# ON CONFIGURATION SPACES: PART 1

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# INTRODUCTION AND OUTLINE

The purpose of these notes is to give definitions together with basic, classical properties of configuration spaces.

(1) Definitions.

(2) Examples.

(3) Basic properties.

(4) Where and how do these fit ?

## DEFINITIONS

The configuration space of ordered $m$-tuples of $k$ distinct points in a space $M$ is denoted

$$Conf(M, k)$$

and is the subspace of

$$M^k$$

given by

$$\{(m_1, m_2, \cdots, m_k) | m_i \neq m_j \text{ for all } i \neq j\}.$$

# PROJECTIONS

Observe that the natural projection maps

$$p_i : M^k \to M^{k-1}$$

which deletes the $i$-th coordinate restricts to a map on the level of configuration spaces

$$p_i : Conf(M, k) \to Conf(M, k-1).$$

In what follows below, let

$$Q_j = \{x_1, x_2, \cdots, x_j\}$$

denote a set of $j$ distinct points in $M$.

# CONFIGURATION SPACES OF MANIFOLDS

**Theorem 0.1.** *If $M$ is a manifold without boundary, the natural projection map*

$$p_i : Conf(M, k) \to Conf(M, k-1)$$

*is a fibration with fibre*

$$M - Q_{k-1}.$$

**Remarks:**

(1) This theorem was stated and proven in a classical paper by Fadell-Neuwirth. The result also follows from an earlier result of R. Palais who was addressing a different question.

(2) This theorem as well as variations provide tools for analyzing features of configuration spaces of manifolds. What can be said about configuration spaces of singular spaces ? Configuration spaces of graphs are studied in work of Ghrist, Farley and others.

## OTHER PROJECTIONS

Analogous natural composite projection maps

$$p_I : Conf(M,k) \to Conf(M,k-j)$$

defined by

$$p_{i_1} \circ p_{i_2} \circ \cdots \circ p_{i_j}$$

for $i_1 < i_2 < \cdots < i_j$ are fibrations with fibre

$$Conf(M - Q_{k-j}, j).$$

## CLASSICAL EXAMPLES FROM SURFACES

This section is restricted to configuration spaces of surfaces $S$.

Recall that if $S_g$ is a closed orientable surface of genus $g$ with $g \geq 2$, then $S_g$ is a quotient of the hyperbolic plane $\mathbb{H}^2$ by an action of a discrete group $\Gamma_g$ acting freely and properly discontinuously on $\mathbb{H}^2$. Thus

$$S_g = \mathbb{H}^2/\Gamma_g.$$

In the special case of $g = 1$, then

$$S_1 = \mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}.$$

Thus if $g \geq 1$, these spaces are all $K(\pi, 1)$'s. Similarly

$$S_g - Q_i$$

is homotopy equivalent to a wedge of circles

$$\vee_{2g+i-1} S^1,$$

and so

$$S_g - Q_i$$

are all $K(\pi, 1)$'s if $i \geq 1$.

By the fibering theorem of Fadell-Neuwirth, Palais, the next theorem follows at once.

**Theorem 0.2.** *If $M$ is surface which is not homeomorphic to either the 2-sphere $S^2$ or the projective plane $\mathbb{RP}^2$, then*

$$Conf(M, k) \text{ and } Conf(M, k)/\Sigma_k$$

*are $K(\pi, 1)$ (open) manifolds.*

The cases for which $S$ is either the 2-sphere $S^2$ or the projective plane $\mathbb{RP}^2$ are special and are addressed later.

# BRAID GROUPS OF A SURFACE

The **definition of the $n$-stranded braid group as well as the $n$-stranded pure braid group for a surface** $\mathbb{S}$ is given next.

The $n$-stranded braid group for a surface $\mathbb{S}$ denoted

$$B_n(\mathbb{S})$$

is the fundamental group of $Conf(\mathbb{S}, n)/\Sigma_n$.

The $n$-stranded pure braid group for a surface $\mathbb{S}$ denoted

$$P_n(\mathbb{S})$$

is the fundamental group of $Conf(\mathbb{S}, n)$.

**Theorem 0.3.** *If $\mathbb{S}$ is surface which is not homeomorphic to either the 2-sphere $S^2$ or the projective plane $\mathbb{RP}^2$, then*

$$Conf(\mathbb{S}, n)/\Sigma_n = K(B_n(\mathbb{S}), 1),$$

*and*

$$Conf(\mathbb{S}, n) = K(P_n(\mathbb{S}), 1)$$

*are $K(\pi, 1)$ (open) manifolds.*

## SPECIAL CASES

If $\mathbb{S}$ is either the 2-sphere $S^2$ or the projective plane $\mathbb{RP}^2$, the associated configuration spaces are not $K(\pi,1)$'s. The following variation gives the appropriate $K(\pi,1)$.

Recall that the rotation group $SO(3)$ acts naturally on $S^2$ as well as $\mathbb{RP}^2$.

Since the group of unit quaternions

$$S^3$$

double covers $SO(3)$, this group acts naturally on $S^2$ and $\mathbb{RP}^2$ as well as on their configuration spaces.

**Theorem 0.4.** *If $\mathbb{S}$ is either the 2-sphere $S^2$ or the projective plane $\mathbb{RP}^2$ and $n \geq 3$, then*

$$ES^3 \times_{S^3} Conf(S^2, n)/\Sigma_n = K(B_n(S^2), 1),$$

*and*

$$ES^3 \times_{S^3} Conf(\mathbb{RP}^2, n)/\Sigma_n = K(B_n(\mathbb{RP}^2), 1).$$

## REMARKS

(1) The groups

$$P_n(S^2)$$

are basic as will be seen in Jie Wu's tutorials.

(2) Using techniques of Hopf algebras, it is possible to extend the definition of braid groups to give "braid groups" for any manifold $M$. Even in case of

$$M = R^n,$$

the structure of these groups is sensitive to the dimension of $M$. Introductory remarks are given in the next section.

## LOOP SPACES

Recall that if a space $X$ has a base-point denoted $*$, then the (pointed) loop space is the space of continuous functions

$$f : [0,1] \longrightarrow X$$

such that $f(*) = *$. This space is called the loop space of $X$ and is denoted

$$\Omega(X).$$

An element $f$ in $\Omega(X)$ is called a loop.

# BRAIDS IN CONFIGURATION SPACES

The graph of a loop in $Conf(\mathbb{R}^2, k)$

$$f : [0,1] \longrightarrow Conf(\mathbb{R}^2, k)$$

$$graph(f) : [0,1] \longrightarrow [0,1] \times Conf(\mathbb{R}^2, k)$$

is a braid ( where a picture should appear on the board soon ).

## BRAIDS AS LOOPS

The classical isomorphism

$$\pi_0(\Omega(X)) \to \pi_1(X)$$

for path-connected spaces $X$ induces an isomorphisms

$$\pi_0(\Omega(Conf(\mathbb{S},k)) \to \pi_1(Conf(\mathbb{S},k))$$

as well as

$$\pi_0(\Omega(Conf(\mathbb{S},k)/\Sigma_k) \to \pi_1(Conf(\mathbb{S},k)/\Sigma_k).$$

The identification of $\pi_0(\Omega(Conf(\mathbb{S},k)/\Sigma_k)$ as the braid group for the surface $\mathbb{S}$ is precisely the isotopy classes of braids ( with end-points fixed ).

## GENERATORS

Pictures of generators for one choice of generators for

$$B_n(\mathbb{R}^2)$$

as well as for

$$P_n(\mathbb{R}^2)$$

are as follows:

# BRAID GROUPS FOR GENERAL MANIFOLDS

Notice that if $M$ is a simply-connected manifold of dimension at least 3, then the fundamental group

$$\pi_1(Conf(M, k))$$

is trivial ( by the Fadell-Neuwirth fibration theorem ). Thus to construct non-trivial analogues of braid groups for general manifolds, an alternative construction is required.

There exist groups analogous to classical braid groups which reflect the feature that a braid on a surface can be identified as a loop in a configuration space. The analogue of a braid group for manifolds $M$ of dimension at least 3 arise from (1) suitably compatible, (2) parameterized families of loops for configuration spaces the manifold $M$. The structure of Hopf algebras provides a setting for this construction.

A provisional definition of the pure k-stranded braid group for a simply-connected manifold $M$ of dimension at least 3 is given next ( with natural modifications for the non-simply-connected case omitted ).

$$P_k(M) = Hom^{coalgebra}(H_*(\Omega S^2), H_*\Omega(Conf(M,k)).$$

These groups were investigated in a paper by T. Sato and the author ( available by request ).

# PURE BRAID GROUPS FOR $\mathbb{R}^n$

**Theorem 0.5.** *Assume that $m, n \geq 2$. Then*

*(1) the groups*

$$P_k(\mathbb{R}^m)$$

*and*

$$P_k(\mathbb{R}^n)$$

*are isomorphic if and only if $n = m$, and*

*(2) the Malĉev completions of*

$$P_k(\mathbb{R}^m)$$

*and*

$$P_k(\mathbb{R}^n)$$

*are isomorphic if and only if $m + n$ is even.*

(i) The Malĉev completion (roughly) replaces $\mathbb{Z}$ by $\mathbb{Q}$.

(ii) Properties of the "pure braid groups" $P_k(M)$ are similar to the classical braid groups but are measuring a different "flavor of linking phenomena".

(iii) These groups also "fit" with Jie Wu's lectures in that they also give simplicial groups.

## FURTHER CLASSICAL EXAMPLES

This section gives a partial list of some classical configuration spaces as well as homotopy equivalent spaces.

(1) If $G$ is a topological group, then there is a homeomorphism
$$Conf(G, k) \to G \times Conf(G - e, k - 1)$$
gotten by shearing.

(2) There is a homeomorphism
$$S^{n-1} \times \mathbb{R}_+ \to Conf(\mathbb{R}^n, 2).$$

(3) If $k \geq 3$, there are homeomorphisms
$$PGL(2, \mathbb{C}) \to Conf(S^2, 3)$$
as well as
$$PGL(2, \mathbb{C}) \times Conf(S^2 - Q_3, k - 3) \to Conf(S^2, k).$$

(4) Let $V(n, 2) = SO(n)/SO(n-2)$ denoted the Stiefel manifold of ortho-normal 2-frames in $\mathbb{R}^{n+1}$. There is a fibre homotopy equivalence
$$V(n, 2) \to Conf(S^{n-1}, 3).$$

# HOMOGENEOUS SPACES

Let $M$ denote a manifold without boundary. Let

$$Top(M)$$

denote the group of homeomorphisms of $M$ which leave the complement of a compact set fixed. Let

$$Top(M,k)$$

denote the subgroup of $Top(M)$ which point-wise fixes a given set of $k$ distinct points point-wise

$$Q_k = \{m_1, m_2, ..., m_k\}$$

in $M$. Topologize $Top(M)$ by the compact-open topology.

There is a natural diagonal action of $Top(M)$ on $Conf(M,k)$

$$\Theta_k : Top(M) \times Conf(M,k) \to Conf(M,k)$$

defined by the equation

$$\Theta_k(f, (m_1, m_2, \cdots, m_k)) = (f(m_1), f(m_2), \cdots, f(m_k)).$$

A classical theorem gives that configuration spaces of manifolds behave like homogeneous spaces in the following sense.

**Theorem 0.6.** *Assume that $M$ is a path-connected manifold. Then*

*(1) $Top(M,k)$ is a closed subgroup of $Top(M)$ and*

*(2) the induced natural quotient map*

$$\bar{\Theta}_k : Top(M)/Top(M,k) \to Conf(M,k)$$

*is a homeomorphism.*

## A PROBLEM

Assume that

$$Q_\infty = \{m_1, m_2, \cdots, m_n, \cdots\}$$

is a totally ordered countable dense subset of a manifold $M$. The configuration space $Conf(M, k)$ can be thought of as the space of (ordered) embeddings

$$\tilde{E}mb(Q_k, M)$$

where

$$Q_k = \{m_1, m_2, \cdots, m_k\}.$$

The projection maps

$$p_k : Conf(M, k) \to Conf(M, k-1)$$

can be thought of as the map

$$p_k : \tilde{E}mb(Q_k, M) \to \tilde{E}mb(Q_{k-1}, M)$$

which "deletes" the image of $m_k$.

There are induced natural maps

$$
\begin{array}{ccc}
Top(M) & \xrightarrow{\Theta_k} & Conf(M,k) \\
\Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle p_k} \\
Top(M) & \xrightarrow{\Theta_{k-1}} & Conf(M,k-1) \\
\Big\downarrow{\scriptstyle ...} & & \Big\downarrow{\scriptstyle ...} \\
Top(M) & \xrightarrow{\Theta_2} & Conf(M,2) \\
\Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle p_2} \\
Top(M) & \xrightarrow{\Theta_1} & Conf(M,1)
\end{array}
$$

together with an induced map to the inverse limit

$$\Theta : Top(M) \to \varinjlim Conf(M,k)$$

which is a continuous bijection.

**Problem:** What ( if anything ) can be said about the structure of $Top(M)$ via this limit ?