Dimension of Torelli groups

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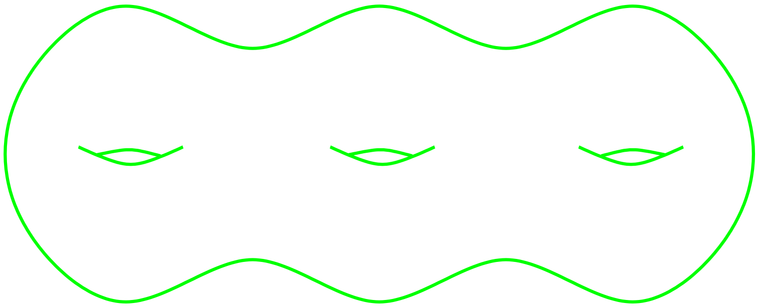
The Torelli group

S_g = surface of genus g



$\mathrm{Mod}(S_g) = \pi_0(\mathrm{Homeo}^+(S_g))$

Definition of Torelli group $\mathcal{I}_g$:

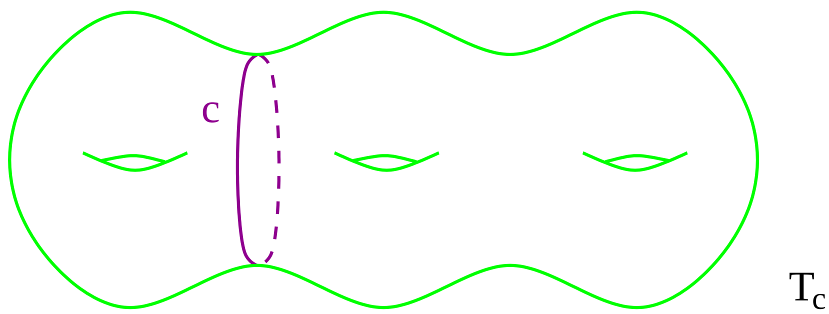
$$1 \to \mathcal{I}_g \to \mathrm{Mod}(S_g) \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to 1$$

Compare

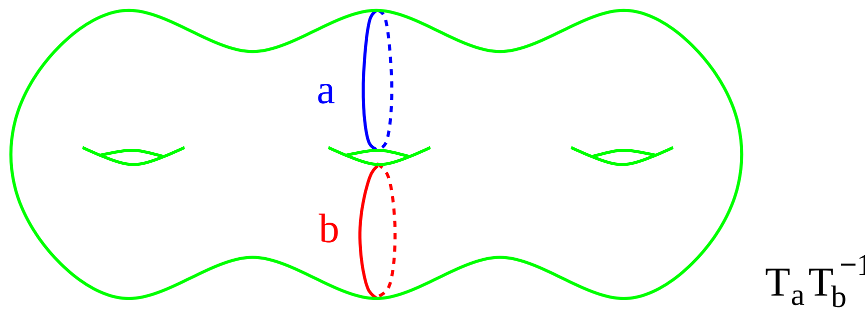
$$1 \to PB_n \to B_n \to \Sigma_n \to 1$$

Elements of the Torelli group

Dehn twists about separating curves



Bounding pair maps



Thm (Birman '71 + Powell '78). *These elements generate $\mathcal{I}_g$.*

Known finiteness properties for $\mathcal{I}_g$

Dehn 1920's: $\mathcal{I}_1 = 1$

Johnson 1983: $\mathcal{I}_g$ finitely generated $g \geq 3$

McCullough–Miller 1986, Mess 1992: $\mathcal{I}_2 \cong F_\infty$

Johnson–Millson–Mess 1992: $H_3(\mathcal{I}_3, \mathbb{Z})$ not f.g.

Akita 2001: $H_\star(\mathcal{I}_g, \mathbb{Z})$ not f.g., $g \geq 7$

Big Question: Finitely presented?

Cohomological dimension

$G =$ discrete group

$H^\star(G, M) = H^\star(K(G,1), M)$

$\mathrm{cd}(G) = \sup_n \{H^n(G, M) \neq 0 \quad \text{some } M\}$

Eilenberg–Ganea (+ Stallings, Swan):
If $\mathrm{cd}(G) \neq 2$, then $\mathrm{cd}(G)$ equals the smallest dimension of a $K(G,1)$.

Examples: $\mathrm{cd}(\mathbb{Z}^n) = n$, $\mathrm{cd}(B_n) = n - 1$

Main Theorem. *Let $g \geq 2$.*

$$\mathrm{cd}(\mathcal{I}_g) = 3g - 5$$

Mess 1990: $\mathrm{cd}(\mathcal{I}_g) \geq 3g - 5$

BBM 2007: $\mathrm{cd}(\mathcal{I}_g) \leq 3g - 5$

We also recover:

Thm (Mess '92). *$\mathcal{I}_2$ is a free group, with one Dehn twist generator for each homology splitting.*

A principle of Quillen

$G =$ group

$X =$ contractible CW-complex

$G \circlearrowleft X$

$$\mathrm{cd}(G) \leq \sup\{\mathrm{cd}(\mathrm{Stab}(\sigma)) + \dim(\sigma)\}$$

The supremum is over simplices σ of X.

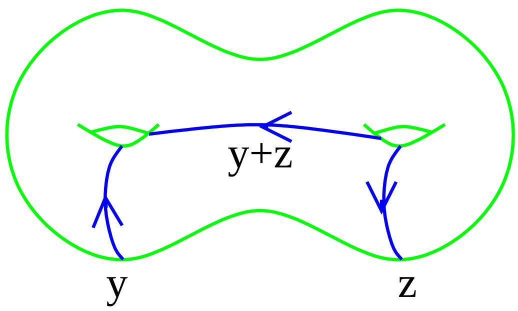
An idea for a complex

Fix some $x \in H_1(S, \mathbb{Z})$

Pick a hyperbolic metric on S

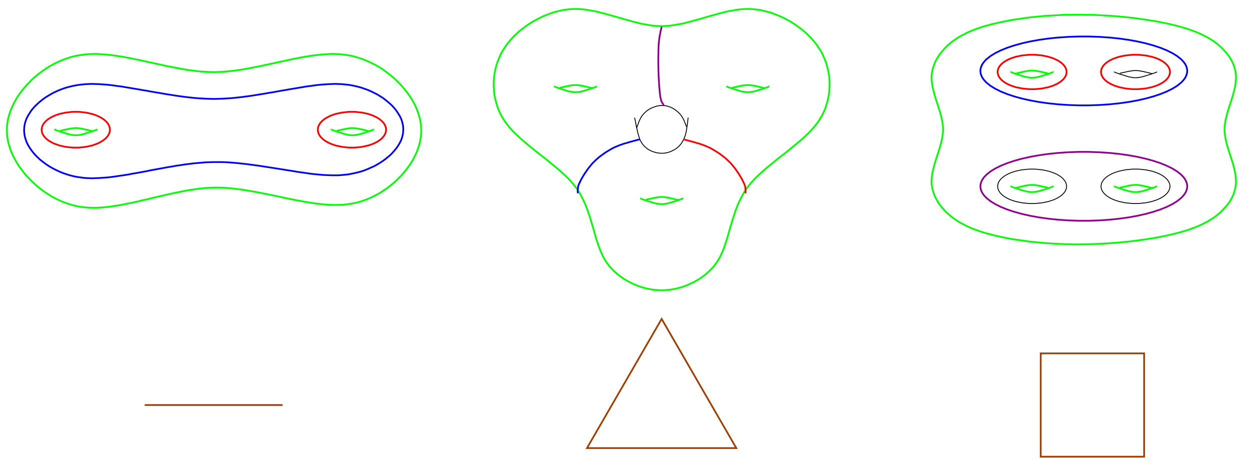
Look at the *space* of shortest 1-cycles in S representing x.

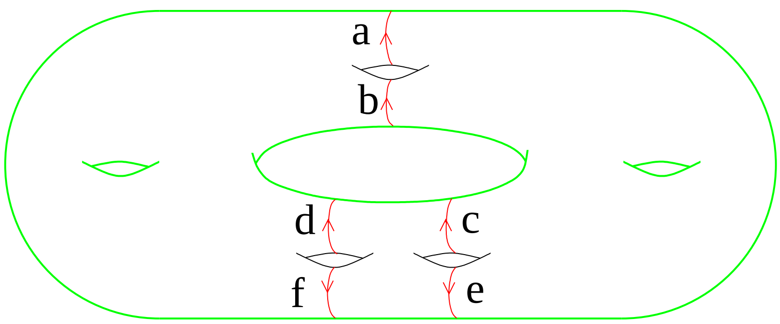
Example

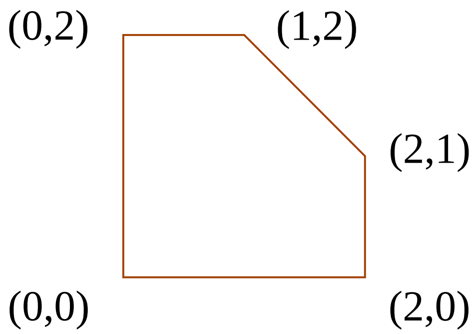


Get a cell!

Examples of cells







$$x = [d] + 2[e] + [f]$$

Minimizing cycles

Again, fix some $x \in H_1(S, \mathbb{Z})$ and pick a hyperbolic metric on S

A *(real) minimizing cycle* is a shortest real-valued 1-chain in S which represents x

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How do we know that minimizing cycles exist?

Minimizing cycles II

Why do minimizing cycles exist?

Step 1. Integral minimizing cycles

Integral minimizing cycles exist.

Step 2. Rational minimizing cycles

You cannot make a shorter cycle by using rational coefficients instead of integers (clear denominators).

Step 3. Real minimizing cycles

You still cannot make a shorter cycle if you try real coefficients ("borrow" until all coefficients are rational).

As indicated earlier, the space of real minimizing cycles for a given x and a given hyperbolic metric is always a compact polytope.

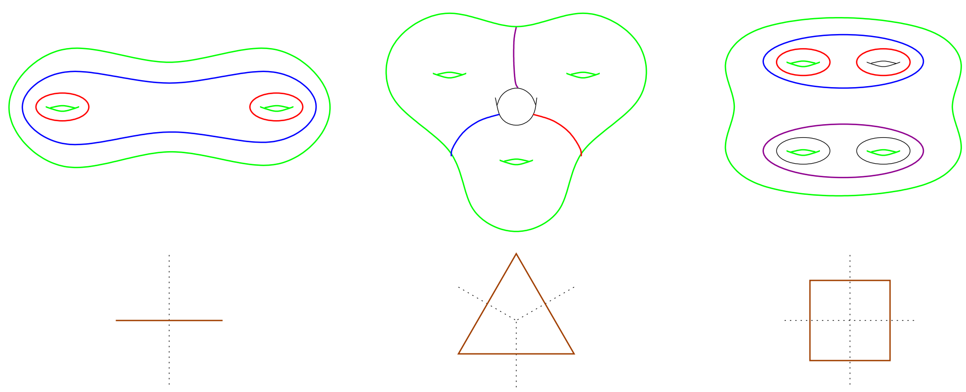
The complex of minimizing cycles

Teich(S) = space of hyperbolic metrics on S

A *chamber* of Teich(S) is the set of points with the same polytope

Proposition: Chambers are contractible.

Definition of the complex: chambers and polytopes are posets, by inclusion. There is a natural functor. The geometric realization is a complex $\mathcal{B}$, on which $\mathcal{I}_g$ acts.



Theorem: $\mathcal{B}$ is contractible.

Proof: $\mathcal{B}$ models Teich(S) $\cong \mathbb{R}^{6g-6}$

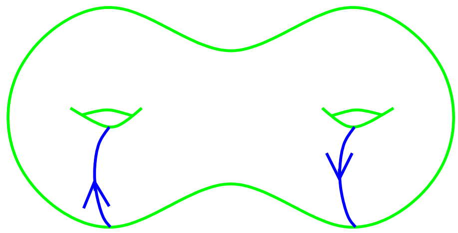
Stabilizers

Recall Quillen condition:

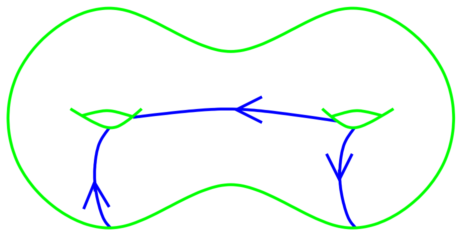
$$\mathrm{cd}(\mathcal{I}_g) \leq \sup\{\mathrm{cd}(\mathrm{Stab}(\sigma)) + \dim(\sigma)\}$$

Idea of proof: stabilizer of a cell of $\mathcal{B}$ is the stabilizer in $\mathcal{I}_g$ of a multicurve, i.e., the Torelli group of a simpler surface. Apply induction.

In genus 2, stabilizers of vertices are 1-dimensional

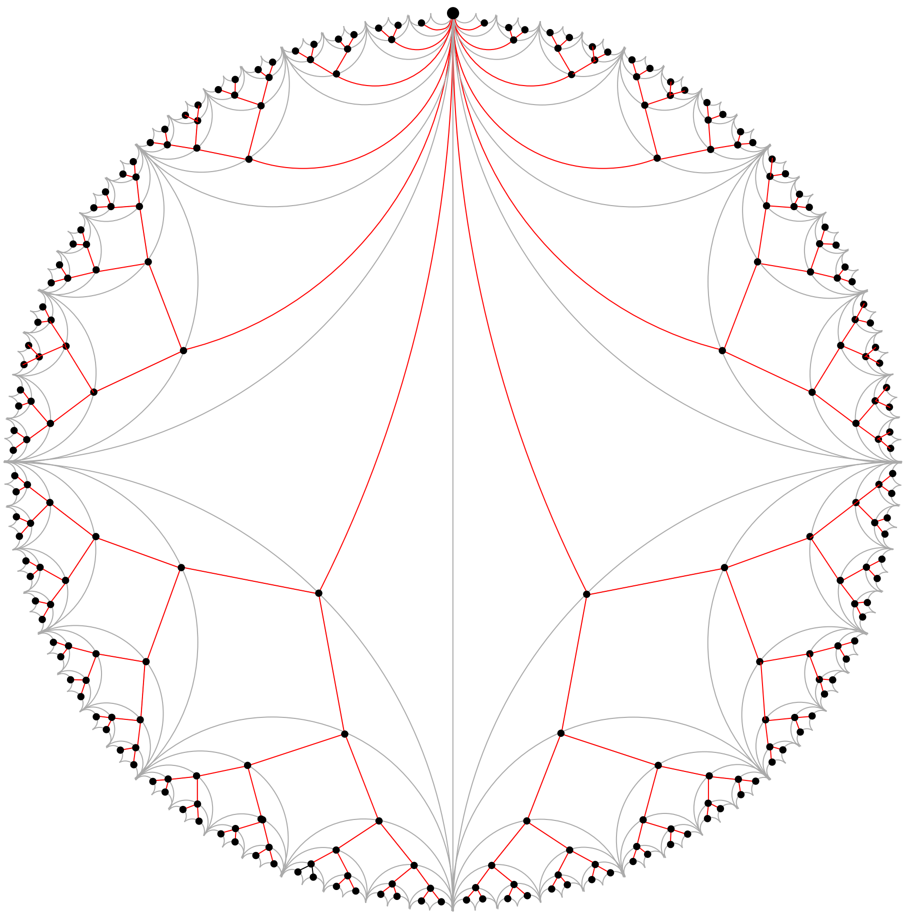


and stabilizers of edges are trivial (0-dimensional)



Genus 2

The quotient of $\mathcal{B}$ by $\mathcal{I}_2$ is a tree, which is infinitely many copies of



glued along their distinguished vertices.