Representing braids by hypergeometric integrals

Toshitake Kohno

(The University of Tokyo)

(1) Classical hypergeometric functions, uniformization of orbifolds and Burau-Gassner representations of braids.

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(2) Comparison between monodromy of KZ equation and hypergeometric integrals – relation to Lawrence-Krammer-Bigelow representations.

(3) Description of a basis of the space of conformal blocks by means of hypergeometric integrals with respect to regularizable cycles.

1 Hypergeometric functions

The hypergeometric series

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \, \frac{z^n}{n!}, \qquad |z| < 1$$

where $(a,n) = \prod_{i=0}^{n-1}(a+i)$ was introduced by **Euler** in 1778 as a solution of the differential equation

$$z(z-1)u'' + \{c - (a+b+1)z\}u' - abu = 0$$

There is an integral representation

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b;c;z)$$

$$= \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a}\ du.$$

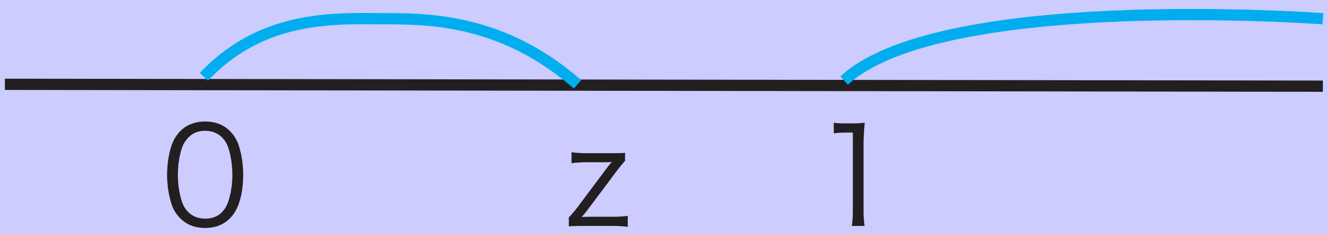
By change of variable $u = t^{-1}$ we have

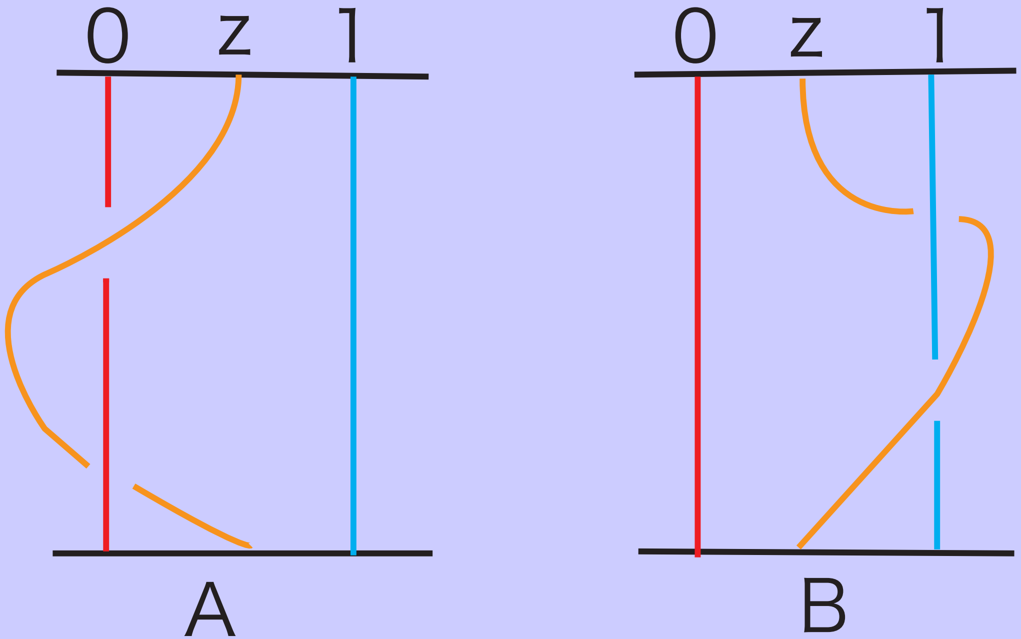
$$u(z) = \int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-z)^{-a}\ dt$$

Basis of solutions of the hypergeometric differential equation is given by the integrals:

$$u_1(z) = \int_0^z t^{a-c}(1-t)^{c-b-1}(t-z)^{-a}\,dt$$

$$u_2(z) = \int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-z)^{-a}\,dt$$





• Describe the monodromy representations of $(u_1(z), u_2(z))$ w.r.t.analytic continuation for the above braids.

• This gives linear representations of the pure braid group P_3 modulo the full twist Δ^2.

By putting $t = uz$, we have

$$u_1(z) = z^{1-c} \int_0^1 u^{a-c}(1-u)^{-a}(1-uz)^{c-b-1}\, du.$$

This implies that the monodromy matrix A w.r.t. the basis $u_1(x)$, $u_2(x)$ is expressed as

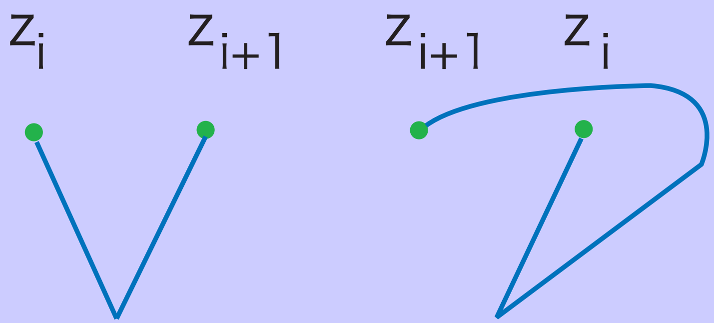
$$A = \begin{pmatrix} e^{2\pi i(1-c)} & 0 \\ 0 & 1 \end{pmatrix}$$

The matrix B is more complicated.

Braid action on the integrals

$$u_j(z) = \int_{C_j} (t - z_1)^{-\mu_1} \cdots (t - z_n)^{-\mu_n} \, dt$$

is described as follows.



$$u_i \mapsto \tilde{u}_{i+1}, \quad u_{i+1} \mapsto (1 - \xi_i)\tilde{u}_{i+1} + \xi_i \tilde{u}_i,$$

$(\xi_i = e^{-2\pi i \mu_i})$ Gassner representation

The monodromy of hypergeometric differential equation is Gassner representation

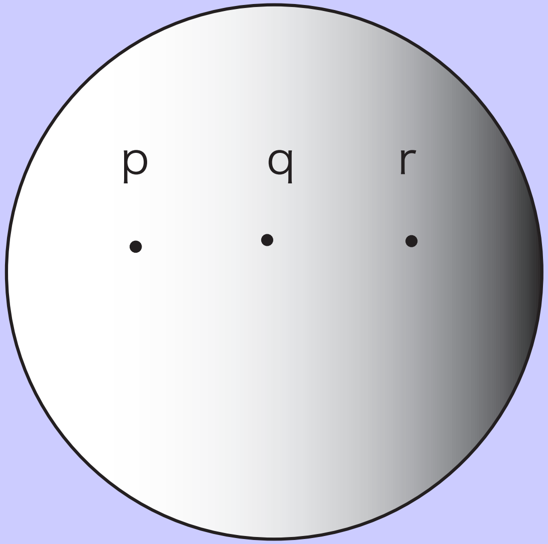
$$P_3 \longrightarrow GL(\mathbf{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}])$$

at special values. Suppose

$(1-c)^{-1} = p,\ (c-a-b)^{-1} = q,\ (b-a)^{-1} = r$

are positive integers or ∞. The monodromy matrices have relations (with C monodromy around ∞): $A^p = B^q = C^r = ABC = I.$

2 Uniformization of orbifolds



Gauss-Schwarz theory. Hypergeometric differential equation uniformaizes the orbifold $S(p, q, r)$. The universal branched covering is isomorphic to :

Sphere, Euclidean plane, Hyperbolic plane

according as

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1, \quad \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1, \quad \frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$$

• The monodromy group (image of Gassner representation) is identified with orientation preserving Schwarz triangle group $\Gamma^+(p,q,r)$.

• This describes the kernel of Gassner representations at these special values.
The kernel is the normal subgroup generated by

$$\gamma_{12}^p, \quad \gamma_{23}^q, \quad (\gamma_{12}\gamma_{23})^r, \quad \Delta^2$$

3 Deligne-Mostow and P_4

Generalization of hypergeometric functions due to Picard, Appell, ...

$$F(z) = \int_C t^{-\mu_0}(t-1)^{-\mu_1} \prod_{2\leq i\leq d+1} (t-z_i)^{-\mu_i} dt$$

- We assign the exponent at ∞ by
$\mu_0 + \cdots + \mu_{d+1} + \mu_\infty = 2.$
- We suppose $\mu_i > 0$ for $0 \leq i \leq d+1$ and $i = \infty$.
- The monodromy defines linear representation of P_n modulo Δ^2.

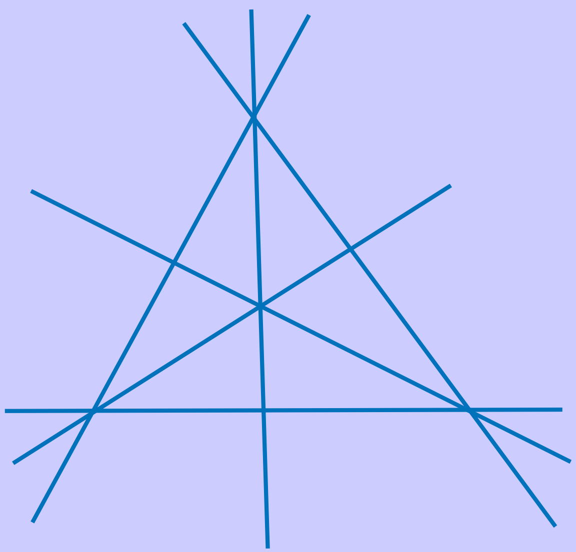
INT condition $\boxed{(1-\mu_i-\mu_j)^{-1} \in \mathbf{Z}}$ for all $i \neq j$ such that $\mu_i + \mu_j < 1$

Theorem [Deligne-Mostow]. Under the above INT condition, the monodromy of hypergeometric functions is a lattice in $PU(1,d)$.

These are **Burau-Gassner** representations at special values.

In the case of $d = 2$ this construction gives the universal branched covering of an orbifold with 6 lines in $\mathbf{C}P^2$ as 2-dimensional complex ball, where branch index is given by

$$n_{ij} = (1 - \mu_i - \mu_j)^{-1}$$



• Blow-up at triple points.

• The kernel of Gassner representation at these special values is the normal subgroup of P_4 generated by Δ^2 and

$$\gamma_{ij}^{n_{ij}} \quad (\gamma_{ij}\gamma_{jk}\gamma_{ki})^d, (d = \mathsf{lcm}(n_{ij}, n_{jk}, n_{ki}))$$

• The image is identified with a subgroup of the automorphism of 2-dimensional complex ball. $Aut(\mathbf{D}^2)$

• In the work of Delign-Mostow there is a list of 27 such differential equations.

4 KZ equation

$\mathfrak{g}$: semi-simple Lie algebra.

$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.

$\Omega = \sum_\mu I_\mu \otimes I_\mu$

$r_i : \mathfrak{g} \to End(V_i)$, $1 \leq i \leq n$ representations.

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Ω_{ij} : the action of Ω on the i-th and j-th components of $V_1 \otimes \cdots \otimes V_n$.

$$\omega = \frac{1}{\kappa} \sum_{i,j} \Omega_{ij} d \log(z_i - z_j), \quad \kappa \in \mathbf{C} \setminus \{0\}$$

ω defines a flat connection for a trivial vector bundle over the configuration space

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j, \ i \neq j\}$$

with fiber $V_1 \otimes \cdots \otimes V_n$ since we have

$$\boxed{\omega \wedge \omega = 0}$$

As the holonomy we have representations

$$\theta_\kappa : P_n \longrightarrow GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

$$\theta_\kappa : B_n \longrightarrow GL(V^{n\otimes}).$$

[Drinfeld-K Theorem].
These representations are described by means of quantum R matrices. (description by quantum groups)

We shall express the solutions of KZ equation $d\varphi = \omega\varphi$ by hyergeometric integrals.

5 Solutions of KZ equation

Consider the case $\mathfrak{g} = sl_2(\mathbf{C})$ with basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

V_λ : highest weight representation of $sl_2(\mathbf{C})$ with highest weight vector v :

$$Hv = \lambda v,\ \ Ev = 0$$

$V_{\lambda_1} \otimes \cdots V_{\lambda_n}$: tensor product of highest weight representations of $sl_2(\mathbf{C})$

Set $\lambda = \lambda_1 + \cdots + \lambda_n$.
For a non-negative integer ℓ put

$$W[\lambda - 2\ell] = \{x \in V_{\lambda_1} \otimes \cdots V_{\lambda_n} \ ; \ Hx = (\lambda - 2\ell)x\},$$

$$N[\lambda - 2\ell] = \{x \in W[\lambda - 2\ell] \ ; \ Ex = 0\}.$$

The KZ connection ω commutes with the diagonal action of $\mathfrak{g}$ on $V_{\lambda_1} \otimes \cdots V_{\lambda_n}$, hence it leaves invariant the space of null vectors $N[\lambda - 2\ell]$.

$\pi : X_{n+m} \longrightarrow X_n$: projection defined by $(z_1, \cdots, z_n, t_1, \cdots, t_m) \mapsto (z_1, \cdots, z_n)$.

$X_{n,m}$: fiber of π.

$$\Phi = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{\kappa}} \prod_{1 \leq i \leq m, 1 \leq \ell \leq n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}}$$

$$\times \prod_{1 \leq i < j \leq m} (t_i - t_j)^{\frac{2}{\kappa}}$$

(multi-valued function on X_{n+m}).

Construct solutions of KZ equation with values in $N[\lambda - 2\ell]$.

Example. (the case $\ell = 1$)
$W[\lambda - 2]$ is spanned by

$$v_1 \otimes \cdots \otimes F v_j \otimes \cdots \otimes v_n, \quad 1 \leq j \leq m$$

with highest weight vectors $v_1, \cdots, v_n$ for $V_{\lambda_1}, \cdots V_{\lambda_n}$. $N[\lambda - 2\ell]$ is a codimension one linear subspace.

The solutions are

$$\varphi = \sum I_j v_1 \otimes \cdots \otimes F v_j \otimes \cdots \otimes v_m$$

with

$$I_j = \int_\Delta \eta_j, \quad \eta_j = \Phi \frac{dt}{t - z_j}$$

$E\varphi = 0$ implies $\lambda_1 I_1 + \cdots + \lambda_n I_n = 0$ which is a relation among de Rham cohomology classes. The action of P_n on $N[\lambda - 2\ell]$ is identified with Gassner representation.

6 Hypergeometric pairing

Put $Y_{n,m} = X_{n,m}/S_m$.

$\mathcal{L}$: local system on $Y_{n,m}$ associated with the multi-valued function Φ.

The twisted de Rham complex $(\Omega^*(Y_{n,m}), \bigtriangledown)$ is defined by

$$\bigtriangledown\omega = d\log\Phi \wedge \omega.$$

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The twisted de Rham complex $(\Omega^*(Y_{n,m}), \nabla)$ is defined by

$$\nabla\omega = d\log\Phi \wedge \omega.$$

Hypergeometric pairing:

$$H_m(Y_{n,m}, \mathcal{L}^*) \times H^m(\Omega^*(Y_{n,m}), \nabla) \to \mathbf{C}$$

defined by

$$(c, w) \mapsto \int_c \Phi w.$$

There is a map

$$\rho : N[\lambda - 2m] \to \Omega^m(Y_{n,m})$$

with $\rho(w) = R_w(t, z) dt_1 \wedge \cdots \wedge dt_m$ a rational form so that the following theorem holds.

Let w_J be a basis of $N[\lambda - 2\ell]$.

Theorem [Schechtman-Varchenko...].

$$\sum_J \int_\Delta \Phi \rho(w_J)$$

is a solution of the KZ equation, where Δ is a cycle in $H_m(Y_{n,m}, \mathcal{L}^*)$.

Theorem. For generic λ, κ, there is an isomophism

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \cong N[\lambda - 2m]^*$$

where ϕ is defined by

$$\langle \phi(c), w \rangle = \int_c \Phi \rho(w).$$

This gives a basis of the solution of KZ equation with values in $N[\lambda - 2m]$.

Moreover, the following two representations of pure braid groups are equivalent:

(1) Action of P_n on the twisted homology $H_m(Y_{n,m}, \mathcal{L}^*)$.

(2) Holonomy representation of the KZ equation with valued in $N[\lambda - 2m]$.

In the case $\lambda_1 = \cdots = \lambda_n$ and $m = 2$ they are LKB representations.

Remark. For generic λ, κ,

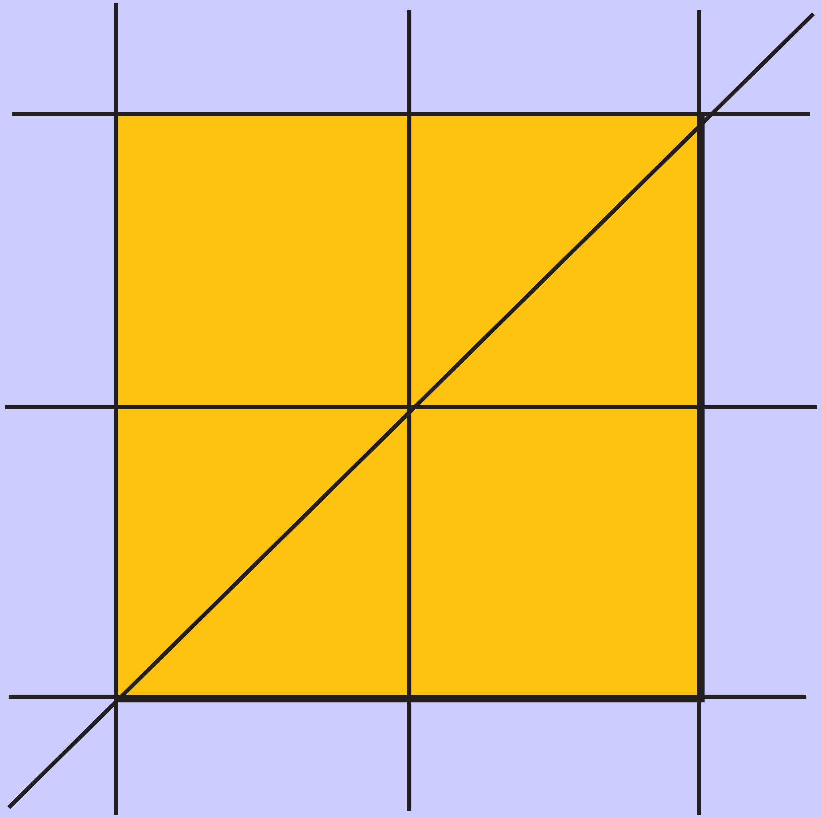
$$H_j(Y_{n,m}, \mathcal{L}^*) \cong 0, \quad j \neq m$$

and we have an isomorphism

$$H_m(Y_{n,m}, \mathcal{L}^*) \cong H_m^{lf}(Y_{n,m}, \mathcal{L}^*)$$

(homology with locally finite chains)

The above homology is spanned by bounded chambers.



bounded chambers : basis of twisted homology (the case $n = 3, m = 2$).

7 Space of conformal blocks

Put $\kappa = K + 2$ (K a positive integer). Suppose $0 \leq \lambda_1, \cdots, \lambda_{n+1} \leq K$.

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$: affine Lie algebra.

$p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$ with $p_{n+1} = \infty$

Assign highest weights $\lambda_1, \cdots, \lambda_{n+1} \in \mathbf{Z}$ to $p_1, \cdots, p_{n+1}$.

$\mathcal{H}_j$: irreducible representations of $\widehat{\mathfrak{g}}$ with highest weight λ_j at level K.

Ref. [T. Kohno] Conformal Field Theory and Topology, Monograph, AMS 2002

The space of conformal blocks is defined as

$$\mathcal{H}(p,\lambda) = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_{n+1}}/(\mathfrak{g} \otimes \mathcal{M}_p)$$

where $\mathcal{M}_p$ is the set of meromorphic functions on $\mathbf{C}P^1$ with poles at most at $p_1, \cdots, p_{n+1}$.

$\mathcal{H}(p,\lambda)$ is identified with a quotient space of $N[\lambda_{n+1}]$ and there is a map

$$\rho : \mathcal{H}(p,\lambda) \longrightarrow H^m(\Omega^*(Y_{n,m}), \nabla).$$

so that the map

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \to \mathcal{H}(p, \lambda)^*$$

defined by

$$\langle \phi(c), w \rangle = \int_c \rho(w)$$

is surjective ([Feigin-Schechtman-Varchenko]).

Consider the natural map

$$\alpha : H_m(Y_{n,m}, \mathcal{L}^*) \longrightarrow H_m^{lf}(Y_{n,m}, \mathcal{L}^*)$$

and put $\mathrm{Im}(\alpha) = H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg}$
(the set of regularizable cycles).

Theorem. ϕ induces an isomorphism

$$H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg} \cong \mathcal{H}(p, \lambda)^*$$

equivariant under the action of braids.

Final Remarks

• Equivalence of two flat bundles over the configuration space – the space of conformal blocks and regularizable cycles.

• Positive definite hermitian form invariant under the action of braid groups by Hodge theory for cohomology with local systems. Geometric structure of the representations of braid groups.

• Possible generalization to mapping class groups of surfaces.