Singular Hecke algebras, Markov traces, and link invariants

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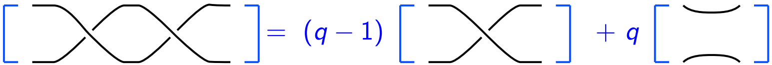
joint with **Loic RABENDA**

B_n is the braid group on n strands.
$\mathbb{K} = \mathbb{C}(q)$.

Definition. The Hecke algebra of the symmetric group, $\mathcal{H}(B_n)$, is the quotient of $\mathbb{K}[B_n]$ by the relations

$$\sigma_i^2 = (q-1)\sigma_i + q\,, \quad 1 \leq i \leq n-1.$$



Definition. A Markov trace on $\{\mathcal{H}(B_n)\}_{n=1}^{+\infty}$ is a collection of $\mathbb{K}$ linear maps

$$\mathrm{tr}_n\colon \mathcal{H}(B_n) \to \mathbb{K}(z), \quad n \geq 1\,,$$

such that

- $\mathrm{tr}_n(\alpha\beta) = \mathrm{tr}_n(\beta\alpha)$ for all $\alpha, \beta \in B_n$;
- $\mathrm{tr}_{n+1}(\alpha) = \mathrm{tr}_n(\alpha)$ for all $\alpha \in B_n$;
- $\mathrm{tr}_n(\alpha\sigma_n) = z \cdot \mathrm{tr}_n(\alpha)$ for all $\alpha \in B_n$.

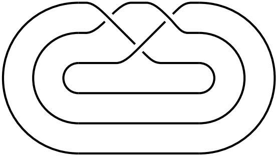
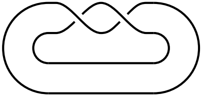
Remark. Markov traces form a $\mathbb{K}(z)$-vector space, TR_0.

Theorem (Ocneanu). There exists a unique Markov trace $T = \{\mathrm{tr}_n\}_{n=1}^{+\infty}$ such that $\mathrm{tr}_1(1) = 1$.

Corollary. TR_0 is a 1-dimensional vector space spanned by the Ocneanu trace.

Definition. To a braid β one can associate a link, $\widehat{\beta}$, called the closure of β, as follows

$\beta =$ 

$\hat{\beta} =$  $\simeq$ 

$\beta \in B_n$ is denoted by (β, n).

Definition. Two braids (α, n) and (β, m) are connected by a Markov move if either

- $n = m$, $\alpha = \gamma_1\gamma_2$, and $\beta = \gamma_2\gamma_1$, for some $\gamma_1, \gamma_2 \in B_n$;
- $n = m + 1$ and $\alpha = \beta\sigma_n^{\pm 1}$;
- $m = n + 1$ and $\beta = \alpha\sigma_n^{\pm 1}$.

Theorem (Markov). Let (α, n) and (β, m) be two braids. $\widehat{\alpha}$ is isotopic to $\widehat{\beta}$ if and only if (α, n) and (β, m) are connected by a finite sequence of Markov moves.

Let $T = \{\mathrm{tr}_n\}_{n=1}^{+\infty}$ be a trace.
Let $\varepsilon : B_n \to \mathbb{Z}$ be the homomorphism defined by $\varepsilon(\sigma_i) = 1$ for all $1 \leq i \leq n-1$.
Set

$$z = \frac{q-1}{1-qy} \quad \Leftrightarrow \quad y = \frac{z-q+1}{qz}\,.$$

Definition. For $\beta \in B_n$ set

$$I_T(\beta) = \left(\frac{q-1}{1-qy}\right)^{-n+1} \cdot \sqrt{y}^{\varepsilon(\beta)-n+1} \cdot \mathrm{tr}_n(\beta)\,.$$

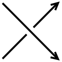
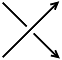
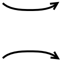
Theorem (Jones). Let (α, n) and (β, m) be two braids. If $\widehat{\alpha}$ is isotopic to $\widehat{\beta}$ then $I_T(\alpha) = I_T(\beta)$.

Definition. Let $\mathcal{L}_0$ be the set of links.
Let $L \in \mathcal{L}_0$. Choose a braid (α, n) such that $\widehat{\alpha} = L$. Set

$$I_T(L) = I_T(\alpha)\,.$$

Then $I_T : \mathcal{L}_0 \to \mathbb{K}(\sqrt{y})$ is a well-defined invariant.

Definition. Let A be an abelian group, $I : \mathcal{L}_0 \to A$ an invariant, and $t, x \in A$. We say that I satisfies the (t, x) skein relation if

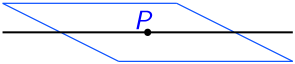
$t^{-1} \cdot I \Big($  $\Big) - t \cdot I \Big($  $\Big) = x \cdot I \Big($  $\Big)$

Theorem (Many people). There exists a unique invariant $I : \mathcal{L}_0 \to \mathbb{C}[t^{\pm 1}, x^{\pm 1}]$ which satisfies the (t, x) skein relation and takes the value 1 on the unknot.

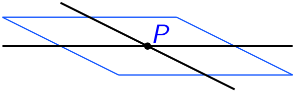
Definition. The above is the HOMFLY polynomial.

Theorem (Jones). If T is the Ocneanu trace, then I_T is the HOMFLY polynomial (up to change of variable).

Definition. A singular link on n components is an immersion $L : \mathbb{S}^1 \cup \cdots \cup \mathbb{S}^1 \to \mathbb{R}^3$ such that each point P of L has a neighborhood of the form



Regular point



Singular point

Definition. A singular braid is defined with the same local model. The singular braids form a monoid SB_n.

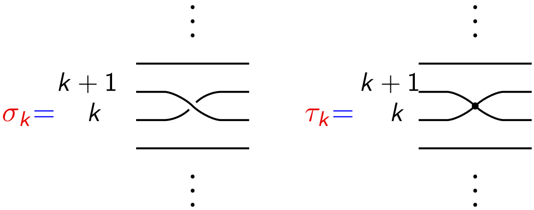
Theorem (Baez, Birman). SB_n has a presentation with generators

$$\sigma_1, \ldots, , \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}, \tau_1, \ldots, \tau_{n-1}\,,$$

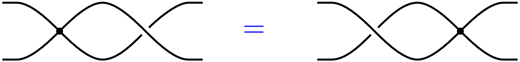
and relations

$$\begin{aligned}
\sigma_k \sigma_k^{-1} = \sigma_k^{-1} \sigma_k &= 1 && \text{for } 1 \le k \le n-1\,,\\
\sigma_k \tau_k &= \tau_k \sigma_k && \text{for } 1 \le k \le n-1\,,\\
\sigma_k \sigma_l \sigma_k &= \sigma_l \sigma_k \sigma_l && \text{if } |k-l| = 1\,,\\
\sigma_k \sigma_l \tau_k &= \tau_l \sigma_k \sigma_l && \text{if } |k-l| = 1\,,\\
\sigma_k \sigma_l &= \sigma_l \sigma_k && \text{if } |k-l| \ge 2\,,\\
\sigma_k \tau_l &= \tau_l \sigma_k && if |k-l| \ge 2\,,\\
\tau_k \tau_l &= \tau_l \tau_k && \text{if } |k-l| \ge 2\,.
\end{aligned}$$

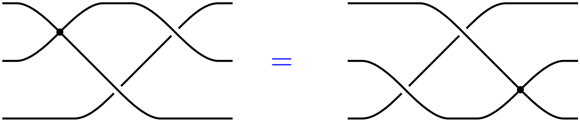
Generators.



Some relations.

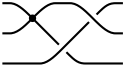


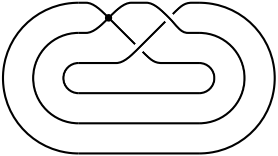
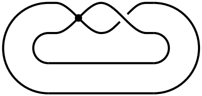
$\tau_i \quad \sigma_i \quad = \quad \sigma_i \quad \tau_i$



$\tau_i \quad \sigma_j \quad \sigma_i \quad = \quad \sigma_j \quad \sigma_i \quad \tau_j$

Definition. To a singular braid β one can associate a singular link, $\widehat{\beta}$, called the closure of β, as follows

$\beta =$ 

$\hat{\beta} =$  $\simeq$ 

Fact. Alexander and Markov theorems hold for singular braids. (due to Birman and Gemein).

Definition. The singular Hecke algebra $\mathcal{H}(SB_n)$ is the quotient of $\mathbb{K}[SB_n]$ by the relations

$$\sigma_i^2 = (q-1)\sigma_i + q\,, \quad 1 \le i \le n-1\,.$$

Let $S_d B_n$ be the singular braids with d singular points.
$\mathcal{H}(S_d B_n)$ the linear subspace of $\mathcal{H}(SB_n)$ spanned by $S_d B_n$.
We have the graduation

$$\mathcal{H}(SB_n) = \bigoplus_{d=0}^{+\infty} \mathcal{H}(S_d B_n) .$$

Proposition (P., Rabenda). $\mathcal{H}(S_d B_n)$ is of finite dimension.

Definition. A Markov trace on $\{\mathcal{H}(S_d B_n)\}_{n=1}^{+\infty}$ is a collection of $\mathbb{K}$ linear maps

$$\mathrm{tr}_n^d \colon \mathcal{H}(S_d B_n) \to \mathbb{K}(z), \quad n \geq 1\,,$$

such that

- $\mathrm{tr}_n(\alpha\beta) = \mathrm{tr}_n(\beta\alpha)$ for all suitable α, β;
- $\mathrm{tr}_{n+1}(\alpha) = \mathrm{tr}_n(\alpha)$ for all $\alpha \in S_d B_n$;
- $\mathrm{tr}_n(\alpha\sigma_n) = z \cdot \mathrm{tr}_n(\alpha)$ for all $\alpha \in S_d B_n$.

Let $T = \{\mathrm{tr}_n^d\}_{n=1}^{+\infty}$ be a Markov trace on $\{\mathcal{H}(S_d B_n)\}_{n=1}^{+\infty}$.
Let $\varepsilon : SB_n \to \mathbb{Z}$ be the homomorphism defined by $\varepsilon(\sigma_i) = 1$ and $\varepsilon(\tau_i) = 0$.
Set

$$z = \frac{q-1}{1-qy} \quad \Leftrightarrow \quad y = \frac{z-q+1}{qz}\,.$$

Definition. For $\beta \in S_d B_n$ set

$$I_T(\beta) = \left(\frac{q-1}{1-qy}\right)^{-n+1} \cdot \sqrt{y}^{\varepsilon(\beta)-n+1} \cdot \mathrm{tr}_n^d(\beta)\,.$$

Proposition (P., Rabenda). Let (α, n) and (β, m) be two singular braids with d singular points. If $\widehat{\alpha}$ is isotopic to $\widehat{\beta}$ then $I_T(\alpha) = I_T(\beta)$.

Definition. Let $\mathcal{L}_d$ be the set of singular links with d singular points.
Let $L \in \mathcal{L}_d$. Choose a singular braid (α, n) such that $\widehat{\alpha} = L$. Set

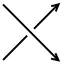
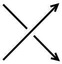
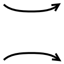
$$I_T(L) = I_T(\alpha)\,.$$

Then $I_T : \mathcal{L}_d \to \mathbb{K}(\sqrt{y})$ is a well-defined invariant.

Proposition (P., Rabenda).

- I_T satisfies the (t, x) skein relation for $t = \sqrt{q}\sqrt{y}$ and $x = \sqrt{q} - \frac{1}{\sqrt{q}}$.
- If $I : \mathcal{L}_d \to \mathbb{C}(\sqrt{q}, \sqrt{y})$ is an invariant which satisfies the (t, x) skein relation, then $I = I_T$ for some Markov trace T.

Recall. The (t, x) skein relation is:

$t^{-1} \cdot I\Big($  $\Big) - t \cdot I\Big($  $\Big) = x \cdot I\Big($  $\Big)$

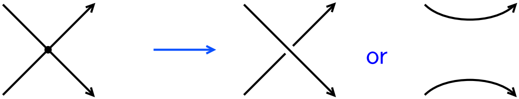
Let TR_d be the set of traces on $\{\mathcal{H}(S_d B_n)\}_{n=1}^{+\infty}$.
TR_d is a $\mathbb{K}(z)$-vector space.

Recall (Ocneanu). $\dim TR_0 = 1.$

Theorem (P., Rabenda). $\dim \mathrm{TR}_d = d + 1.$

Proof.

Part 1. Construct $d+1$ linearly independent traces in TR_d using desingularization arguments :



determines

$$\mathcal{H}(S_{d+1}B_n) \to \mathcal{H}(S_dB_n)$$

which determines

$$\mathrm{TR}_d \to \mathrm{TR}_{d+1} \, .$$

Part 2. Prove that $\dim \mathrm{TR}_d \leq d+1$ with a long a tedious calculation.

Set

$$\begin{aligned}\omega_1 &= [\tau_1^{a_1}\tau_3^{a_2}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ \omega_2 &= [\tau_1^{a_1+a_3}\tau_3^{a_2}(\sigma_3-\sigma_1)\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ \omega_3 &= [\tau_1^{a_1+a_2}B_{1\,2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\,.\end{aligned}$$

Obviously, $\omega_2,\omega_3\in\mathrm{Span}(\mathcal{D}_{l-1,\infty})$. On the other hand, by Lemma 4.11,

$$\begin{aligned} & [(\sigma_3-\sigma_1)^2\tau_1^{a_1}\tau_3^{a_2}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ = & [(\sigma_3-\sigma_1)^2\tau_1^{a_1}\tau_2^{a_2+a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ & +[(\sigma_3-\sigma_1)^2\tau_3^{a_2}\tau_2^{a_1+a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ & +[(\sigma_3-\sigma_1)(B_{1\,2}-B_{2\,3})\tau_2^{a_1+a_2+a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ \in & \ \mathrm{Span}(\mathcal{D}_{l-1,\infty})\,.\end{aligned}$$

Moreover, by Lemma 4.10,

$$\begin{aligned} & [C_{1\,3}\tau_1^{a_1}\tau_3^{a_2}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ = & [C_{1\,3}\tau_1^{a_1+a_3}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ \in & \ \mathrm{Span}(\mathcal{D}_{l-1,\infty})\,.\end{aligned}$$

Hence, by Lemma 4.10,

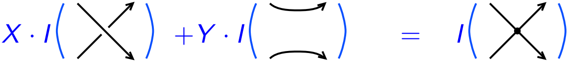
$$\begin{aligned}\omega_1 &= (q+1)^{-2}[(\sigma_3-\sigma_1)^2\tau_1^{a_1}\tau_3^{a_2}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &\quad +(q+1)^{-2}[C_{1\,3}\tau_1^{a_1}\tau_3^{a_2}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &\in \mathrm{Span}(\mathcal{D}_{l-1,\infty})\,.\end{aligned}$$

Finally, by (**??**),

$$\begin{aligned}\omega &= [(z-\sigma_1)\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &= [(\sigma_3-\sigma_1)\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &= [\tau_1^{a_1}\sigma_3\tau_2^{a_2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]-[\tau_1^{a_1}\sigma_1\tau_2^{a_2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &= [\tau_1^{a_1}\tau_3^{a_2}\sigma_2\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]+[\tau_1^{a_1+a_3}\tau_3^{a_2}(\sigma_3-(q-1))\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &\quad -[\tau_1^{a_1+a_2}B_{1\,2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &= \omega_1+\omega_2-\omega_3\in\mathrm{Span}(\mathcal{D}_{l-1,\infty})\,.\end{aligned}$$

Let $\mathcal{L}$ be the set of all singular links.

Definition. Let A be an abelian group, $I : \mathcal{L} \to A$ an invariant, and $X, Y \in A$. We say that I satisfies the (X, Y) desingularization relation if



Theorem (P., Rabenda). There exists a unique invariant

$$\hat{I} : \mathcal{L} \to \mathbb{C}[t^{\pm 1}, x^{\pm 1}, X, Y]$$

such that

- $\hat{I}$ satisfies the (t, x) skein relation.
- $\hat{I}$ satisfies the (X, Y) desingularization relation.
- $\hat{I}(\text{unknot}) = 1$.

Theorem (P., Rabenda). Let $L, L' \in \mathcal{L}_d$. We have $\hat{I}(L) = \hat{I}(L')$ if and only if $I(L) = I(L')$ for every invariant $I : \mathcal{L}_d \to \mathbb{C}[t^{\pm 1}, x^{\pm 1}]$ which satisfies the (t, x) skein relation.

First Announcement

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Group Theory

Messigny et Vantoux
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Lie algebras associated to discrete groups

Cédric BONNAFE (Universit de Besançon)
Polynomial invariants of finite groups, and reflection groups

Goulnara ARJANTSEVA (Universit de Genève)
Random groups

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