps for all choices of $k \in I$ and $\{b_i\}_{i\in I, i\ne k}$ with $b_i \in X_i$ for all $i \ne k$.

(b) Show that the projection maps $\pi_{X_k} : \prod_{i\in I} X_i \to X_k$ are continuous open maps.

(c) Prove that the product topology is the coarsest topology on $\prod_{i\in I} X_i$ relative to which every projection map π_{X_k} is continuous, in the sense that any topology on $\prod_{i\in I} X_i$ that makes every π_{X_k} continuous must contain the product topology as a subset.

29. Let $f_i : A_i \to X_i$ functions between topological spaces for each $i \in I$. Define the function $\prod_{i\in I} f_i : \prod_{i\in I} A_i \to \prod_{i\in I} X_i$ by

$$\prod_{i\in I} f_i : (a_i)_{i\in I} \mapsto (f_i(a_i))_{i\in I}.$$

Show that $\prod_{i\in I} f_i$ is a continuous function with respect to the usual box (respectively product) topologies on $\prod_{i\in I} A_i$ and $\prod_{i\in I} X_i$ if and only if f_i is continuous for all $i \in I$.

30. Define the diagnoal inclusion map $\sigma_X : X \to \prod_{i\in I} X_i$ by $\sigma_X(x) = (x_i)_{i\in I}$, where $x_i = x$ for all $i \in I$. Prove that σ_X is continuous if $\prod_{i\in I} X_i$ is endowed with the product topology.

31. Let $f_i : A \to X_i$ be a function for each $i \in I$. Define the function $f : A \to \prod_{i\in I} X_i$ by

$$f(a) = (f_i(a))_{i\in I}.$$

Let $\prod_{i\in I} X_i$ be endowed with the product topology. Show that f is continuous if and only if every f_i is.

32. Let A be a subset of a topological space X. Prove the following:
    (a) $\mathrm{Cl}_X(A) = X \setminus \mathrm{Int}_X(X \setminus A)$.
    (b) $\mathrm{Int}_X(A) = X \setminus \mathrm{Cl}_X(X \setminus A)$.
    (c) $\mathrm{Int}_X(A)$ and $\mathrm{Bd}_X(A)$ are disjoint, and $\mathrm{Cl}_X(A) = \mathrm{Int}_X(A) \cup \mathrm{Bd}_X(A)$.
    (d) $\mathrm{Bd}_X(A) = \emptyset$ if and only if A is both open and closed.

33. Let Y be a subspace of X and A be a subset of Y. Show the following:
    (a) $\mathrm{Cl}_Y(A) = \mathrm{Cl}_X(A) \cap Y$.
    (b) $\mathrm{Int}_Y(A) = \mathrm{Int}_X(A) \cap Y$.

34. Let X and Y be topological spaces, and let $f : X \to Y$ be a function. Show that the following are equivalent:
    (a) f is continuous.
    (b) For every subset A of X, we have $f(\mathrm{Cl}_X(A)) \subseteq \mathrm{Cl}_Y(f(A))$.
    (c) For every closed set B in Y, the set $f^{-1}(B)$ is closed in X.

35. Let $f : X \to Y$ be a bijective function between topological spaces. Show that the following are equivalent:
    (a) f is a homeomorphism.
    (b) A subset V of Y is open in Y if and only if $f^{-1}(V)$ is open in X.
    (c) A subset V of Y is closed in Y if and only if $f^{-1}(V)$ is closed in X.
    (d) For any subset A of X, $f(\mathrm{Cl}_X(A)) = \mathrm{Cl}_Y(f(A))$.

36. Let X be a topological space with basis $\mathcal{B}$, and let A be a subset of X. Show that the following are equivalent for an element $x \in X$:
    (a) $x \in \overline{A}$.
    (b) Every open set of X that contains x intersects A nontrivially.
    (c) Every basis set in $\mathcal{B}$ that contains x intersects A nontrivially.

37. Let A be a subset of a topological space X. Prove that $\overline{A} = A \cup A'$. Deduce that a subspace A is closed if and only if A contains all its limit points.

38. Let $\{A_i\}_{i\in I}$ be a family of subsets of X, and let $A_1, A_2, ..., A_n$ be a finite collection within this family. Show that the following hold:
    (a) $\bigcup_{i\in I} A_i' \subseteq (\bigcup_{i\in I} A_i)'$.
    (b) $\bigcap_{i\in I} A_i' \supseteq (\bigcap_{i\in I} A_i)'$.
    (c) $\bigcup_{j=1}^{n} \overline{A_j} = \overline{\bigcup_{j=1}^{n} A_j}$.
    (d) $\bigcup_{i\in I} \overline{A_i} \subseteq \overline{\bigcup_{i\in I} A_i}$.
    (e) $\bigcap_{i\in I} \overline{A_i} \supseteq \overline{\bigcap_{i\in I} A_i}$.

39. Let $\{A_i\}_{i\in I}$ be a family of subsets of X, and let $A_1, A_2, ..., A_n$ be a finite collection within this family. Show that the following hold:
    (a) $\bigcap_{j=1}^{n} A_j^\circ = (\bigcap_{j=1}^{n} A_j)^\circ$.
    (b) $\bigcap_{i\in I} A_i^\circ \supseteq (\bigcap_{i\in I} A_i)^\circ$.
    (c) $\bigcup_{i\in I} A_i^\circ \subseteq (\bigcup_{i\in I} A_i)^\circ$.

40. Let A_i be a subset of X_i for each $i \in I$. Show that

$$\mathrm{Cl}_{\prod_{i\in I} X_i}\left(\prod_{i\in I} A_i\right) = \prod_{i\in I} \mathrm{Cl}_{X_i}(A_i),$$

if $\prod_{i\in I} X_i$ is endowed with the box topology or product topology.

41. Let X be a topological space, A be a set, and $p : X \to A$ be a surjective function. Show that the quotient topology on A induced by p is the finest topology on A relative to which p is continuous, in the sense that if $\Im'$ is any topology on A that makes p continuous, then $\Im'$ is contained within the quotient topology as a subset.

42. Suppose that A has the quotient topology with respect to the surjective mapping $p : X \to A$. Prove that a set C is A is closed if and only if $p^{-1}(C)$ is closed in X.

43. Let X be a topological space and let $f : X \to Y$ be a surjective function. Suppose that Y is given the quotient topology with respect to f. Show that a function $g : Y \to Z$ from Y to a topological space Z is continuous if and only if the composite $g \circ f$ is continuous.

44. Show that the composite of two quotient maps is again a quotient map.

45. Let X and Y be topological spaces. Prove that X and Y are compact if and only if $X \times Y$ is compact with respect to the product topology.

46. Show that $[0, 1]^n$ is a compact subset of $\mathbb{R}^n$ for all positive integers n.

47. (Heine-Borel) Show that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

48. Let X and Y be topological spaces. Show the following:
    (a) If X is Hausdorff, then any subspace of X is Hausdorff.
    (b) X and Y are Hausdorff if and only if $X \times Y$ is Hausdorff with the product topology.

49. Let $f : X \to Y$ be a continuous map. Suppose that X is compact and Y is Hausdorff. Show that f is a closed map. Hence deduce that a bijective continuous map from a compact space to a Hausdorff space is a homeomorphism.

50. Let $f : X \to Y$ be a quotient map. Suppose that X is Hausdorff. Show that if f is a closed map and $f^{-1}(y)$ is compact for any $y \in Y$, then Y is Hausdorff.

51. Let X, Y be topological spaces. Prove the following:
    (a) If $p : X \to Y$ is a quotient map and Z is a locally compact Hausdorff space, then $p \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is a quotient map.
    (b) If A is a compact subspace of X and $p : X \to X/A$ is the canonical quotient map, then for any space Z, $p \times \mathrm{id}_Z : X \times Z \to (X/A) \times Z$ is a quotient map.

52. Show that if $p : A \to B$ and $q : C \to D$ are quotient maps and A, D are locally compact Hausdorff spaces, then $p \times q : A \times B \to C \times D$ is a quotient map.

53. Suppose that X is a G-space. Prove that the canonical projection $X \to X/G$ is an open map.

54. Let X be a compact Hausdorff space. Prove the following:
    (a) If G is a finite group and X is a G-space, then X/G is a compact Hausdorff space.
    (b) If A is closed subspace of X, then X/A is compact Hausdorff.

55. Show that if U is a connected subspace of X and $U \subseteq V \subseteq \overline{U}$, then V is connected.

56. Suppose $X = \bigcup_{i\in I} A_i$, where each A_i is connected, and $\bigcap_{i\in I} A_i \neq \emptyset$. Show that X is connected.

57. Show that every quotient of a connected (resp. path-connected) space is connected (resp. path-connected).

58. Show that every finite product of a family of connected (resp. path-connected) spaces is connected (resp. path-connected).

59. For a space Z, define the fold map $\bigtriangledown : Z \vee Z \to Z$ by $\bigtriangledown : (z, *) \mapsto z$ and $\bigtriangledown : (*, z) \mapsto z$. Show that the fold map is continuous.

60. Let X and Y be pointed spaces with base points x_0 and y_0 respectively. Show that $X \vee Y$ is homeomorphic to the subspace $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ of $X \times Y$.

61. Show that $(X \vee Y) \vee Z \cong X \vee (Y \vee Z)$ for any pointed spaces X, Y, Z.

62. Show that $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$ for any pointed spaces X, Y, Z.

63. Given three pointed spaces X, Y, Z, show that $(X \vee Y) \wedge Z$ is homeomorphic to $(X \wedge Z) \vee (Y \wedge Z)$.

64. Show that $S^n \wedge S^m \cong S^{n+m}$ for any nonnegative integers n, m.

65. Show that for pointed spaces X, Y, we have $\Omega^n(X \times Y) \cong \Omega^n(X) \times \Omega^n(Y)$ and for nonnegative integers n, m, we have $\Omega^{n+m}(X) \cong \Omega^n(\Omega^m(X))$.

66. Show that for nonnegative integers n, m, we have $\Sigma^{n+m}(X) \cong \Sigma^n(\Sigma^m(X))$, and for any pointed spaces X, Y, we have
$$\mathrm{Map}_*(\Sigma^n(X), Y) \cong \mathrm{Map}_*(X, \Omega^n(Y)).$$

67. Show that for a pointed space X with base point x_0, $\Sigma(X) \cong (X \times I)/((X \times \{0\}) \cup (X \times \{1\}) \cup (\{x_0\} \times I))$, where $I = [0, 1]$.

68. Let p_0 be any point of S^n and let $f : S^n \to Y$. Show that the following are equivalent:
    (a) f is null-homotopic.
    (b) f has a continuous extension to D^{n+1}, where we identify S^n as the boundary of D^{n+1} in the natural way.
    (c) f is null-homotopic relative to p_0.

    Deduce that any continuous map from S^n to a contractible space has a continuous extension over D^{n+1}.

69. Show that $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$ as sets for any $n \geq 0$.

70. Show that $\pi_n(X) \cong \pi_0(\Omega^n(X)) \cong \pi_1(\Omega^{n-1}(X))$ as sets for all $n \geq 1$.

71. Let X be a pointed space with base point x_0. Show that $\pi_0(X) = [S^0, X]$ and the equivalence classes of X under path-connectedness are equivalent as sets. In particular any n-connected space X is path-connected if and only if $\pi_0(X)$ is the one-point set.

72. Show that a subspace A of X is a weak retract of X if and only if $i^* : [X, A] \to [A, A]$ is a surjective function, where $i : A \hookrightarrow X$ is the inclusion map.

73. Show that if (X, A) has the homotopy extension property with respect to A, then A is a weak retract (resp. weak deformation retract) of X if and only if A is a retract (resp. deformation retract) of X.

74. Show that if $(X \times I, ((X \times \{0\}) \cup (X \times \{1\}) \cup (A \times I)))$ has the homotopy extension property with respect to X and A is closed in X, then A is a deformation retract of X if and only if A is a strong deformation retract of X.

75. If A is contractible and the pair (X, A) satisfies the homotopy extension property with respect to X, show than the canonical quotient $X \to X/A$ is a homotopy equivalence.

76. Let $A \subseteq B \subseteq X$ be subspaces. Suppose that the inclusions $A \hookrightarrow B$ and $B \hookrightarrow X$ are cofibrations. Show that the inclusion $A \hookrightarrow X$ is a cofibration.

77. Show that the natural inclusion $S^n \hookrightarrow D^{n+1}$ is a cofibration.

78. Let (X, A) be a pair of spaces. Suppose that the inclusion $i : A \hookrightarrow X$ is a cofibration. Show that $p = (\mathrm{id}_Z)^i : \mathrm{Map}(X, Z) \to \mathrm{Map}(A, Z)$ is a fibration for any space Z.

79. Let X be a pointed space with base point x_0. We say that the base point x_0 is **nondegenerate** if the inclusion $\{x_0\} \hookrightarrow X$ is a cofibration. Let X be a pointed space with nondegenerate basepoint x_0. Prove that the evaluation map $\mathrm{Map}(X, Y) \to Y$ defined by $f \mapsto f(x_0)$ is a fibration.

80. If $p : E \to B$ is a fibration, show that $p^{\mathrm{id}_Z} : \mathrm{Map}(Z, E) \to \mathrm{Map}(Z, B)$ is a fibration for any locally compact space Z.

81. Let $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ be covering projections. Show that $p \times q : \tilde{X} \times \tilde{Y} \to X \times Y$ is a covering projection.

82. Let $p : \tilde{X} \to X$ be a covering projection, and let $B \subseteq X$. Let $\tilde{B} = p^{-1}(B) \subseteq \tilde{X}$, and let $p' = p|_{\tilde{B}} : \tilde{B} \to B$ be the covering projection of B induced by p. Suppose that $\tilde{X}, X, B$ are all path-connected, and that the map $\pi_1(B) \to \pi_1(X)$ induced by the inclusion $B \hookrightarrow X$ is surjective. Show that $\tilde{B}$ is also path-connected.

83. Let X be path-connected and let Y be simply connected. Suppose that there exist small contractible open neighbourhoods of the base points x_0, y_0 of X, Y respectively, and that $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a universal covering projection. Let
$$Z = \{(\tilde{x}, y) \in \tilde{X} \times Y \ : \ (p(\tilde{x}), y) \in X \vee Y\}$$
and let $p' = (p \times \mathrm{id}_Y)|_Z : Z \to X \vee Y$. Show that $p' : Z = \widetilde{X \vee Y} \to X \vee Y$ is a universal covering projection.

84. (Borsuk-Ulam) Show that there does not exist any nonzero continuous map $f : S^2 \to S^1$ such hat $f(-x) = -f(x)$ for all $x \in S^2$. Deduce that no subspace of $\mathbb{R}^2$ is homeomorphic to S^2.

85. Show that the map $f : S^{m+n} = S^n \wedge S^m \to S^m \wedge S^n = S^{n+m}$ given by $f : x \wedge y \mapsto y \wedge x$ has degree $(-1)^{mn}$.

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