# ON SPACES OF HOMOMORPHISMS

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# INTRODUCTION

This lecture is based on joint work with Jon Lopez, Stratos Prassidis, and Marston Conder.

The motivation arises from work of Toshitake Kohno, Alex Adem, Enrique Torres-Giese, Jon Berrick, Yan Loi Wong as well as **many projects** with Jie Wu and the speaker.

# SEMIDIRECT PRODUCTS

Let

$$G$$

denote a discrete group such that

(1) $\pi$ is a normal subgroup of $G$,
(2) the natural quotient homomorphism

$$p : G \to G/\pi = \Gamma$$

is split.

This means that there is a homomorphism

$$\sigma : G/\pi \to G$$

such that

$$p \circ \sigma = \text{identity} : G/\pi \to G/\pi.$$

In this case

**$G$ is called a semi-direct product of $\pi$ and $\Gamma$.**

This structure is frequently written ( in the language of extensions ) as a **split** short exact sequence of groups:

$$1 \longrightarrow \pi \xrightarrow{\;i\;} G \xrightarrow{\;p\;} \Gamma \longrightarrow 1.$$

# EXAMPLES

**Example 1**: The product

$$G = \Gamma \times \pi.$$

**Example 2**: The dihedral group of order $2n$,

$$D_{2n},$$

is given by the split extension

$$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{i} G \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

**Example 3**: The pure braid group on $n$-strands

$$P_n = P_n(\mathbb{R}^2)$$

is given by the split extension

$$1 \longrightarrow F_{n-1} \xrightarrow{i} P_n \xrightarrow{p} P_{n-1} \longrightarrow 1.$$

**Example 4** : The extension

$$1 \longrightarrow \mathbb{Z} \xrightarrow{i} P_3(S^2) \xrightarrow{p} P_2(S^2) \longrightarrow 1$$

is not split where $P_n(S)$ denotes the n-stranded pure braid group for the surface $S$.

# MAIN HYPOTHESES

Consider the split short exact sequence of groups:

$$1 \longrightarrow \pi \xrightarrow{\ i\ } G \xrightarrow{\ p\ } \Gamma \longrightarrow 1.$$

**Assume that both $\pi$ and $\Gamma$**

**are subgroups of $GL(m, \mathbb{R})$ for some integer $m$.**


# MAIN PROBLEM

**Identify conditions for $G$ which guarantee**

**that $G$ is isomorphic to a subgroup of $GL(N, \mathbb{R})$**

**for some integer $N$ .**

# <u>REMARKS</u>

**Remark 1**: The main results here are partially successful. The methods as well as a few examples are illustrated.

**Remark 2**: A basic example due to Formanek and Procesi implies that some (strong) hypotheses are required. Their extension, the "poison subgroup", is a split extension

$$1 \longrightarrow F_3 \overset{i}{\longrightarrow} H \overset{p}{\longrightarrow} F_2 \longrightarrow 1$$

where $F_n$ is a free group on $n$ letters and the group $H$ admits the following presentation:

$$\langle a_1, a_2, a_3, \phi_1, \phi_2 \mid \phi_i a_j \phi_i^{-1} = a_j, \phi_i a_3 \phi_i^{-1} = a_3 a_i, i, j = 1, 2 \rangle.$$

**Remark 3**: In this case, the Lie algebra attached to the descending central series for the group $H$ "collapses around it's ears".

# THE BASIC CONSTRUCTION

Recall that a semi-direct product

$$1 \to \pi \to G \to \Gamma \to 1$$

is determined uniquely by a homomorphism

$$\rho : \Gamma \to Aut(\pi)$$

where

$$Aut(\pi)$$

denotes the automorphism group of $\pi$.

In particular, the universal semi-direct product is the so-called holomorph, $Hol(G)$, a group which dates back to Burnside and is given by

$$1 \to \pi \to Hol(\pi) \to Aut(\pi) \to 1.$$

The group $Hol(\pi)$, **as a set** is the product

$$Aut(\pi) \times \pi.$$

The product structure is defined by the formula

$$(f, x) \cdot (g, y) = (f \cdot g, g^{-1}(x) \cdot y)$$

for $f, g$ in $Aut(\pi)$, and $x, y$ in $\pi$. This "twisted" product structure appears (implicitly) in many of the lectures at this conference.

A semidirect product is given by a pull-back as follows:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi & \longrightarrow & G & \longrightarrow & \Gamma & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \rho} & & \\
1 & \longrightarrow & \pi & \longrightarrow & \mathrm{Hol}(\pi) & \longrightarrow & Aut(\pi) & \longrightarrow & 1
\end{array}
$$

**Remark 4**: These group extensions correspond in a natural way to certain fibre bundles. The methods for addressing these extensions are sometimes more accessible geometrically, sometimes more accessible algebraically depending on the context.

The group $\mathrm{Hol}(F_n)$ is naturally a subgroup of $Aut(F_{n+1})$. Furthermore, this construction provides natural quotients of $B_n$ as well as a mildly different proof that Artin's representation of $B_n$ is faithful when restricted to $\mathrm{Hol}(F_n)$ as observed in math.AT/0409307 by Jie Wu and the speaker.

# FILTRATIONS

The methods here arise by considering "bite-size" pieces of a group and arise formally by considering filtrations of a group and how to fit these together with the map

$$\rho : \Gamma \rightarrow Aut(\pi)$$

which determines the extension.

That is, the methods here are to **develop properties of a particular homomorphism** rather than trying to directly construct representations.

## A filtration of a group $\pi$

is a descending chain of normal subgroups

$$\cdots \subseteq L_j(\pi) \subseteq \cdots \subseteq L_1(\pi) \subseteq L_0(\pi) = \pi$$

for $j \geq 0$ such that $\cap_{j \geq 1} L_j(\pi) = \{1\}$.

# A bounded $p$-congruence system

## for the group $\pi$

is a filtration of $\pi$

$$\cdots \subseteq L_j(\pi) \subseteq \cdots \subseteq L_1(\pi) \subseteq L_0(\pi) = \pi$$

for $j \geq 0$ such that

(1) $\pi/L_1(\pi)$ is finite,

(2) $L_1(\pi)/L_{1+j}(\pi)$ is a finite $p$-group

  for all $j \geq 0$ and

(3)
$$d(L_i(\pi)/L_j(\pi)) \leq d$$

  for all $0 \leq i < j$ where $d(G)$ denotes the minimal number of generators of the group $G$.

# A THEOREM OF A. LUBOTZKY

A. Lubotzky exhibited a condition which identifies whether finitely generated discrete group $H$ admits a faithful finite dimensional representation in $GL(n, \mathbb{F})$ where $\mathbb{F}$ denotes a field of characteristic zero.

**Theorem 0.1.** *A finitely generated group $H$ admits a faithful finite dimensional representation over a field of characteristic zero if and only if there exists a prime p together with a bounded p-congruence system for the group $H$.*

# TWO FILTRATIONS

The main approach here is to intertwine filtrations for both $\pi$ and $\Gamma$ which "glue together" to give a filtration of the semi-direct product of groups

$$1 \to \pi \to G \to \Gamma \to 1.$$

The ingredients are **two filtrations**

(1) $F_*(\Gamma)$ given by

$$\cdots \subseteq F_j(\Gamma) \subseteq \cdots \subseteq F_1(\Gamma) \subseteq F_0(\Gamma) = \Gamma$$

with $j \geq 0$ for the group $\Gamma$, and

(2) $L_*(\pi)$ given by

$$\cdots \subseteq L_j(\pi) \subseteq \cdots \subseteq L_1(\pi) \subseteq L_0(\pi) = \pi$$

with $j \geq 0$ for the group $\pi$.

# STABLE EXTENSIONS

The group extension

$$1 \longrightarrow \pi \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

together with the two filtrations $F_*(\Gamma)$ and $L_*(\pi)$ is said to be **stable** provided

for every
$$(g, y) \in F_{r+s}(\Gamma) \times L_{r+s}(\pi)$$

and for every
$$(f, x) \in F_r(\Gamma) \times L_r(\pi)$$
the following properties are satisfied:

$$(1)\ f(y) \in L_{r+s}(\pi)\ \text{and}$$

$$(2)\ g(x) = x \cdot \gamma_x\ \text{for}\ \gamma_x \in L_{r+s}(\pi).$$

# A THEOREM ON STABLE EXTENSIONS

**Theorem 0.2.** *Assume that the split extension*

$$1 \longrightarrow \pi \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

*satisfies the following properties.*

(1) *the groups $\Gamma$ and $\pi$ are filtered by bounded $p$-congruence systems, and*
(2) *the extension is stable with respect to these two filtrations.*

*Then $G$ admits a faithful finite dimensional representation in $GL(N, \mathbb{R})$ for some $N$.*

**Corollary 0.3.** *Assume that*

*(1) there is a split extension*

$$1 \longrightarrow \pi \longrightarrow G \longrightarrow \Gamma \longrightarrow 1,$$

*(2) the groups $\Gamma$ and $\pi$ are filtered by bounded p-congruence systems,*

*(3) the extension is stable with respect to these two filtrations, and*

*(4) the filtration quotients $gr_*(\pi)$ and $gr_*(\Gamma)$ admit the structure of Lie algebras ( induced by the commutator ).*

*Then $G$ admits a filtration such that the associated graded is a Lie algebra*

$$gr_*(G)$$

*which gives a split short exact sequence of Lie algebras*

$$0 \longrightarrow gr_*(\pi) \xrightarrow{i} gr_*(G) \xrightarrow{p} gr_*(\Gamma) \longrightarrow 0.$$

Are there any examples ?

How do these structures fit ?

# A CLASSICAL EXAMPLE

Consider the natural quotient map $\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$ together with the induced kernel of the quotient map

$$SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/q\mathbb{Z})$$

denoted

$$\Gamma_q.$$

If $p$ denotes a prime, then the "tower" of groups

$$\cdots \subseteq \Gamma_{p^r} \subseteq \cdots \subseteq \Gamma_{p^2} \subseteq \Gamma_p$$

is a bounded p-congruence system for the free group on $1 + p(p^2 - 1)/12$ letters with

$$\Gamma_{p^r}/\Gamma_{p^{r+1}} = \oplus_3 \mathbb{Z}/p\mathbb{Z}$$

for $p$ an odd prime.

The Lie algebras associated to principal congruence subgroups associated to $SL(n, \mathbb{Z})$ as well as related linear groups such as $SL(n, \mathbb{Z}[t^{\pm 1}])$ have been analyzed in the thesis of Jon Lopez.

# AUTOMORPHISMS OF TOWERS

Consider a filtration of a group $\pi = \pi_p$ ( also called a "tower")

$$\cdots \longrightarrow \pi_{p^3} \longrightarrow \pi_{p^2} \longrightarrow \pi_p = \pi$$

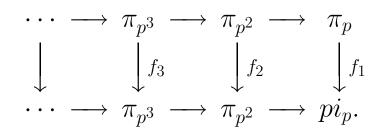The automorphism group of this tower denoted

$$Aut(\pi^*)$$

means the group of automorphisms of $\pi$ given by those isomorphisms

$$f_1 : \pi_p \to \pi_p$$

which restrict to automorphisms

$$f_r : \pi_{p^r} \to \pi_{p^r}$$

for every $r$ as pictured on the next page.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \pi_{p^3} & \longrightarrow & \pi_{p^2} & \longrightarrow & \pi_p \\
& & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 \\
\cdots & \longrightarrow & \pi_{p^3} & \longrightarrow & \pi_{p^2} & \longrightarrow & pi_p.
\end{array}$$

**Theorem 0.4.** *Assume that*

$$\longrightarrow \pi_{p^3} \longrightarrow \pi_{p^2} \longrightarrow \pi_p = \pi$$

*is a p-congruence system for the group $\pi = \pi_p$ with automorphism group of the tower given by $Aut(\pi^*)$.*

*If the split extension*

$$1 \longrightarrow \pi \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

*is classified by a map*

$$\Gamma \xrightarrow{\rho} Aut(\pi, *)$$

*and $\Gamma$ is a subgroup of $GL(m, \mathbb{R})$, then*

*$G$ is a subgroup of $GL(N, \mathbb{R})$ for some $N$.*

# A QUESTION

Consider the automorphism of a free group on $n$ letters,

$$Aut(F_n).$$

Let

$$IA_n$$

denote the kernel of the natural map

$$Aut(F_n) \to GL(n, \mathbb{Z}).$$

Is it the case that all of the elements in $IA_n$ are realized by the automorphisms of the tower for a p-congruence system for the group $F_n$ ?

# A WILD GUESS

Fix a basis $\{x_1, x_2, \cdots, x_n\}$ for $F_n$ and consider the subgroup generated by the automorphisms of $F_n$

$$\chi_{k,i}$$

defined by

$$\chi_{k,i}(x_j) = \begin{cases} x_j & \text{if } k \neq j, \\ (x_i^{-1})(x_k)(x_i) & \text{if } k = j. \end{cases}$$

This subgroup is known as the **"group of loops"** as defined by Craig Jensen, Jon McCammond and John Meier, **"McCool's group"** , or **"the group of basis conjugating automorphisms"**.

Is it the case that all of the elements in $M_n$ are realized by the automorphisms of the tower for a p-congruence system for the group $F_n$ ? Guess: yes.

# EXAMPLES AND NON-EXAMPLES

**Example 1**:

It is as yet far from clear whether the methods imply Bigelow and Krammer's result for Artin's braid group: it is unclear whether the required monodromy can be realized as an automorphism of a tower ( which suffices ).

**Example 2**:

Some examples arise by automorphisms of towers given by principle congruence subgroups. These do not suffice to show that Artin's braid group is linear. They do apply to show that some choices of subgroups of McCool's group admit faithful finite dimensional representations.

Further applications from other automorphisms arise from surfaces of genus greater than 2 and will be addressed elsewhere. One example is on the next page.

**Example 3**: Consider the extension

$$F[a_1, a_2, \cdots , a_n, b] \longrightarrow G_n \longrightarrow F[x, y]$$

for which the action of $F[x, y]$ is given as follows.

The action of $x$ is given by
(1) $x(a_q) = a_{q+1}$ if $1 \leq q < n$ with $x(a_n) = b \cdot a_1 \cdot b^{-1}$
and
(2) $x(b) = b$.

The action of $y$ is given by
(1) $y(a_q) = a_1 \cdot a_q \cdot a_1^{-1}$ and
(2) $y(b) = a_1 \cdot b \cdot a_1^{-1}$.

The group $G_n$ admits a faithful finite dimensional representation by the above theorem. After knowing this, it is then direct to construct a concrete faithful representation in $GL(8, \mathbb{R})$ ( although it may be easy to lower the value "8" ).

# CONNECTION TO DALE'S LECTURES

Consider the split extension as filtered by Theorem 0.2

$$1 \longrightarrow \pi \longrightarrow G \longrightarrow \Gamma \longrightarrow 1.$$

If the group $\Gamma$ is residually nilpotent and satisfies the conditions that

- $gr_*(G) = \oplus_{g \geq 0} F_j(G)/F_{j+1}(G)$ is torsion free,
- $\pi = F_n$, and
- the monodromy is in $IA_n$,

then Dale proves that $G$ is orderable.

## Is this an accident ?

## Does $G$ admit a faithful finite dimensional representation ?

# JOAN,


## We all wish you


# A VERY HAPPY BIRTHDAY !