On Global and Braid Index

A TALK PRESENTED BY

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1. Introduction

(a) A braid is a set of n strings stretching between two parallel planes. Berger (2002) observed that many invariants of knots and links have their counterparts in braid theory. His review demonstrated how integrals over the braid path can yield topological invariants.

(b) Braids can be organised into groups, this group is denoted by B_n.

(c) When α is the plane, the braid can be closed, say, $\widehat{\alpha}$ i.e. corresponding ends can be connected in pairs, to form a link, i.e. a possibly intertwined union of possibly knotted loops in three dimensions. The number of components of the links can be anything from 1 to n, depending on the permutation of strands determined by the link.

(d) Alexander (1928) observed that every link can be obtained in this way from a braid. Different braids can give rise to the same link, just as different crossing diagrams can give rise to the same knot. Birman (1976) defined a link to be the union of $\alpha \geq 1$ mutually disjoint simple closed polygonal curves, embedded in E^3 . The case $\alpha = 1$ is referred to as a knot.

The Jones polynomial which was discovered by Jones in 1985 is defined, a priori, as a braid invariant and then shown to depend on the class of the closed braid (1987). Therefore, the closure of a braid $\alpha \in B_n$ is the oriented link $\widehat{\alpha}$ obtained by tying the top end of each string to the same position on the bottom of the braid.

(e) Consider n strings, each oriented vertically from a lower to an upper "bar". If this is the least number of strings needed to make a closed braid representation of a link, n is called the braid index. That is, the braid index is the fewest number of strings required to represent the link by a braid. The braid index is equal to the least number of Seirfet circles in any projection of a knot, (Yamada 1987).

Jones(1987) gave a table of braid word and polynomials for knots up to 10 crossings. Jones polynomials for knots are given in Adams (1994) and for oriented links up to nine crosings by Doll and Hoste(1991).

2. The Global Index

von Neumann algebra is a branch of algebra directly related to quantum theory and to statistical mechanics. In the work of Murray and von Neumann, the dimensions of certain geometric hilbert spaces are measured by a von Neumann algebra. This notion is used in this work. A W*-algebra is a weakly closed self-adjoint unital algebra of operators on a hilbert space, H. A von Neumann algebra is a factor if its centre consists only of the scalar multiples of the identity.

The factor is type 11_1 if it admits a linear functional, called a trace, $tr : M \to C$, which satisfies the following conditions:

i $tr(xy) = tr(yx) \forall x, y \in M$

ii $tr(1) = 1$

iii $tr(xx^*) > 0$, where x^* is the adjoint of x.

The type 11_1 has a trace and each projection e can be written as the sum of two projections e_1, e_2.

If N is a subfactor of M, the number $\frac{dim_N(H)}{dim_M(H)}$ is called the (global) index of N in M and written [M:N]. The index is defined in general as $dim_N(L^2(M,tr))$ where N is the subfactor of M and tr is the trace of M. Let N $\subseteq M$ be a 11_1 factor and let $p \in N' \bigcap M$ be a projection. The index of N at p is

$$[M_p : N_p] = [M : N]_p.$$

Before giving the propositions, we recall the rules of calculation associated with $dim_{M'}$ [see Dixmier (1969), Jones (1983)]

Rules

i $dim_M(H) \geq 0$

ii $dim_M(H) = (dim_{M'}(H))^{-1}$

iii If e is a projection in M', $dim_{Me}(eH) = tr_{M'}(e) dim_M(H)$

iv If e is a projection in M, $dim_{Me}(eH) = (tr_M(e))^{-1} dim_M(H))$

v If $M \otimes 1$ is the amplification of M on $H \otimes H$,
$dim_M(H \otimes H) = dim_C(H) dim_M(H)$

vi $dim_M(H) = 1$ iff M is standard on H i.e. there is a cyclic trace vector for M, $dim_M = \infty$ if M' is infinite.

Proposition 2.1: *Let N_1 and N_2 be subfactors of the finite factors M_1 and M_2 respectively. Then $N_1 \otimes N_2$ is a subfactor of $M_1 \otimes M_2$ and $[M_1 \otimes M_2 : N_1 \otimes N_2] = [M_1 : N_1][M_2 : N_2]$.*

Proof: $M_1 \otimes M_2$ is standard on $H_1 \otimes H_2$
and by commutation theorem for tensor products
$(N_1 \otimes N_2)\,' = N_1\,' \otimes N_2\,'$.

Supposing e_1, e_2 are projections,
$tr(e_1 \otimes e_2)_{(N_1 \otimes N_2)\,'} = tr_{N_1\,'}(e_1) tr_{N_2\,'}(e_2)$.
Therefore, $[M_1 \otimes M_2 : N_1 \otimes N_2]$
$= dim_{N_1 \otimes N_2}(L^2(M_1 \otimes M_2) tr_{(N_1 \otimes N_2)\,'}(e_1 \otimes e_2)$
$= [dim_{N_1}(L^2(M_1, tr_{N_1\,'}(e_1)))][dim_{N_2}(L^2(M_2, tr_{N_2\,'}(e_2)))]$
$= [M_1 : N_1][M_2 : N_2]$.

Proposition 2.2 : *The index at p and the global index are related by the formula*
$[M_1 \otimes M_2 : N_1 \otimes N_2]_p = [M_1 : N_1][M_2 : N_2] tr_{M_1(p)} tr_{N_1'}(p) tr_{M_2(p)} tr_{N_2'}(p)$.

Proof: By (iv), $dim_{M_p}(pH) = tr_{M_1}(p)^{-1}$.
By definition,
$dim_{N_1}(H) = [M_1 : N_1]$.
By (iii), $dim_{N_1 p}(pH) = tr_{N_1'}(p) dim_{N_1}(H) = [M_1 : N_1] tr_{N_1'}(p)$.

Similar result follows for $[M_2 : N_2]$.
Thus by proposition 2.1

$$[M_1 \otimes M_2 : N_1 \otimes N_2]_p = [M_1 : N_1]_p [M_2 : N_2]_p$$

$$= \frac{dim_{N_1 p}(pH) dim_{N_2 p}(pH)}{dim_{M_1 p}(pH) dim_{M_2 p}(pH)},$$

hence the desired result.

3. The Braid Index

The question immediately arose by Jones, what posible values can the index take? The answer is $\left(4cos^2\frac{\pi}{k}|k = 3, 4, \ldots\right)$. He represents M on $L^2(M, tr)$ and considers the extension e_N to $L^2(M, tr)$. He defines $\langle M, e_N\rangle$ to be 11_1 factor generated by M and e_N on $L^2(M, tr)$. The index of M in $\langle M, e_N\rangle$ is the same as that of N in M. Thus he iterated this extension process and obtained a sequence of 11_1 factors, each one obtained from the previous one by adding a projection. The inductive limit gives a 11_1 factor and the projections in the construction are numbered $e_1,\ e_2, \ldots$

Lemma 3.1 [Jones, 1983] : Let M be a von Neumann algebra with faithful normal normalised trace, tr. Let $[e_i|,\ i = 1, 2, \ldots,]$ be a projection in M satisfying

a. $e_i e_{i\pm 1} e_i = \tau e_i$ for some $\tau \leq 1$

b. $e_i e_j = e_j e_i$ for $|i - j| \geq 2$

c. $tr(we_i) = \tau tr(w)$ if w is a word on 1, $e_1,\ e_2, \ldots, e_{i-1}$

Then if P denotes the von Neumann algebra generated by the $e_i's$

i $P \cong R$ (the hyperfinite II_1 factor)

ii $P_\tau = \{e_2,\ e_3, \dots,\}''$ is a subfactor of P with $[P : P_\tau] = \tau^{-1}$

iii $\tau \leq \frac{1}{4}$ or $\tau = \frac{1}{4}\frac{sec^2\pi}{k}, k = 3, 4, \dots.$

Remarks:

The index for a subfactor $[M:N]=\tau^{-1}$, that is $[M:N]^{-1}=\tau=\frac{1}{4}sec^2\frac{\pi}{k}$, where τ is $tr(e_N)$, the trace turn into the new knot polynomials.

Jones, while investigating the index of a subfactor of a type II_1 factor, he analysed certain finite dimensional von Neumann algebra A_n generated by an identity 1 and n projections which he call $e_1,\ e_2,\dots\ e_n$.

satisfying

i $e_i^2 = e_i,\ e_i^* = e_i$

ii $e_i e_{i+1} e_i = \frac{t}{(1+t)^2} e_i$

iii $e_i e_j = e_j e_i$ if $|i - j| \geq 2$

iv $tr(ab) = tr(ba),\ tr(1) = 1$

v $tr(we_{n+1}) = \frac{t}{(1+t)^2} tr(w)$ if w is in A_n,

vi $tr(a^*a) > 0$ if $a \neq 0$.

Here, $\frac{t}{(1+t)^2} = \tau$. Putting $\tau = 1$, $t = 1 + 2t + t^2$ and dividing through by t we have, $1 = t^{-1} + 2 + t$ that is $2 + t + t^{-1} = 1$. We claim that $2 + e^{\frac{2\pi i}{3}} + e^{\frac{-2\pi i}{3}} = 1$, t is a complex number and is equal to $e^{\frac{2\pi i}{3}}$. An arbitrarily large family of such projections can only exist if t is either real and positive or $t = e^{\frac{\pm 2\pi i}{k}}$ for some $k = 3, 4, 5, \ldots$.

This t now replaces the index τ.

The similarity between relations (ii) and (iii) and Artin's representation of the n-string braid group,

$$\left\{ s_1,\ s_2,\ \ldots,\ s_n :\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i;\ |i-j| \geq 2 \right\}$$

was first pointed out by Hatt and de la Harpe. It transpires that if one defines $g = \sqrt{t}[te_i - (1 - e_i)]$, the g_i satisfies the relations, and one obtains representations r_t of B_n by sending s_i to g_i .

If L is a tame oriented classical link, the trace invariant V_L is defined by

$$V_L(t) = \left(-\frac{(t+1)}{\sqrt{t}} \right)^{n-1} tr(r_t(\alpha)) \quad ,$$

for any (α, n) such that $\widehat{\alpha} = L$.

Theorem 3.1: *For* $t = e^{\frac{2\pi i}{k}}, k = 3,\ 4,\ 5, \ldots$
$v_{\widehat{\alpha}}(t) = (-2cos\frac{\pi}{k})^{n-1}$, *if and only if* $\alpha \in kerr_t$ *(for* $\alpha \in B_n$ *).*
Furthermore, for $t = e^{i\theta}$, $v_{\widehat{\alpha}}(\theta) = (-2cos\frac{\theta}{2})^{n-1}$

Proof: $\widehat{\alpha} = L$.
By definition, $V_{\widehat{\alpha}}(t) = \left(-\frac{(t+1)}{\sqrt{t}}\right)^{n-1} tr(r_t(\alpha))$ and since $\alpha \in kerr_t$
it implies that $V_{\widehat{\alpha}}(t) = \left(-\frac{(t+1)}{\sqrt{t}}\right)^{n-1}$.
We note that $\tau = \frac{t}{(1+t)^2}$ or $\tau^{-1} = \frac{(1+t)^2}{t}$, $\Rightarrow \sqrt{\tau^{-1}} = \frac{(1+t)}{\sqrt{t}}$.

Also by **lemma 3.1** (iii), we have $\tau = \frac{1}{4}sec^2\frac{\pi}{k}$, $k = 3,\ 4,\ \ldots$ and
$\tau^{-1} = 4cos^2\frac{\pi}{k}$, $k = 3,\ 4,\ \ldots \Rightarrow \sqrt{\tau^{-1}} = 2cos\frac{\pi}{k} = \frac{(1+t)}{\sqrt{t}}$.

Hence,

$$V_L(t) = \left(-2cos\frac{\pi}{k}\right)^{n-1}. \tag{1}$$

Suppose $t = e^{i\theta}$, it implies that $\theta = \frac{2\pi}{k} \Rightarrow \frac{\pi}{k} = \frac{\theta}{2}$. Substituting θ in (1) we have

$$V_L(\theta) = \left(-2cos\frac{\theta}{2}\right)^{n-1}. \tag{2}$$

4. Conclusions

We give a formula to calculate the least number of strings needed to form a braid via von Neumann algebra. Also the index at p and the global index are related by a formula.

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