Twisted Reidemeister torsion for twist knots

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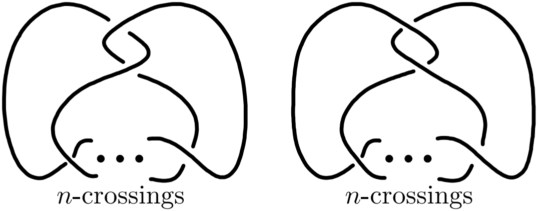

Figure: Twist knots $J(2, n)$ and $J(-2, n)$, $n > 0$.

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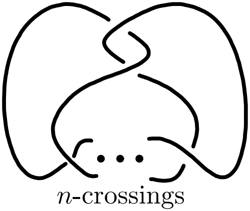

Figure: Twist knots $J(2, n)$ and $J(-2, n)$, $n > 0$.

Example

In D. Rolfsen's table the trefoil knot is $3_1 = J(2, 2)$, the figure eight knot is $4_1 = J(2, -2)$, $5_2 = J(2, 4)$, $6_1 = J(2, -4)$.

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where w is the word $[y, x^{-1}] = yx^{-1}y^{-1}x$.

$$\pi_1(J(2,2m+1)) = \langle x, y \mid w^m x = y w^m \rangle$$

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- except the trefoil, are hyperbolic.
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- representations of the fundamental group to $\mathrm{SL}_2(\mathbb{C})$ are described by Riley's method.

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- Sign-refined twisted torsion (in analog to V. Turaev's sign-refined torsion) was proposed by J. Dubois (2005).

Torsion of a chain complex

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Let $C_* = (0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \to 0)$ be a chain complex of finite dimensional vector spaces over $\mathbb{C}$.

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Choose a basis $\mathbf{c}^i$ for C_i, and a basis $\mathbf{h}^i$ for $H_i = H_i(C_*)$.
Let $\mathbf{b}^i$ be a sequence of vectors in C_i such that $d_i(\mathbf{b}^i)$ is a basis of $B_{i-1} = \mathrm{im}(d_i)$.
Let $\widetilde{\mathbf{h}}^i$ be a lift of $\mathbf{h}^i$ in $Z_i = \ker(d_i)$.

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The *sign-determined Reidemeister torsion* of C_* is

$$\mathrm{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) = (-1)^{|C_*|} \cdot \prod_{i=0}^{n} [d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i \mathbf{b}^i / \mathbf{c}^i]^{(-1)^{i+1}} \in \mathbb{C}^*.$$

where $|C_*| = \sum_{k \geqslant 0} \alpha_k(C_*)\beta_k(C_*)$, $\alpha_i(C_*) = \sum_{k=0}^{i} \dim C_k$, $\beta_i(C_*) = \sum_{k=0}^{i} \dim H_k$.

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Here:
$C_*(\widetilde{W}; \mathbb{Z})$ is the complex of the universal cover with integer coefficients which is a $\mathbb{Z}[\pi_1(W)]$-module,
$Ad \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C})), A \mapsto Ad_A$ is the adjoint representation,
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Let

$$\tau_0 = \mathrm{sgn}\left(\mathrm{Tor}(C_*(W;\mathbb{R}), c^*, h^*)\right) \in \{\pm 1\}.$$

Define the twisted Reidemeister torsion of W to be

$$\mathrm{TOR}(W;\mathfrak{sl}_2(\mathbb{C})_\rho, \mathbf{h}^*, \mathfrak{o}) = \tau_0 \cdot \mathrm{Tor}(C_*(W;\mathfrak{sl}_2(\mathbb{C})_\rho), \mathbf{c}^*_{\mathcal{B}}, \mathbf{h}^*) \in \mathbb{C}^*.$$

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We say that $\rho \in R^{\mathrm{irr}}(\pi_1(K); \mathrm{SL}_2(\mathbb{C}))$ is *regular* if $\dim H_1^{\rho}(E_K) = 1$.
For a regular representation ρ, we have

$$\dim H_1^{\rho}(E_K) = 1, \ \dim H_2^{\rho}(E_K) = 1 \text{ and } H_j^{\rho}(E_K) = 0 \text{ for all } j \neq 1, 2.$$

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Let λ be the longitude of K. We say that an irreducible representation $\rho : \pi_1(K) \to \mathrm{SL}_2(\mathbb{C})$ is λ-*regular*, if (J. Porti 1997):

1. the inclusion $\iota \colon \lambda \hookrightarrow E_K$ induces a *surjective* map
$$\iota^* \colon H_1^{\rho}(\lambda) \to H_1^{\rho}(E_K),$$
2. if $\mathrm{trace}(\rho(\pi_1(\partial E_K))) \subset \{\pm 2\}$, then $\rho(\lambda) \neq \pm \mathbf{1}$.

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Let $\rho\colon \pi_1(K) \to \mathrm{SL}_2(\mathbb{C})$ be a λ-regular representation. There is a canonical way to choose a basis $\mathfrak{o}$ for the homology with real coefficients, and a basis $\{h^{\rho}_{(1)}(\lambda), h^{\rho}_{(2)}\}$ for the twisted homology.

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$$\mathbb{T}^K_\lambda(\rho) = \mathrm{TOR}\left(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho, \{h^\rho_{(1)}(\lambda), h^\rho_{(2)}\}, \mathfrak{o}\right) \in \mathbb{C}^*.$$

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This torsion (defined by Porti, Dubois) is a numerical invariant, associated with not necessary *acyclic* (i.e. exact) chain complexes, generally not easy to compute. It has role in the asymptotic expansions of the colored Jones polynomial (Dubois-Kashaev, 2007).

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$$C_*(W; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho) = C_*(\widetilde{W}; \mathbb{Z}) \otimes_{Ad\circ\rho\otimes\alpha} (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t)).$$

The sign–defined Reidemeister torsion of W with respect to this $\widetilde{\mathfrak{sl}_2}(\mathbb{C})$-twisted chain complex is defined to be

$$\mathrm{TOR}(W; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \mathbf{h}^*, \mathfrak{o}) = \tau_0 \cdot \mathrm{Tor}(C_*(W; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \mathbf{c}^*_{\mathcal{B}}, \mathbf{h}^*)) \in \mathbb{C}(t)^*.$$

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If ρ is λ-regular, then all homology groups $H_*(E_K; \widetilde{\mathfrak{sl}_2(\mathbb{C})}_\rho)$ vanishes, the chain $C_*(W; \widetilde{\mathfrak{sl}_2(\mathbb{C})}_\rho)$ is acyclic (Y. Yamaguchi, 2005), and we define the twisted Reidemeister torsion polynomial at ρ to be

$$\mathcal{T}^K_\lambda(\rho) = \mathrm{TOR}(W; \widetilde{\mathfrak{sl}_2(\mathbb{C})}_\rho, \emptyset, \mathfrak{o}) \in \mathbb{C}(t)^*.$$

The torsion is also determined up to a factor t^m where $m \in \mathbb{Z}$.

Theorem (Yamaguchi (2005))

The derivative with respect to t of $\mathcal{T}_{\lambda}^{K}(\rho)$ *at* $t = 1$ *is equal to* $-\mathbb{T}_{\lambda}^{K}(\rho)$.

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F. Waldhausen proved that the Whitehead group of a knot group is trivial. As a result, W_K has the same simple homotopy type as E_K. So, the CW–complex W_K can be used to compute the twisted Reidemeister torsion polynomial.

The twisted complex $C_*(W_K; \widetilde{\mathfrak{sl}_2(\mathbb{C})_\rho})$ becomes:

$$0 \longrightarrow (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t))^{k-1} \stackrel{\partial_2}{\longrightarrow} (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t))^{k} \stackrel{\partial_1}{\longrightarrow} \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t) \longrightarrow 0.$$

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Where, writing Φ for $(Ad \circ \rho) \otimes \alpha$:

$$\partial_1 = (\Phi(x_1 - 1), \Phi(x_2 - 1), \ldots, \Phi(x_k - 1)).$$

and ∂_2 is expressed using the Fox's free differential calculus

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Let $A^1_{K, Ad\circ\rho}$ denote the $3(k-1) \times 3(k-1)$–matrix obtained from the matrix of ∂_2 by deleting its first row.

The torsion polynomial $\mathcal{T}_\lambda^K(\rho)$ can be described, up to a factor t^m $(m \in \mathbb{Z})$ as:

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This rational function has the first order zero at $t = 1$. The twisted Reidemeister torsion $\mathbb{T}_\lambda^K(\rho)$ is expressed as

$$\mathbb{T}_\lambda^K(\rho) = -\lim_{t\to 1} \frac{\mathcal{T}_\lambda^K(\rho)}{(t-1)} = -\lim_{t\to 1} \left(\tau_0 \cdot \frac{\det A^1_{K,Ad\circ\rho}}{(t-1)\det(\Phi(x_1 - 1))} \right).$$

Formulas for the torsion of twist knots

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Using Riley's method we can parametrize a non–abelian $\mathrm{SL}_2(\mathbb{C})$-representation ρ by two parameters u and s as follows:

$$\rho(x) = \begin{pmatrix} \sqrt{s} & 1/\sqrt{s} \\ 0 & 1/\sqrt{s} \end{pmatrix}, \rho(y) = \begin{pmatrix} \sqrt{s} & 0 \\ -\sqrt{s}u & 1/\sqrt{s} \end{pmatrix}.$$

Let $W = \rho(w)$. Then s and u satisfy Riley's equation $\phi_{J(2,2m)}(s,u) = W_{1,1} + (1-s)W_{1,2} = 0$.

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Let $\xi_{\pm}$ are the eigenvalues of W, given by explicit expressions in terms of s and u.

Theorem

Let m be a positive integer.

1. *The Reidemeister torsion* $\mathbb{T}_{\lambda}^{J(2,2m)}(\rho)$ *is:*

$$\frac{\tau_0}{s+s^{-1}-2}\left[C_1(m)\xi_+^{m-1}t_m + C_2(m)\xi_-^{m-1}t_m + C_3(m)\right].$$

2. *Similarly,* $\mathbb{T}_{\lambda}^{J(2,-2m)}(\rho)$ *is*

$$\frac{\tau_0}{s+s^{-1}-2}\left[-C_1(-m)\xi_+^{-m-1}t_m - C_2(-m)\xi_-^{-m-1}t_m + C_3(-m)\right].$$

Where $C_1(m), C_2(m), C_3(m), t_m, \xi_+, \xi_-$ *are explicit expressions in terms of* s, u, m *(the formulas are available in our paper).*

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But formulas for the twisted Reidemeister torsion at holonomy representations are simpler (could be efficiently computed using computer) .
Every twist knots except the trefoil knot are hyperbolic.
The exterior of a hyperbolic knot admits a hyperbolic structure which determines a unique discrete faithful representation of the knot group in $\mathrm{PSL}_2(\mathbb{C})$, called the *holonomy representation*.
Such a representation lifts to $\mathrm{SL}_2(\mathbb{C})$ and determines two representations in $\mathrm{SL}_2(\mathbb{C})$.
Such lifts are λ-regular representations.

Lemma

Let K be a hyperbolic two–bridge knot and suppose that its knot group admits a presentation $\pi_1(K) = \langle x, y \mid wx = yw \rangle$.
If ρ_0 denotes a lift in $\mathrm{SL}_2(\mathbb{C})$ of the holonomy representation, then ρ_0 is given by, up to conjugation,

$$x \mapsto \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \pm \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix},$$

where u is a root of Riley's equation $\phi_K(1, u) = 0$ of K.

Theorem

Let $m > 0$, then

1.

$$\mathbb{T}_{\lambda}^{J(2,2m)}(\rho_u) = \frac{-\tau_0}{u^2+4}\Big[\big(4+m(u^2-4u+8)\big)\,t_m(\xi_+^m+\xi_-^m) \\ + \big(t_m(\xi_+^{m-1}+\xi_-^{m-1})-1\big)\,(u^2-4)m \\ +(-5u^2-8u+4)t_m^2\Big]\,,$$

2.

$$\mathbb{T}_{\lambda}^{J(2,-2m)}(\rho_u) = \frac{-\tau_0}{u^2+4}\Big[\big(-4+m(u^2-4u+8)\big)\,t_m(\xi_+^m+\xi_-^m) \\ + \big(t_m(\xi_+^{m+1}+\xi_-^{m+1})+1\big)\,(u^2-4)m \\ +(-5u^2-8u+4)t_m^2\Big]\,.$$

Asymptotic behavior of torsion at the holonomy

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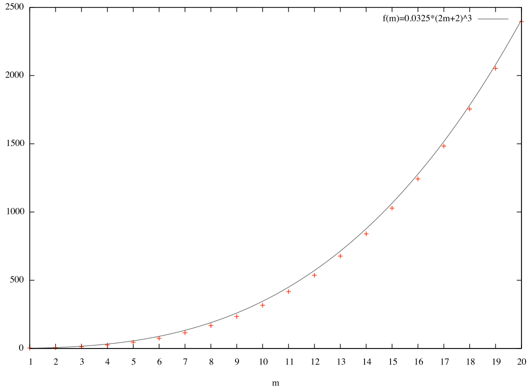

Figure: Graph of $|\mathbb{T}_\lambda^{J(2,-2m)}(\rho_0)|$ and $f(m) = C(\sharp J(2,-2m))^3$, where $\sharp K$ is the number of crossings of K.

Observation

The sequence $\left(|\mathbb{T}_{\lambda}^{J(2,-2m)}(\rho_0)|\right)_{m\geqslant 1}$ has the same behavior as the sequence $\left(C(\sharp J(2,-2m))^3\right)_{m\geqslant 1}$, for some constant C.

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Thank you!