Densely ordered braid subgroups

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A group G is left-ordered if there exists a strict total ordering $<$ of its elements such that $g < h \Rightarrow fg < fh$.

Given such an ordering the positive cone

$$P = \{g \in G | 1 < g\} \quad \text{satisfies}$$

- $P \cdot P \subset P$ and

- For every $g \in G$, exactly one of $g = 1$, $g \in P$, or $g \in P^{-1}$ holds.

Conversely, if a group has a subset P satisfying the above, then a left-ordering may be defined by $g < h \Leftrightarrow g^{-1}h \in P$.

Left-orderings of a group can be either

Discrete: every element has an immediate successor and predecessor. Equivalently, the positive cone has a least element.

or

Dense: whenever $f < h$, there exists g with $f < g < h$.

A discretely left-ordered group $(G,<)$ can contain a subgroup H such that $(H,<)$ is densely ordered by the SAME ordering!

Example. *Consider $\mathbb{Q}\times\mathbb{Z}$, with the lexicographic order:*

$$(p,m)<(q,n) \Longleftrightarrow p<q \text{ or else } p=q \text{ and } m<n.$$

Then the ordering is discrete, with least positive element $(0,1)$, but the subgroup $\mathbb{Q}\times\{0\}$ is densely ordered by the restriction of the lexicographic ordering.

We will see that a similar phenomenon occurs rather naturally in the braid groups B_n.

Recall that B_n has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$ and

$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ if $|i - j| = 1$.

For $1 \leq i \leq n$ there are natural inclusions $B_i \hookrightarrow B_n$.

Note that if the generators are allowed to commute, they all become equal. That is: $\dfrac{B_n}{[B_n, B_n]} \cong \mathbb{Z}$ and the commutator subgroup $[B_n, B_n]$ is precisely the set of words in the generators with total exponent zero.

Theorem (Dehornoy). *B_n is left-orderable.*

The Dehornoy ordering has a positive cone $P \subset B_n$ as follows: A word in the generators $\sigma_1, \cdots, \sigma_i$ is said to be i-positive if the generator σ_i occurs at least once, and with only positive exponents. A braid $\beta \in B_n$ is said to be i-positive if $\beta \in \langle \sigma_1, \cdots, \sigma_i \rangle$ and it admits at least one i-positive representative braid word.

The positive cone in B_n is

$$P = \{\beta \in B_n : \beta \text{ is } i\text{-positive for some } i\}.$$

A key fact established by Dehornoy is that for any braid $\beta \in \langle \sigma_1, \cdots, \sigma_i \rangle \subset B_n$ we have exactly one of the following:

- β is i-positive

- β is i-negative, i.e. β^{-1} is i-positive

- β is i-neutral, meaning $\beta \in \langle \sigma_1, \cdots, \sigma_{i-1} \rangle$.

Proposition. *The Dehornoy ordering of B_n is discrete, with least positive element σ_1.*

Proof. Clearly $\sigma_1 > 1$. If $1 < \beta < \sigma_1$, then β is i-positive for some i.

- If $i > 1$, then $\sigma_1^{-1}\beta$ is still i-positive, so

$$1 < \sigma_1^{-1}\beta \Rightarrow \sigma_1 < \beta.$$

- If $i = 1$, then $\beta = \sigma_1^k$, yet $\beta < \sigma_1$. Hence $k < 1$.

□

Let $C(r)$ = the centralizer of B_{r-1} in B_r.

Lemma. *Suppose N is a non-trivial normal subgroup of B_n, with $N \cap B_{n-1} = \{1\}$ and $n \geq 3$. Then if N is discretely ordered by the Dehornoy ordering, the least positive element lies in $C(n)$.*

Proof: Choose $\beta \in N \setminus C(n)$, and choose $\gamma \in B_{n-1}$ not commuting with β. We'll find $\alpha \in N$ with $1 < \alpha < \beta$, by considering 3 cases:

Case 1: $\beta\gamma\beta^{-1}$ is $(n-1)$ -neutral: Then $\beta\gamma\beta^{-1}\gamma^{-1} \in N \cap B_{n-1} = \{1\}$, contradiction – this case doesn't occur.

Case 2: $\beta\gamma\beta^{-1}$ is $(n-1)$ -positive. Then choose $\alpha := \beta\gamma\beta^{-1}\gamma^{-1}$.

Case 3: $\beta\gamma\beta^{-1}$ is $(n-1)$ -negative, choose $\alpha := \beta\gamma^{-1}\beta^{-1}\gamma$.

Proof of case 2: ($\beta\gamma\beta^{-1}$ is $(n-1)$ -positive)

We've chosen $\alpha := \beta\gamma\beta^{-1}\gamma^{-1}$.

Since $\beta\gamma\beta^{-1}$ is $(n-1)$-positive, the braid $\beta\gamma\beta^{-1}\gamma^{-1}$ is also $(n-1)$-positive (recall $\gamma \in B_{n-1}$ so it does not contain σ_{n-1}).

Since β is $(n-1)$-positive, $\gamma\beta^{-1}\gamma^{-1}$ is $(n-1)$-negative, so

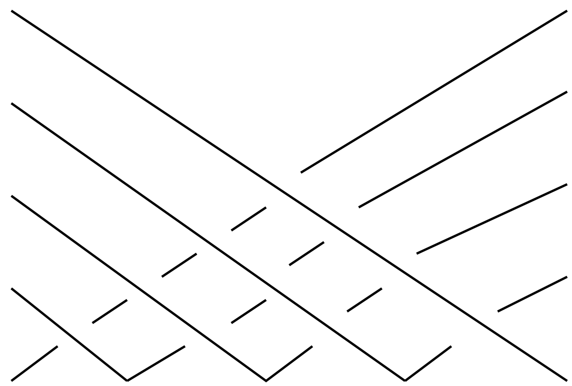
$$\gamma\beta^{-1}\gamma^{-1} < 1 \Rightarrow \beta\gamma\beta^{-1}\gamma^{-1} < \beta.$$

Together,

$$1 < \alpha < \beta.$$

The Garside Half-twist is the braid

$$\Delta_k := (\sigma_1\sigma_2\cdots\sigma_{k-1})(\sigma_2\cdots\sigma_{k-1})\cdots(\sigma_{k-2}\sigma_{k-1})(\sigma_{k-1}).$$



The braid Δ_5

The **centre** ZB_n, $n \geq 3$, is infinite cyclic, generated by Δ_n^2. The ordering of B_n, restricted to ZB_n is necessarily discrete, and its least element is Δ_n^2, which is of course in $C(n)$. ($ZB_2 = B_2$, generated by $\Delta_1 = \sigma_1$)

Theorem. *Suppose N is a discretely ordered nontrivial normal subgroup of B_n. Then the least positive element of N is either a power of σ_1, or lies in $C(r)$, where r is the largest integer such that $3 \leq r \leq n-2$ and $N \cap B_{r-1}$ is trivial.*

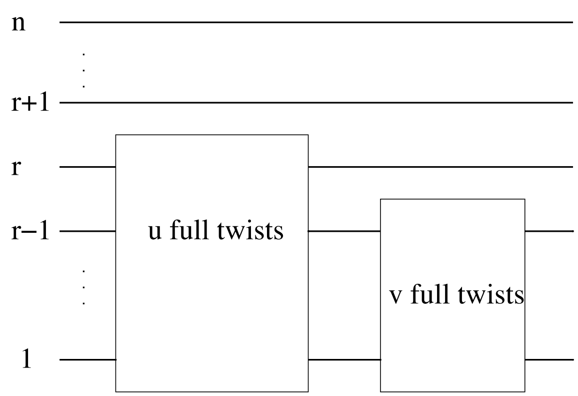
Proof. (Sketch) If $\sigma_1^m \in N$ for some $m > 0$, then some positive power of σ_1 is the smallest positive element.

Otherwise, $N \cap B_{r-1}$ is trivial for some r, while $N \cap B_r$ is not, and apply the lemma with r replacing n to get the smallest positive element of $N \cap B_r$. Finally, argue this is in fact the smallest positive element of N. □

Proposition (Fenn, Rolfsen, Zhu 1996). *For $r \geq 3$, the centralizer $C(r)$ of B_{r-1} in B_r consists of the elements*

$$\Delta_3^{2u}\sigma_1^v \quad \textit{if } r = 3, \qquad \Delta_r^{2u}\Delta_{r-1}^{2v} \quad \textit{if } r > 3,$$

where $u, v \in \mathbb{Z}$.



Typical element of $C(r)$

Note that in each case, the two parts of the word commute, so $C(r) \cong \mathbb{Z} \times \mathbb{Z}$.

Proposition. *Let N be a discretely ordered nontrivial normal subgroup of B_n, and r the greatest integer such that $N \cap B_{r-1}$ is trivial. Then the least positive element $\beta \in N$ is of the form:*

$$\beta = \Delta_r^{2u} \quad \text{or} \quad \beta = \sigma_1^u,$$

where $u \in \mathbb{Z}$ is positive.

Proof. Set $\beta = \Delta_r^{2u} \Delta_{r-1}^{2v}$. Then if $v > 0$, we show that

$$1 < \beta \sigma_{r-1} \beta^{-1} \sigma_{r-1}^{-1} < \beta,$$

and if $v < 0$, then

$$1 < \beta^{-1} \sigma_{r-1} \beta \sigma_{r-1}^{-1} < \beta.$$

Therefore, unless $v = 0$ the braid β is not minimal positive. $\square$

Recall $[B_n, B_n] \subset B_n$ contains exactly those braids having a representative word with zero exponent sum. Neither Δ_r^{2u} nor σ_1^u, $u \geq 1$, have zero exponent sum, so we conclude:

- If $n \geq 3$, $[B_n, B_n]$ is densely ordered.

- Any nontrivial $N \subset [B_n, B_n]$ that is normal in B_n ($n \geq 3$) is densely ordered under the Dehornoy ordering.

Further examples:

- $[P_n, P_n] \subset [B_n, B_n]$ is normal in B_n, and so densely ordered.

- The Burau representation

$$\rho_n : B_n \to GL_n(\mathbb{Z}[t, t^{-1}]),$$

defined on generators by

$$\sigma_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1},$$

satisfies $\ker(\rho_n) \subset [B_n, B_n]$, so the kernel is densely ordered for values of n for which the representation is unfaithful.

- The kernel of $h : B_4 \to B_3$ defined by

$$h(\sigma_1) = \sigma_1, \quad h(\sigma_2) = \sigma_2, \quad h(\sigma_3) = \sigma_1$$

is the normal closure of $\sigma_1 \sigma_3^{-1}$ in B_4 and therefore is densely ordered.

A braid $\beta \in B_n$ is said to be Brunnian if, for each strand, deleting that strand from β results in the trivial $(n-1)-$braid.

- For $n \geq 3$ the subgroup of Brunnian braids in B_n is densely ordered.

- For $n \geq 3$ and $1 \leq k < n-1$ the subgroup of k-Brunnian braids in B_n is densely ordered.

A k-Brunnian braid is one which becomes trivial upon removal of an arbitrary set of k strands. The set of k-Brunnian braids is a normal subgroup of B_n and is nontrivial provided $1 \leq k < n-1$.

Clearly none of the candidates for least positive element is k-Brunnian.

A braid $\beta \in B_n$ is homotopically trivial if there is a homotopy from β to the trivial braid, as a disjoint mapping of strings in $\mathbb{R}^2 \times \mathbb{R}$ with endpoints fixed. Thus the strings may deform to non-braids and cross themselves, but cannot cross each other during the homotopy. Goldsmith showed that the set of homotopically trivial braids in B_n, $n \geq 3$, is a nontrivial normal subgroup of the pure braid subgroup, thus answering a question of Artin.

- For $n \geq 3$, the subgroup of homotopically trivial braids in B_n is densely ordered.

This is clear, for similar reasons to the above.

Thanks for listening.....