The Dimension Of
The Second Bounded Cohomology
(as a vector space over R)

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I. History

- Hirsh-Thurston
- P. Trauber
  - Bounded cohomology of an amenable group is zero.
- M. Gromov
  - Volume and bounded cohomology (1985)
  - Bounded cohomology of topological spaces
  - $\widehat{H}^*(G) \stackrel{\text{def}}{=} \widehat{H}^*(K(G,1))$
  - Applied to Riemannian Geometry
  - Simplicial Multicomplexes - Difficulty

- R. Brooks
  - Relative homological Algebra
    - Strong relatively injective G-resolution with trivial G-module $\mathbf{R}$.
  - Norm is not clear
  - The second bounded cohomology of $\mathbf{Z} * \mathbf{Z}$ is infinite.

- N. Ivanov
  - Relative homological algebra
    - Relatively Injective G- Resolution of the trivial G-module $\mathbf{R}$
  - Canonical seminorm
  - Theorem: For a topological space X equipped with a universal covering, $\widehat{H}^*(X) \cong \widehat{H}^*(\pi_1(X))$.

II. Definition

(1) Bounded cohomology of a (discrete) group G

- $C^n(G) = \{f \colon G^n \to \mathbf{R}\}$.
- $H^*(G) = H^*(G, \mathbf{R})$:
  Cohomology of $\{C^n(G) \xrightarrow{d_n} C^{n+1}(G)\}$, where
  $$d_n(f)(g_1, \cdots, g_{n+1}) = f(g_2, \cdots, g_{n+1})$$
  $$+ \sum_{i=1}^{n} (-1)^i f(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1})$$
  $$+ (-1)^{n+1} f(g_1, \cdots, g_n).$$
- $B^n(G)$
  $= \{f \in C^n(G) \mid \|f\| = \sup_{x \in G^n} \mid f(x) \mid < \infty\}$.
  This bounded cochain group is a Banach space.
- The bounded cohomology of G, $\widehat{H}^*(G)$, is the cohomology of $\{B^n(G), d_n\}$.
  $\widehat{H}^n(G)$ has a seminorm in general.
  Especially, $\widehat{H}^{\mathbf{2}}(G)$ is a Banach space.

(2) Bounded cohomology of a topological space X

- $C^n(X) = \{f \colon S_n(X) \to \mathbf{R}\}$, where $S_n(X)$ is the set of n-dimensional singular simplices in X.
- $B^n(X)$
  $= \{f \in C^n(X) \mid \|f\| = \sup_{\sigma \in S_n(X)} |f(\sigma)| < \infty\}$.
- The bounded cohomology of X, $\widehat{H}^*(X)$, is the cohomology of $\{B^*(X), \partial_*\}$.

(3) A group G is amenable

- Definition: The group G has the right invariant mean on $B(G)$, that is, there is a linear functional $m \colon B(G) \to \mathbf{R}$ such that
  $\inf_{x \in G} f(x) \le m(f) \le \sup_{x \in G} f(x)$ and
  $m(g \cdot f) = m(f)$ for all $g \in G$.
- Examples
  Finite groups, Abelian groups, Solvable groups, the homomorphic image of an amenable group.

III. Known Results

- Bounded cohomology of a simply connected space is zero.
- $\widehat{H}^*(X) \cong \widehat{H}^*(\pi_1(X))$
- If $X \simeq Y$, then $\widehat{H}^*(X) = \widehat{H}^*(Y)$
- For an amenable normal subgroup $N \subset G$, $\widehat{H}^*(G/N) \cong \widehat{H}^*(G)$.
- Bounded cohomology of an amenable group is zero.
- For a free group F with rank ≥ 2, $\widehat{H}^2(F)$ is infinite.
- For a normal subgroup N of G, there is a five-term exact sequence
  $0 \to \widehat{H}^2(G/N) \to \widehat{H}^2(G) \to \widehat{H}^2(N)^{G/N} \to \widehat{H}^3(G/N) \to \widehat{H}^3(G)$.

IV. Bounded vs Ordinary cohomology

- The inclusion map $B^*(G) \to C^*(G)$ induces the homomorphism $\widehat{H}^*(G) \to H^*(G)$ which is neither injective nor surjective, in general.
- Eilenberg-Steenrod Axioms
  - True for $\widehat{H}^*(X)$
    Identity Axiom, Composition, Natural transformation,
    Long exact sequence, homotopy axiom, Dimension Axiom
  - $\widehat{H}^*(X)$ does NOT hold Excision Axiom

- Examples:
  - $\pi_1(T) = \mathbf{Z} \oplus \mathbf{Z}$ is amenable
    * $\widehat{H}^n(T) = \widehat{H}^n(\pi_1(T)) = 0$ for $n > 0$
    * $H^2(T) = \mathbf{R}$, while $H^2(\pi_1 T)) = 0$
  - For a sphere S^2,
    * $\widehat{H}^2(S^2) = 0$
    * $H^2(S^2) = \mathbf{R}$
  - For a Torus T_g with genus $g > 1$,
    * $\widehat{H}^2(T_g)$ is infinite
    * $H^2(T_g) = \mathbf{R}$
  - For a free group F with rank ≥ 2
    * $\widehat{H}^2(F)$ is infinite
    * $H^2(F) = 0$

V. Why the second bounded cohomology?

- $\widehat{H}^0(G) = \mathbf{R}$

- $\widehat{H}^1(G) = 0$
  $0 \to \mathbf{R} \xrightarrow{d_0=0} B(G) \xrightarrow{d_1} B^2(G) \xrightarrow{d_2} \cdots$, and
  $\widehat{H}^1(G) = \ker d_1$ is the group of bounded homomorphisms.

- Fujiwara's Conjecture:

  - If $\widehat{H}^2(G) \neq 0$, then $\widehat{H}^2(G)$ is infinite dimensional as a vector space over $\mathbf{R}$.

  - Counterexamples
    $\widehat{H}^2(SL(2, \mathbf{R})) = \mathbf{R}$
    $\widehat{H}^2(\mathrm{Homeo}_+(S^1)) = \mathbf{R}$

  - Common property in counterexamples?
    Linear groups $\sim$ perfect groups

VI. $\widehat{H}^2$(hypoabelian group)

A. Definitions

- The transfinite derived series of G is
  an extension of its derived series to higher ordinals
  defined by the rules
  $G^{(\alpha)} = [G^{(\alpha-1)}, G^{(\alpha)-1}]$ and $G^{(\lambda)} = \bigcap_{\beta<\lambda} G^{(\beta)}$,
  where $\alpha \geq 1$ is a nonlimit ordinal and
  λ is a limit ordinal.

- A group G is said to be residually solvable
  if for each $g \in G$ with $g \neq e$
  there is a normal subgroup $N \trianglelefteq G$ of G such that
  $g \notin N$ and the quotient G/N is solvable.

  A group G is residually solvable if and only if
  $G^{(\omega)} = \bigcap_{n<\omega} G^{(n)}$ is trivial.

  E. g. Free groups of rank ≥ 2 are residually solvable.

- A group G is said to be hypoabelian
  if its maximal perfect subgroup is trivial.

  A group G is hypoabelian if and only if
  $G^{(\alpha)}$ is trivial for some ordinal α.

B. Theorems for Residually Solvable groups

- Let G be residually solvable.
  Then $\widehat{H}^2(G)$ is either zero or infinite dimensional.

  (1) Let $G = F/K$ for a free group F.
  If exact sequence for $G^{(n)} = F^{(n)}/(F^{(n)} \cap K)$
  $1 \to F^{(n)} \cap K \to F^{(n)} \to F^{(n)}/(F^{(n)} \cap K) \to 1$
  is trivial for some finite ordinal n,
  then $\widehat{H}^2(G)$ is zero or infinite dimensional.

  Idea: This is the case that either $G^{(n)} = e$ or
  $F^{(n)} \cap K = e$ so that $G^{(n)} = F^{(n)}$ is free.

  (2) Let $G = F/K$ be residually solvable.
  If the exact sequence induced from $G^{(n)} = F^{(n)}K/K$,
  $0 \to K \to F^{(n)}K \to F^{(n)}K/K = G^{(n)} \to 0$,
  has the first trivial one at the limit ordinal ω,
  $0 \to K \to \cap(F^{(n)}K) \to \cap_{n<\omega}(F^{(n)}K/K) \to 0$,
  then the homomorphism $\varphi^*\colon \widehat{H}^2(F) \to \widehat{H}^2(K)^G$
  induced from the inclusion homomorphism
  $K \xrightarrow{\varphi} F$ is injective.
  Furthermore, $\widehat{H}^2(G) = \widehat{H}^2(F/K)$ is zero.

C. Theorems for Hypoabelian groups

- Let G be hypoabelian.
  Then $\widehat{H}^2(G)$ is either zero or infinite dimensional.
  Idea: It is done by transfinite induction.

  (1) Let $N \trianglelefteq G$ and N be residually solvable.
  Suppose $\widehat{H}^2(G/N) = 0$.
  Then $\widehat{H}^2(G)$ is either zero or infinite dimensional.

  (2) If $G^{(\alpha)} = e$ for a nonlimit ordinal, G contains an abelian normal subgroup $G^{(\alpha-1)}$.

  (3) If $G^{(\delta)} = e$ for a limit ordinal,
  $\delta = \omega k$ for some finite $k < \omega$ or $\delta = \omega\omega$, etc.

- If the dimension $\dim_{\mathbf{R}} V$ of a Banach space V
  as a vector space over $\mathbf{R}$ is not less than countably infinite,
  then the dimension $\dim_{\mathbf{R}} V$ is at least that of the continuum.

- Let G be hypoabelian and $\widehat{H}^2(G) \neq 0$.
  Then the dimension of $\widehat{H}^2(G)$, which is a Banach space, as a vector space over $\mathbf{R}$ is the continuum.

D. Examples

- The infinite dihedral group
  $D_\infty = < x, y \mid x^2 = 1, y^2 = 1 >$
  is residually solvable. Also D_∞ is isomorphic to the semidirect product of cyclic group of order 2 as the active group and infinite cyclic group as the passive group. Its second bounded cohomology is zero.

- Let G be a surface group or a free product of cyclic groups. Then its first commutator subgroup $G' = [G, G]$ is free, and so G is residually solvable.
  Its second bounded cohomology is infinite.

E . Currently...

How about the groups which are not hypoabelian?
The second bounded cohomology of a perfect group.

(1) The dimension of the second bounded cohomology of a perfect group could be anything.

- The second bounded cohomology of a finite perfect group is zero.
- $\widehat{H}^2(SL(2, \mathbf{R})) = \mathbf{R}$.
- Free product of perfect groups are perfect and its second bounded cohomology is infinite.

(2) If P is a maximal perfect normal subgroup of G, then $\widehat{H}^2(G)$ is infinite dimensional or $\widehat{H}^2(G) \cong \widehat{H}^2(P)^{G/P}$.

(3) The second bounded cohomology of a uniformly perfect group.