M manifold

G finite group, acts freely on M.

For integer $m \geq 0$,

$Gq_1, Gq_2, \dots, Gq_m$ m distinct orbits

containing $q_i \in M$, $Q_m^G = \bigcup_{i=1}^{m} Gq_i$

Def. G - configuration space of M

$$F_{m,n}(M;G) = \{(x_1, \dots, x_n) \mid x_i \in M \setminus Q_m^G, \quad Gx_i \cap Gx_j = \phi \quad i \neq j\}.$$

Ex.

- $m = 0$, $G = 1$. $F_n(M)$.
- $m = 0$, $M = \mathbb{R}^q \setminus \{0\}$, $G = \mathbb{Z}_2$ antipodal.

  $F_{0,n}(\mathbb{R}^q \setminus \{0\}; \mathbb{Z}_2)$ — $F_B(\mathbb{R}^q, n)$.
- $G = 1$ $F_{m,n}(M)$ defined by F.N

Recall.

$$F_{m,n}(M;G)/G^n \approx F_{m,n}(M/G).$$

Then $\quad F_{m,n}(M;G) \longrightarrow F_{m,n}(M/G)$

covering with fiber G^n.

$$\Rightarrow \qquad \pi_i(F_{m,n}(M;G)) \cong \pi_i(F_{m,n}(M;G)).$$

Using F.N. we get easily.

Prop.

a) $$\pi_i(F_{1,n}(M;G)) \cong \bigoplus_{k=1}^{n-1} \pi_i(M \setminus Q_k^G)$$

b) If $\quad F_{0,n}(M/G) \longrightarrow M/G \quad$ has a cross-section

$$\pi_i(F_{0,n}(M;G)) \cong \bigoplus_{k=0}^{n-1} \pi_i(M \setminus Q_k^G)$$

Cor.

$$\pi_i(F_B(\mathbb{R}^q,n)) \cong \bigoplus_{k=0}^{n-1} \pi_i(\underbrace{S^{q-1} \vee \cdots \vee S^{q-1}}_{2k+1})$$

$W_n = \Sigma_n \ltimes \mathbb{Z}_2^n$ acts on $F_B(\mathbb{R}^q, n)$

$$(\sigma, (g_1 \dots g_n)) \times (x_1 \dots x_n) \longmapsto (g_1 x_{\sigma^{-1}(1)}, \dots, g_n x_{\sigma^{-1}(n)})$$

Cor $\pi_1(F_B(\mathbb{R}^q, n)/W_n)$ is ~~K(~~ W_n. for $q \geq 3$

Take direct limit

Cor. $F_B(\mathbb{R}^\infty, n)/W_n$ is a $K(W_n, 1)$.

$\pi_1(F_B(\mathbb{R}^2, n))$.

Compute π_1.

M manifold of dim n

$\mathscr{C}_M$ cell decomposition — "good" conditions

$K \subset M$ sub cell complex of $\mathscr{C}_M$ of dim $n-2$

G acts freely* on M

$\pi_1(M \setminus K/G)$ representation.

generators $\longleftrightarrow$ $(n-1)$-cells

relations $\longleftrightarrow$ $(n-2)$-cells $\in M \setminus K$.

$$F_B(\mathbb{R}^q, n) = \underset{n}{\times} \mathbb{R}^q \setminus \left(\bigcup_{i<j} H_{i,j}^{\varepsilon} \right).$$

Nakamura-Mui decomposition of type B.

for $\underset{n}{\times} \mathbb{R}^q$.

⊙ $\mathbb{R}^q$ ordered by lexico-graphic order.

Each $a \in \underset{n}{\times} \mathbb{R}^q$ - written uniquely as

$a = w.(a_1, \ldots, a_n)$, where $a_1 \geq \ldots \geq a_n \geq 0$, $w \in W_n$.

⊙ Such $(a_1, \ldots, a_n)$ defines a sequence of integers $(r_1, \ldots, r_n; r_{n+1})$ with $r_1 = q$, $0 \leq r_i \leq q$ by

$$\begin{cases} a_{i-1}^j = a_i^j & \text{if } j \leq \rho_i \\ a_{i-1}^{\rho_i+1} > a_i^{\rho_i+1} & \text{if } \rho_i < q \end{cases} \qquad (*)$$

and $r_i = q - \rho_i$.

⊙ Conversely, given a sequence of integers $\alpha = (r_1, \ldots, r_n; r_{n+1})$ with $r_1 = q$, $0 \leq r_i \leq q$ denote α the subset of $\underset{n}{\times} \mathbb{R}^q$ consiting of all $(a_1, \ldots a_n) \in \underset{n}{\times} \mathbb{R}^q$ which satisfy $(*)$.

$\cancel{\alpha \approx \text{disk of dim} \left(\sum_{i=1}^{n+1} r_i - q \right)}$.

$\mathcal{C}_B(q,n) = \{ w.\alpha \mid w \in W_n, \quad \alpha = (r_1 \cdots r_n; r_{n+1})$ with

$r_1 = q \quad , \; 0 \leq r_i \leq q \, . \}$.

$\dim |w\alpha| = \sum_0^{n+1} r_i - q$.

nq-cell $\quad (q, \cdots q; q)$

$(nq-1)$-cell $\quad (q, \cdots \overset{i}{q-1}, \cdots q; q)$

$(nq-2)$-cell $\quad (q \cdots, q-1, \cdots \; q-1, \cdots q; q)$.

When $q \geq 3$ $\quad (q, \cdots, q-2, \cdots q; q)$.

Thm. $\pi_1(F_B(\mathbb{R}^2, n)/W_n)$ - representation

with - generators $\quad g_1 \cdots g_{n-1}; h \qquad$ (I)

- relations

$$g_i g_j = g_j g_i \qquad |i-j| \geq 2$$
$$g_i h = h g_i \qquad u < n-1 \qquad \text{(II)}$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$$
$$(g_{n-1} h)^2 = (h \cdot g_{n-1})^2 .$$

Cor. W_n admits a presentation with $\qquad q > 2$

gen. (I)

relation (II) + $\begin{cases} g_i^2 = 1 \\ h^2 = 1 \end{cases}$.

$F_B(\mathbb{R}^q, n)$ relate $F_n(\mathbb{R}^q)$ $\underline{\text{denoted}}$ $F_A(\mathbb{R}^q, n)$

- Identify $F_A(\mathbb{R}^q, n) \leftrightsquigarrow F_A(\mathbb{R}^q_+, n)$.

- For integers $h, k > 0$. define

$$\bar{f}_{h,k} : F_A(\mathbb{R}^h, n) \times (S^k/\mathbb{Z}_2)^n \longrightarrow F_B(\mathbb{R}^{h+k}, n)/\Sigma_n$$

$$(a_1, \dots, a_n), (\bar{x}_1, \dots, \bar{x}_n) \longmapsto [(b_1, \dots, b_n)]$$

where

$$a_i := \begin{pmatrix} a_i^1 \\ \vdots \\ a_i^h \end{pmatrix} \qquad x_i := \begin{pmatrix} x_i^0 \\ \vdots \\ x_i^k \end{pmatrix} \quad \text{and} \quad b_i = \begin{pmatrix} x_i^k \\ \vdots \\ x_i^1 \\ a_i^1 \\ \vdots \\ a_i^h \end{pmatrix}$$

It induces

$$f_{h,k} : F_A(\mathbb{R}^h, n) \underset{\Sigma_n}{\times} (S^k/\mathbb{Z}_2)^n \longrightarrow F_B(\mathbb{R}^{h+k}, n)/W_n$$

$\underline{\text{Prop.}}$ If $h, k \geqslant 3$, maps $f_{h,k}$ induce isomorphism on π_1.

Using these maps $f_{h,k}$.

$H^*(F_B(\mathbb{R}^q, n)/W_n)$.

- 1984, 1987. N.H.V.H. - Mui.

$$D_K \in H_*(F_A(\mathbb{R}^q, 2^n)/\Sigma_{2^n})$$

$$K \in J(q) = \{(k_0, \dots, k_{n-1}) \mid k_i \geq 0, \ \sum k_i < q \quad n > 0\}.$$

$$H_*(F_A(\mathbb{R}^q, 2^n)/\Sigma_{2^n}) \hookrightarrow H_*(F_*(\mathbb{R}^q, \infty)/W_\infty).$$

- 1987. Mui defines homology operations

$$D_K : H_*(X) \longrightarrow H_*(F_A(\mathbb{R}^q, 2^n) \underset{\Sigma_{2^n}}{\times} X^{2^n})$$

⊙ Take $X = \mathbb{R}P^k = S^k/\mathbb{Z}_2$.

$H^*_*(\mathbb{R}P^k)$ $\mathbb{Z}_2$-basis $1 = x_0, x_1 \dots x_k$.

<u>Define</u>

$$D_{(K;i)} = (f_{h,k})_*(D_K(x_i)) \in H_*(F_B(\mathbb{R}^{h+k}, 2^n)/W_n)$$

$$(K, i) \in J_B(q) = \{(k_0, \dots, k_{n-1}, i) \mid n \geq 0 \quad k_i \geq 0, \ \sum k_s < q - i \quad 0 \leq i \leq q\}$$

Thm

Let $J_B(q) = \{ (k_0, \ldots, k_{n-1}, i) \mid n \geq 0,\ 0 < i \leq q,\ k_i \geq 0,\ \sum k_i < q - i \}$

Then

1) $H_*(F_B(\mathbb{R}^q, \infty)/W_\infty) \cong \mathbb{Z}_2[D_{(K,i)} \,;\, (K,i) \in J_B^+(q)]$

as Hopf algebra, where

$$\Delta D_{(K,i)} = \sum_{\substack{K = K' + K'' \\ i = i' + i''}} D_{(K',i')} \otimes D_{(K'',i'')}$$

2) This comultiplication can be reduced by

$$D_{(0, k_1, \ldots, k_{n-1}, i)} = D^2_{(k_1, \ldots k_{n-1}, i)} .$$

Let $W^{(K,i)}$ dual to $D_{(K,i)}$.

<u>Thm</u>.

We have isomorphism of algebras

$$H^*(F_B(\mathbb{R}^q,\infty)/W_\infty) \cong \mathbb{Z}_2[W^{(K,i)};\ (K,i) \in J_B^+(q)_{od}]/I$$

where $I = \left(\left(W^{(K,i)}\right)^{h(q,K,i)} \right)$ with

$$h(q,K,i) = \min\{h \mid 2^h(k_0 + \dots + k_{n-1} + i) \geq q\}.$$

Thm

$$H^*(F_B(\mathbb{R}^q, n)/W_n) \cong H^*(F_B(\mathbb{R}^q, \infty)/W_\infty)/I_n(q).$$

where $I_n(q)$ is defined as follow.

Put $\mu(w^{(K,i)}) = 2^n$ if $K = (k_0, \ldots, k_{n-1})$.

$$\mu(w^{(K_1,i_1)} \cdots w^{(K_s,i_s)}) = \sum_{j=1}^{s} \mu(w^{(K_j,s_j)})$$

$$\mu(\text{polynomial}) = \min \mu(\text{monomial}).$$

Then $I_n(q) = \{ \xi \in H^*(F_B(\mathbb{R}^q, \infty)/W_\infty) \mid \mu(\xi) > n \}$.