ON CONFIGURATION SPACES: PART 2

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INTRODUCTION

Algebraic properties of configuration spaces frequently reflect properties of various geometric problems which are ubiquitous in nature.

These types of structures are directly connected to several subjects such as knots, links, homotopy groups as well as the structure of function spaces. Some of these connections are illustrated in problems posed at the end of this lecture.

This lecture is an illustration of some of these connections as well as a leisurely stroll through various natural techniques. The mathematics is based on joint work with Jon Berrick, Yan Loi Wong and Jie Wu.

<u>BRAID GROUPS AGAIN</u>

The section addresses a naive construction for the braid groups arising as a "cabling" construction.

This construction is interpreted in later sections in terms of the structure of braid groups as well as Vassiliev invariants of pure braids as developed by T. Kohno and extensions by M. Falk-R. Randell.

Throughout this lecture, Artin's n-stranded pure braid group for the plane is denoted

$$P_n = P_n(\mathbb{R}^2).$$

The pure braid groups P_n will be seen to contain a natural group theoretic "model" for the topological space given by the loop space of the 2-sphere.

FINITELY GENERATED FREE GROUPS

Consider the free group generated by n letters

$$\{y_1, \cdots, y_n\}$$

which is denoted

$$F_n = F_n[y_1, \cdots, y_n].$$

Next consider the elements x_i for $1 \leq i \leq n$ in P_{n+1} given by the naive "cabling" pictured on the board.

The braids x_i for $1 \leq i \leq n$ yield homomorphisms from a free group on n letters $F_n = F_n[y_1, \cdots, y_n]$ to P_{n+1}

$$\Theta_n \colon F_n[y_1, \cdots, y_n] \to P_{n+1}$$

defined on generators y_i in F_n by the formula

$$\Theta_n(y_i) = x_i.$$

ON Θ_n

The map Θ_n is faithful (a monomorphism).

The next few sections address one reason why this map is a monomorphism and where this map fits with other structures.

The method of proof is to appeal to the structure of the Lie algebras obtained from the descending central series for both the source and the target of Θ_n. After, some connections will be drawn.

The proof yields more information than the fact that Θ_n is faithful. The method of proof gives a natural connection of Vassiliev invariants of braids to the homotopy groups of the 2-sphere.

THE DESCENDING CENTRAL SERIES

The descending central series (lower central series) for a group G is the collection of subgroups of G given by

$$G = \Gamma^1(G) \geq \Gamma^2(G) \geq .... \geq \Gamma^n(G) \geq \cdots$$

where $\Gamma^n(G)$ is defined by

$$\Gamma^1(G) = G$$

and

$$\Gamma^n(G)$$

is given by the subgroup of G generated by the commutators

$$[..[g_1, g_2]g_3]...]g_t]$$

where $t \geq n$, and $[x, y] = x^{-1}y^{-1}xy$.

- The group $\Gamma^{n+1}(G)$ is a normal subgroup of $\Gamma^n(G)$.
- The quotient group

$$gr_n(G) = \Gamma^n(G)/\Gamma^{n+1}(G)$$

  is abelian.

- The map of sets induced by the commutator

$$<,>: G \times G \to \ G$$

with $< x, y >= x^{-1}y^{-1}xy$ induces a bilinear pairing

$$[-,-] : gr_n(G) \otimes_{\mathbb{Z}} gr_m(G) \to \ gr_{n+m}(G)$$

which satisfies both the antisymmetry law

$$[a,b] = -[b,a],$$

and the Jacobi identity

$$[a[b,c]] = [[a,b]c] - [[a,c]b]$$

for a Lie algebra.

- Thus

$$gr_*(G) = \oplus_{n\geq 1} gr_n(G)$$

is a Lie algebra.

EXAMPLE 1

(1) Let G denote the free group generated by a set S,

$$G = F[S].$$

Recall that the free Lie algebra generated by a set S denoted $L[S]$ is the smallest sub-Lie algebra of the tensor algebra $T[S]$ generated by S.

A classical result due to P. Hall and E. Witt is the next theorem.

Theorem 0.1.

$$gr_*(F[S]) = L[S]$$

Remark: A non-standard, direct proof of this result follows at once from the theory of Hopf algebras.

EXAMPLE 2

Let

$$G = P_k(\mathbb{R}^2) = P_k.$$

Recall Artin's generators

$$A_{i,j} \text{ for } P_k$$

(defined yesterday for the quiz).

The elements $A_{i,j}$ project to elements

$$B_{i,j}$$

in

$$gr_1(P_k) = P_k/\Gamma^2(P_k).$$

Note: If $1 \leq i < j$, there are analogous braids $A_{j,i}$ It is the case that $A_{j,i} = A_{i,j}$ in P_k.

Artin's relations imply that the elements $B_{i,j} \in gr_1(P_k)$ satisfy

the "horizontal 4T relations"

defined as follows:

HORIZONTAL 4T RELATIONS

The "horizontal 4T relations" or "infinitesimal braid relations" are as follows:

(1) $[B_{i,j}, B_{s,t}] = 0$ if $\{i, j\} \cap \{s, t\} = \phi$,

(2) $[B_{i,j}, B_{i,t} + B_{t,j}] = 0$ for $1 \leq j < t < i \leq k$, and

(3) $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ for $1 \leq j < t < i \leq k$.

Let

$$\mathcal{L}_k$$

denote the quotient of the free Lie algebra generated by the elements

$$B_{i,j} \text{ for } 1 \leq i < j \leq k$$

modulo the "horizontal 4T relations".

The next theorem was proven by T. Kohno and subsequently by M. Falk and R. Randell (in a context which also applies to certain other choices of $gr_*(G)$ for a large family of groups G).

Theorem 0.2.

$$gr_*(P_k) = \mathcal{L}_k.$$

A ”HORRIBLE COMPUTATION“

First, the result of a

”horrible computation“

is the following theorem.

Theorem 0.3. *If $n \geq 1$, then the homomorphism*

$$\Theta_n \colon F_n[y_1, \cdots, y_n] \to P_{n+1}$$

is injective.

The following natural questions arise:

Why is the theorem correct ?

Why is the theorem (possibly) useful ?

Where and how do these fit ? / Problems

These three questions give natural connections between several structures.

<u>WHY IS THE THEOREM CORRECT ?</u>

The following is a horrible computation.

Theorem 0.4.

The induced map of Lie algebras

$$gr_*(\Theta_n) : gr_*(F_n) \longrightarrow gr_*(P_{n+1})$$

is a monomorphism.

Remark: Since F_n is residually nilpotent, the map

$$\Theta_n : F_n \longrightarrow P_{n+1}$$

is a monomorphism.

WHY IS THE THEOREM USEFUL ?

Yesterday, Jie defined a simplicial group. That is a collection of groups

$$\{\Gamma_0, \Gamma_1, \Gamma_2, \cdots\}$$

denoted Γ_* with homomorphisms called face operations

$$d_i \colon \Gamma_n \to \Gamma_{n-1}, 0 \leq i \leq n,$$

together with homomorphisms called degeneracy operations

$$s_j : \Gamma_n \to \Gamma_{n+1}, 0 \leq j \leq n,$$

which satisfy the simplicial identities. (These identities are ones which many of you have been using for standard computations with the classical braid groups.)

AN EXAMPLE OF A SIMPLICIAL GROUP

One basic example of a simplicial group given by setting

$$\Gamma_n = P_{n+1}$$

for $n = 0, 1, 2, 3, \cdots$.

The face operations for $0 \le i \le n$ are induced by the $(n+1)$-projection maps

$$d_i = p_{i+1} : Conf(\mathbb{R}^2, n+1) \to Conf(\mathbb{R}^2, n)$$

on the level of fundamental groups.

The j-th degeneracy operation for $0 \le j \le n$ (with $1 \le j+1 \le n+1$) is induced by the map on fundamental groups obtained by "doubling the $j+1$-st strand"

$$s_j : Conf(\mathbb{R}^2, n+1) \to Conf(\mathbb{R}^2, n+2).$$

Call this simplicial group (which is obtained from the pure braid groups by "projections" and "doubling")

$$AP_*.$$

That

$$\Theta_n : F_n \longrightarrow P_{n+1}$$

is a monomorphism admits the following interpretation.

Theorem 0.5. *The collections of groups* AP_* *is naturally a simplicial groups.*

Furthermore, the smallest simplicial subgroup of AP_* *which contains the braid* $A_{1,2}$ *is Milnor's free group model for* ΩS^2*.*

Remark: The embedding Θ_n is providing a group theoretic "picture" for the loop space of the 2-sphere ΩS^2.

WHERE AND HOW TO THESE FIT ? PROBLEMS

(Problem 1) The braid groups appear to be "encoding" combinatorics for the homotopy groups of the 2-sphere as well as other spheres. Please see Jie's lectures for more information.

How can these properties be made more explicit as well as computationally effective ?

For example, is there a natural way in which the structure of the braid groups imply that

(i) the order of 2-torsion in $\pi_*(S^2)$ is bounded above by 4 and

(ii) the p-torsion is bounded above by p for an odd prime p ?

(Problem 2) The proof above using Lie algebras admits an interpretation in terms of the Adams/Bousfield-Kan spectral sequence on the level of classical homotopy groups.

That is the Lie algebra obtained from the descending central series of the pure braid groups is the E^0-term of the associated spectral sequence.

Do these structures inform on invariants of braids or possibly knots and/or vice versa ? Find a "sensible" explanation for this fact.

(Problem 3) Ryan Budney analyzed much of the stucture of the space of

”long knots in $\mathbb{R}^3$”.

The homotopy groups as well as much of the homology groups for this space are understood.

Dev Sinha, Ismar Volic, Pascal Lambrechts, Viktor Turchin and others have exhibited related structure for the space of ”long knots in $\mathbb{R}^n$” for $n > 3$, however, the homotopy and homology groups are still not well-understood.

The Lie algebra

$$gr_*(P_k)$$

is basic in their work on the homotopy groups of

”long knots in $\mathbb{R}^n$“ for $n > 3$.

It seems natural to try to identify the connection to AP_* more closely as well as to find the homology and homotopy of the space of "long knots in $\mathbb{R}^n$.

(Problem 4) The proof that Θ_n is a monomorphism admits another interpretation:

There is a natural morphism of Lie algebras

$$D : gr_*(P_{n+1}) \to Der(L[V_n])$$

where

$$Der(L[V_n])$$

denotes the Lie algebra of derivations of the free Lie algebra $L[V_n]$ generated by n elements.

The map D arises from the fact that the sub-Lie algebra of $gr_*(P_{n+1})$ generated by the elements

$$B_{i,n} \text{ for } 1 \leq i \leq n$$

is a Lie ideal.

Theorem 0.6. *The composite*

$$gr_*(F_n) \xrightarrow{gr_*(\Theta_n)} gr_*(P_{n+1}) \xrightarrow{D} Der(L[V_n])$$

is an embedding.

Earlier, Y. Ihara constructed an embedding

$$\Lambda : gr_*(Gal(\bar{\mathbb{Q}}/\mathbb{Q})) \to Der(L[V_n \otimes_{\mathbb{Z}} (\prod_{p \text{ is prime}} \mathbb{Z}_p)]).$$

Coincidentally, the images of the two maps

$$\Lambda \text{ and } D \circ gr_*(\Theta_n)$$

are not equal but have an interesting overlap (and induce non-trivial intersections with the group theoretic analogue of ΩS^2 given by $\Theta_n(F_n)$).

Is this connection an "accident" ?

Find a computationally useful "explanation" for this fact.