Dept. of Math. University of Oslo
Pure Mathematics
ISBN 82–553–1298–6 No. 17
ISSN 0806–2439 July 2001

Optimal Control of Stochastic Partial Differential Equations and Application to Partial Observation Control

Bernt Øksendal
Dept. of Mathematics, University of Oslo
Box 1053 Blindern, N–0316 Oslo, Norway
(email: oksendal@math.uio.no)
and
Norwegian School of Economics and Business Administration,
Helleveien 30, N–5045 Bergen, Norway

July 31, 2001

Abstract

We prove a sufficient maximum principle for the optimal control of systems described by a quasilinear stochastic heat equation. The result is applied to give a maximum principle solution method for stochastic control problems with partial (incomplete) information.

1 Introduction

Let $T > 0$ and let G be an open set in $\mathbb{R}^n$ with C^1 boundary ∂G. Suppose that the state $Y(t,x) \in \mathbb{R}$ of a system at time $t \in [0,T]$ and at the point $x \in \bar{G} = G \cup \partial G$ is given by a *quasilinear stochastic heat equation* of the form

$$(1.1)\qquad dY(t,x) = \begin{cases} [LY(t,x) + b\big(t,x,Y(t,x),\nabla Y(t,x),u(t,x)\big)]dt \\ \quad +\sigma\big(t,x,Y(t,x),\nabla Y(t,x),u(t,x)\big)dB(t); \quad (t,x) \in (0,T) \times G \end{cases}$$

$$(1.2)\qquad Y(0,x) = \xi(x)\ ; \qquad x \in \bar{G}$$

$$(1.3)\qquad Y(t,x) = \eta(t,x); \qquad (t,x) \in (0,T) \times \partial G\ .$$

Here $dY(t,x)$ denotes the Itô differential with respect to t, $\nabla Y(t,x) = \nabla_x Y(t,x)$ denotes the gradient of Y with respect to x, while L is a second order partial differential operator acting on x given by

$$(1.4)\qquad L\phi(x) = \operatorname{div}\big(a(x)\nabla\phi(x)\big); \qquad \phi \in C^2(\mathbb{R}^n)$$

where $a(x) = [a_{ij}(x)]_{1\le i,j\le n}$ is a given nonnegative definite symmetric $n \times n$ matrix with entries $a_{ij}(x) \in C^2(G) \cap C(\bar{G})$ for all $i, j = 1, 2, \dots, n$. The process $B(t) = B(t,\omega)$; $t \ge 0$, $\omega \in \Omega$ is a (1-dimentional, 1-parameter) Brownian motion on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P\big)$, while $u(t,x) = u(t,x,\omega)$ is our *control* process. We assume that $u(t,x)$ has values in a given convex set $U \subset \mathbb{R}^k$ and that $u(t,x,\cdot)$ is $\mathcal{F}_t$-measurable for all $(t,x) \in (0,T) \times G$ i.e. that $u(t,x)$ is *adapted* for all $x \in G$. The functions $b : (0,T) \times G \times \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}$ and $\sigma : (0,T) \times G \times \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}$ are given C^1 functions. The boundary value functions $\xi : \bar{G} \to \mathbb{R}$ and $\eta : [0,T] \times \partial G \to \mathbb{R}$ are assumed to be deterministic.

We call the control process $u(t,x)$ *admissible* if the corresponding stochastic partial differential equation (1.1)–(1.3) has a unique, strong solution $Y(\cdot) \in L^2(\lambda \times P)$, where λ is Lebesgue measure on $[0,T] \times \bar{G}$, and with values in a given set $S \subset \mathbb{R}$. The set of admissible controls is denoted by $\mathcal{A}$.

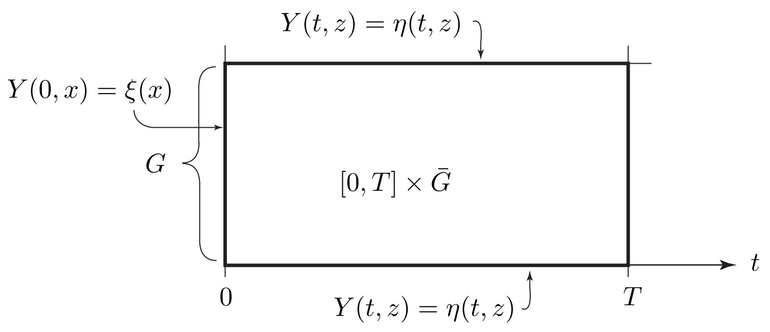

Figure 1: The boundary values of $Y(t,x)$.

Suppose the *performance* $J(u)$ obtained by applying the control $u \in \mathcal{A}$ has the form

$$J(u) = E\Big[\int_0^T \Big(\int_G f\big(t,x,Y(t,x),u(t,x)\big)dx\Big)dt + \int_G g\big(x,Y(T,x)\big)dx\Big] \tag{1.5}$$

where f and g are given lower bounded C^1 functions and E denotes the expectation with respect to P.

We consider the problem to find $J^* \in \mathbb{R}$ and $u^* \in \mathcal{A}$ such that

$$J^* = \sup_{u\in\mathcal{A}} J(u) = J(u^*) \tag{1.6}$$

This is an optimal control problem for the quasilinear stochastic heat equation.

The main purpose of this paper is to prove a maximum principle type of verification theorems for such optimal control problems (Theorems 2.1, 2.2 and 2.3). Then we use the connection between such optimal control problems (with *complete* information) and stochastic control problems with *partial* observation to establish a sufficient maximum principle for partial observation control (Theorem 3.1).

Stochastic control of the stochastic partial differential equations (SPDEs) arizing from partial observation control has been studied by Mortensen [**?**], using a dynamic programming approach, and subsequently by Bensoussan, using a maximum principle method. See [**?**] and the references therein. Our approach differs from the approach of Bensoussan in two ways:

First, we give *sufficient* maximum principle results, not necessary ones. Second, we consider more general quasilinear semielliptic SPDEs.

Here is an outline of the paper: In Section 2 we give 3 versions of a sufficient maximum principle (verification theorem) for optimal control of quasilinear SPDEs. The results are illustrated by an example from optimal harvesting.

In Section 3 we review the (well-known) connection between optimal control of SPDEs (with complete information) and optimal control of SDEs with partial information. We then apply the results of Section 2 to obtain a sufficient maximum principle for partial observation control (Theorem 3.1). Again the results are illustrated by an example.

Necessary maximum principles of various types for partial observation control have been obtained by Paras, Elliott and Kohlmann [?], Bensoussan [B1–3] and Haussmann [?].

2 A Sufficient Maximum Principle

We now formulate a sufficient maximum principle for the optimal control of the problem (1.1)–(1.6).

Define the *Hamiltonian* $H : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times U \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ assosiated to the problem (1.1)–(1.6) by

$$H(t,x,y,\gamma,u,p,q) = f(t,x,y,u) + b(t,x,y,\gamma,u)p + \sigma(t,x,y,\gamma,u)q\ . \tag{2.1}$$

For each $u \in \mathcal{A}$ we consider the following *adjoint backward* SPDE in the two unknown adapted processes $p(t,x)$, $q(t,x)$:

$$\begin{aligned} dp(t,x) = -\Big\{ \Big(\frac{\partial H}{\partial y}\Big)\big(t,x,Y(t,x),\nabla Y(t,x),u(t,x),p(t,x),q(t,x)\big) \\ +Lp(t,x)\Big\}dt + q(t,x)dB(t)\ ; \qquad 0 \le t \le T, \quad x \in G \end{aligned} \tag{2.2}$$

$$p(T,x) = \frac{\partial g}{\partial y}\big(x, Y(T,x)\big); \qquad x \in \bar{G} \tag{2.3}$$

$$p(t,x) = 0\ ; \qquad (t,x) \in (0,T) \times \partial G \tag{2.4}$$

Here $Y(t,x) = Y^u(t,x)$ is the solution of (1.1)–(1.3) corresponding to u.

Theorem 2.1 (Sufficient SPDE maximum principle I)
Let $\hat{u} \in \mathcal{A}$ with corresponding solution $\widehat{Y}$ of (1.1)–(1.3) and let $\hat{p}(t,x)$, $\hat{q}(t,x)$ be a solution of the associated adjoint backward SPDE (2.2)–(2.4). Suppose the following, (2.5)–(2.6):

(2.5) *The functions*
$(y,u) \to H(y,u) := H\big(t,x,y,\nabla\widehat{Y}(t,x),u,\hat{p}(t,x),\hat{q}(t,x)\big)\ ; \quad y \in \mathbb{R},\ u \in U$
and
$y \to g(x,y)\ ; \qquad y \in \mathbb{R}$
are concave*, for all* $(t,x) \in [0,T] \times \bar{G}$

$$\begin{aligned} &H\big(t,x,\widehat{Y}(t,x),\nabla\widehat{Y}(t,x),\hat{u}(t,x),\hat{p}(t,x),\hat{q}(t,x)\big) \\ &\qquad = \sup_{u \in U} H\big(t,x,\widehat{Y}(t,x),\nabla\widehat{Y}(t,x),u,\hat{p}(t,x),\hat{q}(t,x)\big) \end{aligned} \tag{2.6}$$
for all $(t,x) \in [0,T] \times \bar{G}$

Then $\hat{u}(t,x)$ is an optimal control for the stochastic control problem (1.6).

Proof. Let u be an arbitrary admissible control with corresponding solution $Y(t,x) = Y^u(t,x)$ of (1.1)–(1.3). Consider

$$J(\hat{u}) - J(u) = E\Big[\int_0^T \int_G \Big\{\hat{f} - f\Big\} dx\, dt + \int_G \Big\{\hat{g} - g\Big\} dx\Big] \tag{2.7}$$

where

$$\hat{f} = f(t,x,\widehat{Y}(t,x),\hat{u}(t,x))\ , \qquad f = f(t,x,Y(t,x),u(t,x))$$
$$\hat{g} = g(x,\widehat{Y}(T,x)) \qquad \text{and} \qquad g = g(x,Y(T,x))\ .$$

Similarly we put

$$\hat{b} = b(t,x,\widehat{Y}(t,x),\nabla\widehat{Y}(t,x),\hat{u}(t,x))\ , \qquad b = b(t,x,Y(t,x),\nabla\widehat{Y}(t,x),u(t,x))$$
$$\hat{\sigma} = \sigma(t,x,\widehat{Y}(t,x),\nabla\widehat{Y}(t,x),\hat{u}(t,x))\ , \qquad \sigma = \sigma(t,x,Y(t,x),\nabla\widehat{Y}(t,x),u(t,x))$$

and we set

$$\widehat{H} = H(t,x,\widehat{Y}(t,x),\nabla\widehat{Y}(t,x),\hat{u}(t,x),\hat{p}(t,x),\hat{q}(t,x))\ ,$$
$$H = H(t,x,Y(t,x),\nabla\widehat{Y}(t,x),u(t,x),\hat{p}(t,x),\hat{q}(t,x))\ .$$

Then (2.7) can be written

$$J(\hat{u}) - J(u) = I_1 + I_2\ , \qquad \text{where} \tag{2.8}$$

$$I_1 = E\Big[\int_0^T \int_G \Big\{\widehat{H} - H - (\hat{b} - b)\hat{p} - (\hat{\sigma} - \sigma)\hat{q}\Big\} dx\, dt\Big] \tag{2.9}$$

and

$$I_2 = E\Big[\int_G \{\hat{g} - g\} dx\Big]\ . \tag{2.10}$$

By concavity of the function $y \to g(x,y)$ we have

$$g - \hat{g} \le \frac{\partial g}{\partial y}(x,\widehat{Y}(T,x)) \cdot (Y(T,x) - \widehat{Y}(T,x))\ . \tag{2.11}$$

Therefore, writing

$$\widetilde{Y}(t,x) := Y(t,x) - \widehat{Y}(t,x)\ , \tag{2.12}$$

we get

$$\begin{aligned}
I_2 \ge &- E\Big[\int_G \frac{\partial g}{\partial y}\Big(x,\widehat{Y}(T,x)\Big) \cdot \widetilde{Y}(T,x) dx\Big] \\
= &- E\Big[\int_G \hat{p}(T,x) \cdot \widetilde{Y}(T,x) dx\Big] \\
= &- E\Big[\int_G \Big(\hat{p}(0,x) \cdot \widetilde{Y}(0,x) + \int_0^T \big\{\widetilde{Y}(t,x) d\hat{p}(t,x) + \hat{p}(t,x) d\widetilde{Y}(t,x) \\
&\quad + (\sigma - \hat{\sigma}) \cdot \hat{q}(t,x)\big\} dt\Big) dx\Big] \\
= &- E\Big[\int_G \Big(\int_0^T \Big\{\widetilde{Y}(t,x)\Big[-\Big(\frac{\partial H}{\partial y}\Big)^{\wedge} - L\hat{p}(t,x)\Big] \\
&\quad + \hat{p}(t,x)\big[L\widetilde{Y}(t,x) + (b - \hat{b})\big] + (\sigma - \hat{\sigma})\hat{q}(t,x)\Big\} dt\Big) dx\Big]\ ,
\end{aligned} \tag{2.13}$$

where

$$\Big(\frac{\partial H}{\partial y}\Big)^{\wedge} = \frac{\partial H}{\partial y}\big(t, x, \widehat{Y}(t,x), \nabla\widehat{Y}(t,x), \hat{u}(t,x), \hat{p}(t,x), \hat{q}(t,x)\big) .$$

Combining (2.9) and (2.13) we get

$$\begin{aligned} J(\hat{u}) - J(u) = I_1 + I_2 \geq E\Big[\int_0^T \Big(\int_G \Big\{L\hat{p}\cdot\widetilde{Y} - \hat{p}\cdot L\widetilde{Y}\Big\}dx\Big)dt\Big] \\ +E\Big[\int_G \Big(\int_0^T \Big\{\widehat{H} - H + \Big(\frac{\partial H}{\partial y}\Big)^{\wedge}\cdot\widetilde{Y}(t,x)\Big\}dt\Big)dx\Big]. \end{aligned} \tag{2.14}$$

Integration by parts gives that

$$\int_G L\hat{p}\cdot\widetilde{Y}\,dx = \int_{\partial G}(\widetilde{Y}a\nabla\hat{p})^T N dS - \int_G (\nabla\hat{p})^T a\nabla\widetilde{Y}\,dx \tag{2.15}$$

and similarly

$$\int_G L\widetilde{Y}\cdot\hat{p}\,dx = \int_{\partial G}(\hat{p}a\nabla\widetilde{Y})^T N dS - \int_G (\nabla\widetilde{Y})^T a\,\nabla\hat{p}\,dx , \tag{2.16}$$

where $(\cdot)^T$ denotes matrix transposed, $N = N(x)$ is the outward unit normal vector at $x \in \partial G$ and the first integrals on the right are the surface integrals over ∂G.

Subtracting (2.16) from (2.15) we get

$$\int_G \{L\hat{p}\cdot\widetilde{Y} - \hat{p}\cdot L\widetilde{Y}\}dx = \int_{\partial G}\big[a(\widetilde{Y}\nabla\hat{p} - \hat{p}\nabla\widetilde{Y})\big]^T N dS . \tag{2.17}$$

By (1.3) and (2.4) we have $\widetilde{Y}(t,x) = \hat{p}(t,x) = 0$ for all $(t,x) \in (0,T)\times\partial G$. Hence

$$\int_G \{L\hat{p}\cdot\widetilde{Y} - \hat{p}\cdot L\widetilde{Y}\}dx = 0 \qquad \text{for all} \quad t \in (0,T) . \tag{2.18}$$

Therefore (2.14) gives

$$J(\hat{u}) - J(u) \geq E\Big[\int_G \Big(\int_0^T \{\widehat{H} - H + \Big(\frac{\partial H}{\partial y}\Big)^{\wedge}\cdot\widetilde{Y}(t,x)\}dt\Big)dx\Big] . \tag{2.19}$$

Since $H(y,u)$ is concave (by (2.5)), we have

$$H - \widehat{H} \leq \frac{\partial H}{\partial y}(\widehat{Y},\hat{u})\cdot(Y - \widehat{Y}) + \frac{\partial H}{\partial u}(\widehat{Y},\hat{u})(u - \hat{u}) . \tag{2.20}$$

Since $v \to H(\widehat{Y}, v)$ is maximal at $v = \hat{u}$ by (2.6), we have

$$\frac{\partial H}{\partial u}(\widehat{Y},\hat{u})\cdot(u - \hat{u}) \leq 0 . \tag{2.21}$$

Hence by (2.20)

$$H - \widehat{H} - \frac{\partial H}{\partial y}(\widehat{Y},\hat{u})\cdot(Y - \widehat{Y}) \leq 0 \tag{2.22}$$

which by (2.19) gives that

$$J(\hat{u}) - J(u) \geq 0 \ .$$

Since $u \in \mathcal{A}$ was arbitrary the proof is complete. □

In some applications the Hamiltonian function

$$h(t,x,y,u) := H(t,x,y,\nabla\widehat{Y}(t,x),u,\hat{p}(t,x),\hat{q}(t,x)) \tag{2.23}$$

is not concave in both variables (y,u). In such cases it is useful to replace the concavity in (y,u) by a weaker condition, sometimes called the *Arrow condition*:

(2.24) The function $\hat{h}(t,x,y) := \max_{v\in U} h(t,x,y,v)$ exists and is concave in y, for all t,x.

Then we get the following result:

Theorem 2.2 (Sufficient SPDE maximum principle II)
Let $\hat{u},\widehat{Y},\hat{p},\hat{q}$ be as in Theorem 2.1. Suppose that $g(x,y)$ is concave in y and that the maximum condition (2.6) and the Arrow condition (2.24) hold. Then $\hat{u}(t,x)$ is an optimal control for the stochastic control problem (1.6).

Proof. We proceed as in the proof of Theorem 2.1 up to and including (2.19). Then, to obtain (2.22) note that

$$\begin{aligned} H - \widehat{H} - &\frac{\partial H}{\partial y}(\widehat{Y},\hat{u})\cdot(Y-\widehat{Y}) \\ &= h(t,x,Y(t,x),u(t,x)) - h(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) \\ &- \frac{\partial h}{\partial y}(t,x,\widehat{Y}(t,x),\hat{u}(t,x))\cdot(Y(t,x)-\widehat{Y}(t,x)) \end{aligned}$$

This is ≤ 0 by the same argument as in the proof of the Arrow sufficiency theorem for the deterministic case. See [**?**, Theorem 5, p. 107–108]. For completeness we give the details:

Note that by (2.6) we have

$$h(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) = \hat{h}(t,x,\widehat{Y}(t,x)) \ . \tag{2.25}$$

Moreover, by definition of $\hat{h}$ in (2.24) we have

$$h(t,x,y,u) \leq \hat{h}(t,x,y) \qquad \text{for all } \ t,x,y,u \ . \tag{2.26}$$

Therefore, subtracting (2.25) from (2.26) we get

$$\begin{aligned} h(t,x,&y,u) - h(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) \\ &\leq \hat{h}(t,x,y) - \hat{h}(t,x,\widehat{Y}(t,x)) \qquad \text{for all } \ t,x,y,u \ . \end{aligned} \tag{2.27}$$

Hence, to prove (2.25) it suffices to prove that

$$\begin{aligned} &\hat{h}(t,x,Y(t,x)) - \hat{h}(t,x,\widehat{Y}(t,x)) \\ &\quad - \frac{\partial h}{\partial y}(t,x,\widehat{Y}(t,x),\hat{u}(t,x))\cdot(Y(t,x)-\widehat{Y}(t,x)) \leq 0 \qquad \text{for all } \ t,x \ . \end{aligned} \tag{2.28}$$

Fix $(t,x) \in [0,T] \times \bar{G}$.

By concavity of the function $y \to \hat{h}(t,x,y)$ it follows by a standard separating hyperplane argument (see e.g. [**?**, Chapter 5, Section 23]) that there exists a *supergradient* $a \in \mathbb{R}$ for $\hat{h}(t,x,y)$ at $y = \widehat{Y}(t,x)$, i.e.

$$\hat{h}(t,x,y) - \hat{h}(t,x,\widehat{Y}(t,x)) - a \cdot (y - \widehat{Y}(t,x)) \le 0 \qquad \text{for all } \; y \; . \tag{2.29}$$

Define

$$\phi(y) = h(t,x,y,\hat{u}(t,x)) - h(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) - a \cdot (y - \widehat{Y}(t,x)) \; ; \qquad y \in \mathbb{R} \; .$$

Then by (2.27) and (2.29) we have

$$\phi(y) \le 0 \qquad \text{for all } \; y \in \mathbb{R} \; .$$

Moreover, we clearly have

$$\phi(\widehat{Y}(t,x)) = 0 \; .$$

Therefore

$$\phi'(\widehat{Y}(t)) = \frac{\partial h}{\partial y}(t,x,\widehat{Y}(t,x),\hat{u}(t,x)) = a \; .$$

Combining this with (2.29) we obtain (2.28) and the proof is complete. □

Controls which do not depend on x

In some cases, for example in the application to partial observation control (see Section 3), it is of interest to consider only controls $u(t) = u(t,\omega)$ which do not depend on the space variable x. Let us denote the set of such controls $u \in \mathcal{A}$ by $\mathcal{A}_1$. Then the problem corresponding to (1.6) is to find $J_1^* \in \mathbb{R}$ and $u^* \in \mathcal{A}_1$ such that

$$J_1^* = \sup_{u \in \mathcal{A}_1} J(u) = J(u^*) \tag{2.30}$$

where

$$J(u) = E\Big[\int_0^T \Big(\int_G f(t,x,Y(t,x),u(t))dx\Big)dt + \int_G g(x,Y(T,x))dx\Big] \tag{2.31}$$

and $Y(t,x)$ is as before given by (1.1)–(1.3) (but with $u(t,x)$ replaced by $u(t)$).

To handle this situation, we modify Theorem 2.1 as follows:

Theorem 2.3 (Sufficient SPDE maximum principle III)
Let $\hat{u} = \hat{u}(t) \in \mathcal{A}_1$ with corresponding solution $\widehat{Y}(t,x)$ of (1.1)–(1.3) and let $\hat{p}(t,x), \hat{q}(t,x)$ be a solution of the associated adjoint backward SPDE (2.2)–(2.4). Assume that (2.5) and (2.27) hold, where

(2.32) *(Average maximum condition)*

$$\int_G H(t,x,\widehat{Y}(t,x),\nabla\widehat{Y}(t,x),\hat{u}(t),\hat{p}(t,x),\hat{q}(t,x))dx$$
$$= \sup_{u \in U}\Big\{\int_G H(t,x,\widehat{Y}(t,x),\nabla\widehat{Y}(t,x),u,\hat{p}(t,x),\hat{q}(t,x))dx\Big\}$$

Then $\hat{u}(t)$ is an optimal control for the problem (2.25)–(2.26).

Proof of Theorem 2.3. We proceed as in the proof of Theorem 2.1: Let $u \in \mathcal{A}_1$ with corresponding solution $Y(t,x)$ of (1.1)–(1.3). Consider

$$J(\hat{u}) - J(u) = E\Big[\int_0^T \int_G \{\hat{f} - f\}dx\,dt + \int_G \{\hat{g} - g\}dx\Big] \tag{2.33}$$

where

$$\hat{f} = f(t,x,\widehat{Y}(t,x),\hat{u}(t)), \qquad f = f(t,x,Y(t,x),u(t)),$$
$$\hat{g} = g(x,\widehat{Y}(T,x)), \qquad \text{and} \qquad g = g(x,Y(T,x))\ .$$

Using a similar shorthand notation for $b = b(t,x,Y(t,x),\nabla\widehat{Y}(t,x),u(t))$, $\hat{b}$, σ and $\hat{\sigma}$ and setting

$$\widehat{H} = H(t,x,\widehat{Y}(t,x),\nabla\widehat{Y}(t,x),\hat{u}(t,),\hat{p}(t,x),\hat{q}(t,x))\ , \tag{2.34}$$

$$H = H(t,x,Y(t,x),\nabla\widehat{Y}(t,x),u(t),p(t,x),q(t,x)) \tag{2.35}$$

we see that (2.28) can be written

$$J(\hat{u}) - J(u) = I_1 + I_2 \tag{2.36}$$

where

$$I_1 = E\Big[\int_0^T \int_G \{\widehat{H} - H - (\hat{b} - b)\hat{p} - (\hat{\sigma} - \sigma)\hat{q}\}dx\,dt\Big] \tag{2.37}$$

and

$$I_2 = E\Big[\int_G \{\hat{g} - g\}dx\Big]\ . \tag{2.38}$$

By concavity of the function $y \to g(x,y)$ we have

$$\int_G \{g(x,Y(T,x)) - g(x,\widehat{Y}(T,x))\}dx \le \int_G \frac{\partial g}{\partial y}(x,\widehat{Y}(T,x)) \cdot \widetilde{Y}(T,x)dx$$

where

$$\widetilde{Y}(t,x) = Y(t,x) - \widehat{Y}(t,x)\ . \tag{2.39}$$

Therefore we get, as in the proof of Theorem 2.1,

$$\begin{aligned} I_2 \ge -E\Big[\int_0^T \Big(\int_G \{\widetilde{Y}(t,x)\Big[-\Big(\frac{\partial H}{\partial y}\Big)^\wedge - L\hat{p}(t,x)\Big] \\ + \hat{p}(t,x)[L\widetilde{Y}(t,x) + (b - \hat{b})] + (\sigma - \hat{\sigma})\hat{q}(t,x)\}dx)dt] \end{aligned} \tag{2.40}$$

where

$$\Big(\frac{\partial H}{\partial y}\Big)^\wedge = \frac{\partial H}{\partial y}(t,x,\widehat{Y}(t,x),\nabla\widehat{Y}(t,x),\hat{u}(t),\hat{p}(t,x),\hat{q}(t,x))\ .$$

Summing (2.32) and (2.35) we get, as in (2.19),

$$J(\hat{u}) - J(u) = I_1 + I_2 \geq E\Big[\int_0^T \Big(\int_G \Big\{\widehat{H} - H + \widetilde{Y} \cdot \Big(\frac{\partial H}{\partial y}\Big)^\wedge\Big\}dx\Big)dt\Big]. \tag{2.41}$$

where $\widehat{H}$ and H are given (3.29) and (3.30). Since $(y,u) \to H(y,u)$ is concave (by (2.5)), we have

$$H - \widehat{H} \leq \frac{\partial H}{\partial y}(\widehat{Y}, \hat{u}) \cdot (Y - \widehat{Y}) + \frac{\partial H}{\partial u}(\widehat{Y}, \hat{u}) \cdot (u - \hat{u}) \ . \tag{2.42}$$

Combining (2.36) and (2.37) we get

$$\begin{aligned} J(\hat{u}) - J(u) &\geq E\Big[\int_0^T \Big(\int_G -\frac{\partial H}{\partial u}(\widehat{Y}, \hat{u}) \cdot (u - \hat{u})dx\Big)dt\Big] \\ &= -E\Big[\int_0^T (u - \hat{u}) \cdot \frac{\partial}{\partial u}\Big(\int_G H(t,x,\widehat{Y},\nabla\widehat{Y},u,\hat{p},\hat{q})dx\Big)_{u=\hat{u}(t)} dt\Big] \\ &\geq 0 \ , \quad \text{since} \ \ u = \hat{u}(t) \ \ \text{maximizes} \\ &\qquad u \to \int_G H(t,x,\widehat{Y},\nabla\widehat{Y},u,\hat{p},\hat{q})dx \ , \end{aligned}$$

by assumption (2.27). □

Example 2.4 (Optimal harvesting)
Suppose the density $Y(t,x)$ of a population (e.g. fish) at time $t \in (0,T)$ and at the point $x \in G \subset \mathbb{R}^n$ is given by the *stochastic reaction-diffusion equation*

$$dY(t,x) = \big[\tfrac{1}{2}\Delta Y(t,x) + \alpha Y(t,x) - u(t,x)\big]dt + \beta Y(t,x)dB(t) \tag{2.43}$$
$$(\text{where} \ \ \delta = \sum_{i=1}^n \tfrac{\partial^2}{\partial x_i^2} \ \ \text{is the Laplacian})$$

with boundary conditions

$$Y(0,x) = \xi(x) \ ; \qquad x \in \bar{G} \tag{2.44}$$
$$Y(t,x) = \eta(t,x) \ ; \qquad (t,x) \in (0,T) \times \partial G \ . \tag{2.45}$$

Here $u(t,x) \geq 0$ is our *harvesting rate* at (t,x). Suppose we want to maximize a combination of the total expected utility of the consumption and the terminal size of the population, expressed by the performance criterion

$$J(u) = E\Big[\int_0^T \Big(\int_G \frac{u^r(t,x)}{r}dx\Big)dt + \theta \int_G Y(T,x)dx\Big] \tag{2.46}$$

where $r \in (0,1)$ and $\theta > 0$ are given constants. In this case the Hamiltonian (2.1) gets the form

$$H(t,x,y,\gamma,u,p,q) = \frac{u^r}{r} + (\alpha y - u)p + \beta yq \ . \tag{2.47}$$

Therefore the adjoint equation (2.2)–(2.4) becomes

$$
\begin{aligned}
&(2.48) \qquad dp(t,x) = -\big[\alpha p(t,x) + \beta q(t,x) + \tfrac{1}{2}\Delta p(t,x)\big]dt \\
&\qquad\qquad\qquad + q(t,x)dB(t)\ ; \qquad (t,x) \in (0,T)\times G \\
&(2.49) \qquad p(T,x) = \theta\ ; \qquad x \in G \\
&(2.50) \qquad p(t,x) = 0\ ; \qquad (t,x) \in (0,T)\times \partial G\ .
\end{aligned}
$$

Because the boundary conditions and all the coefficients are deterministic, we see that we can choose $q(t,x) = 0$ and solve (2.43) for *deterministic* $p(t,x)$. The equation (2.43) then gets the form

$$
\frac{\partial p}{\partial t}(t,x) + \tfrac{1}{2}\Delta p(t,x) + \alpha p(t,x) = 0\ ; \qquad (t,x) \in (0,T)\times G\ . \tag{2.51}
$$

It is well-known that the boundary value problem (2.44), (2.45), (2.46) has the unique solution

$$
p(t,x) = \theta e^{\alpha T} P\big[W^x(s) \in G \ \text{ for all } \ s \in [t,T]\big]\ , \tag{2.52}
$$

where $W^x(\cdot)$ denotes n-dimensional Brownian motion starting at $x \in \mathbb{R}^n$ with probability law P. (See e.g. [?, Chapter 4] or [?, Chapter 9].)

The function

$$
u \to H(t,x,y,\gamma,v,p,q) = \frac{u^r}{r} + (\alpha y - v)p + \beta y q\ ; \qquad u \geq 0
$$

is maximal when

$$
u = \hat{u}(t,x) = \big(p(t,x)\big)^{\frac{1}{r-1}}, \tag{2.53}
$$

where $p(t,x)$ is given by (2.47).

With this choice of $\hat{u}(t,x)$ we see that all the conditions of Theorem 2.1 are satisfied and we conclude that $\hat{u}(t,x)$ is an optimal harvesting rate.

3 Application to optimal stochastic control with partial observation

In the case of a linear system with a quadratic payoff it is well-known that the stochastic control problem with (linear) partial observation splits into a filtering problem and a stochastic control problem with complete observation. See e.g. [?], [?] and [?]. Some classes of explicitly solvable nonlinear stochastic control problems with partial observation are given in [?]. A recent survey of utility maximization with partial observation can be found in [?].

We now briefly review the well-known connection between optimal control of SPDEs (with complete observation) and optimal control of SDEs with partial observation. We refer to [?] and [?] and the references therein for more information.

Suppose the *state process* $x(t) = x^{(u)}(t) \in \mathbb{R}^n$ of a system and the *observation process* $z(t) \in \mathbb{R}^m$ at time t is given by stochastic differential equations of the form

$$(3.1)\qquad \begin{cases} dx(t) = \alpha(x(t), z(t), u(t))dt + \beta(x(t), z(t), u(t))dv(t)\ ; & t \geq 0 \\ x(0) \ \text{ has density } F(x), \\ \text{i.e. } E[\phi(x(0))] = \int_{\mathbb{R}^n} \phi(x)F(x)dx \text{ for all bounded continuous } \phi \text{ on } \mathbb{R}^n \end{cases}$$

$$(3.2)\qquad \begin{cases} dz(t) = h(x(t))dt + dw(t)\ ; & t \geq 0 \\ z(0) = 0 \end{cases}$$

where $\alpha : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$, $\beta : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^{n \times n}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are given continuous functions, $v(t) = v(t, \omega)$ and $w(t) = w(t, \omega)$; $\omega \in \Omega$ are independent Brownian motions in $\mathbb{R}^n, \mathbb{R}^m$ respectively, on a filtered probability space $(\Omega, \mathcal{F}_t, P)$, $u(t) = u(t, \omega)$ is our control process with values in a closed convex set $U \subset \mathbb{R}^k$. We assume that $u(t, \omega)$ is adapted to the filtration $\mathcal{Z}_t$ generated by the observation process $z(t) \in \mathbb{R}^m$. If the corresponding equation (3.1) has a unique solution $x(t) = x^{(u)}(t)$ with values in a given set $V \subset \mathbb{R}^n$ and with the property that $E\big[\exp\big(\frac{1}{2}\int_0^T h^2(x(t))dt\big)\big] < \infty$ and $z(t)$ given by (3.2) exists, we say that u is *admissible* and write $u \in \mathcal{A}$.

We assume the *performance* of $u \in \mathcal{A}$ is given by

$$(3.3)\qquad J(u) = E\Big[\int_0^T \ell(x(t), z(t), u(t))dt + k(x(T), z(T))\Big]$$

where $\ell : V \times \mathbb{R}^m \times U \to \mathbb{R}$ and $k : V \times \mathbb{R}^m \to \mathbb{R}$ are given lower bounded C^1 functions. We want to find $J^* \in \mathbb{R}$ and $u^* \in \mathcal{A}$ such that

$$(3.4)\qquad J^* = \sup_{u \in \mathcal{A}} J(u) = J(u^*)\ .$$

We now describe how this problem can be transformed into a problem of the type considered in Section 2. To this end, we recall some fundamental results from filtering theory. For proofs and more details we refer to e.g. [?], [?], [?], [?] and [?].

Define the probability measure Q on $\mathcal{F}_T$ by

$$(3.5)\qquad dQ(\omega) = M_T(\omega)dP(\omega) \qquad \text{on } \ \mathcal{F}_T$$

where

$$\begin{aligned} M_t(\omega) &= \exp\Big(-\int_0^t h(x(s))dw(s) - \tfrac{1}{2}\int_0^t h^2(x(s))ds\Big) \\ (3.6)\qquad &= \exp\Big(-\int_0^t h(x(s))dz(s) + \tfrac{1}{2}\int_0^t h^2(x(s))ds\Big); \qquad t \geq 0 \end{aligned}$$

The process M_t is a martingale with respect to P and therefore

$$dQ(\omega) = M_t(\omega)dP(\omega) \qquad \text{on } \ \mathcal{F}_t; \quad t \in [0, T]\ .$$

By the Girsanov theorem $z(t)$ is a Brownian motion with respect to Q. Therefore

$$(3.7)\qquad N_t(\omega) := M_t^{-1}(\omega) = \exp\Big(\int_0^t h(x(s))dz(s) - \tfrac{1}{2}\int_0^t h^2(x(s))ds\Big)$$

is a martingale with respect to Q and we have

$$dP(\omega) = N_t(\omega)dQ(\omega) \qquad \text{on } \mathcal{F}_t \text{ for all } t \in [0,T] \ . \tag{3.8}$$

Define the differential operator $A = A_{z,u}$ by

$$\begin{aligned} A\phi(x) &= A_{z,u}\phi(x) \\ &= \sum_{i=1}^{n} \alpha_i(x,z,u)\frac{\partial \phi}{\partial x_i}(x) + \tfrac{1}{2}\sum_{i,j=1}^{n} \big(\beta\beta^T\big)_{ij}(x,z,u)\frac{\partial^2 \phi}{\partial x_i \partial x_j} \ ; \qquad \phi \in C_0^2(\mathbb{R}^n) \end{aligned} \tag{3.9}$$

i.e., $A_{z,u}$ is the generator of $x(t)$ if we regard z and u as known constants. Let $a = a(x,z,u) = [a_{ij}]_{1\le i,j\le n}$ be the $n \times n$ matrix given by

$$a_{ij}(x,z,u) = \tfrac{1}{2}\big(\beta\beta^T\big)_{ij}(x,z,u) \ ; \qquad 1 \le i,j \le n \ . \tag{3.10}$$

Then we can write

$$A\phi = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{n} a_{ij}\frac{\partial \phi}{\partial x_j}\Big) - \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_i} \cdot \frac{\partial \phi}{\partial x_j} + \sum_{i=1}^{n} \alpha_i \frac{\partial \phi}{\partial x_i} \ . \tag{3.11}$$

Therefore the adjoint A^* of the differential operator A is given by

$$A^*\phi = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{n} a_{ij}\frac{\partial \phi}{\partial x_j}\Big) + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\Big(\frac{\partial a_{ij}}{\partial x_i}\phi\Big) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\alpha_i \phi) \ . \tag{3.12}$$

Let $y(t,x) \in \mathbb{R}$ be the solution of the following stochastic partial differential equation (the Duncan-Mortensen-Zakai equation)

$$\begin{cases} dy(t,x) = A^*_{z(t),u(t)}y(t,x)dt + y(t,x)h(x)dz(t) \ ; & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ y(0,x) = F(x) \end{cases} \tag{3.13}$$

where $F(x)$ is given by (3.1).

Suppose that $A_{z,u}$ (and hence $A^*_{z,u}$) is *uniformly elliptic*, i.e. there exists $\lambda > 0$ such that

$$\zeta^T a(x,z,u)\zeta \ge \lambda|\zeta|^2 \tag{3.14}$$

for all $\zeta \in \mathbb{R}^n$, $x \in V$, $z \in \mathbb{R}^m$ and $u \in U$.

Then $y(t,x)$ is the *unnormalized conditional density of* $x(t)$ *given* $\mathcal{Z}_t$, i.e. for all bounded continuous functions $\phi : \mathbb{R}^n \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} \phi(x)y(t,x)dx = E_Q[\phi(x(t))N_t \mid \mathcal{Z}_t] \ , \tag{3.15}$$

where E_Q denotes expectation with respect to Q.

We now use this to rewrite the performance functional $J(u)$ in (3.3) as follows:

$$\begin{aligned}
J(u) &= E\Big[\int_0^T \ell(x(t),z(t),u(t))dt + k(x(T),z(T))\Big] \\
&= E_Q\Big[\Big\{\int_0^T \ell(x(t),z(t),u(t))dt + k(x(T),z(T))\Big\}N_T\Big] \\
&= E_Q\Big[\int_0^T \ell(x(t),z(t),u(t))N_t dt + k(x(T),z(T))N_T\Big] \\
&= E_Q\Big[\int_0^T E_Q\big[\ell(x(t),z(t),u(t))N_t \mid \mathcal{Z}_t\big]dt + E_Q\big[k(x(T),z(T))N_T \mid \mathcal{Z}_T\big]\Big] \\
&= E_Q\Big[\int_0^T E_Q\big[\ell(x(t),z,v)N_t \mid \mathcal{Z}_t\big]_{z=z(t),u=u(t)}dt + E_Q[k(x(T),z)N_T \mid \mathcal{Z}_T]_{z=z(T)}\Big] \\
&= E_Q\Big[\int_0^T \Big(\int_{\mathbb{R}^n} \ell(x,z,v)y(t,x)dx\Big)_{z=z(t),u=u(t)}dt + \int_{\mathbb{R}^n} k(x,z)_{z=z(T)}\, y(T,x)dx\Big] \\
&= E_Q\Big[\int_0^T \Big(\int_{\mathbb{R}^n} \ell(x,z(t),u(t))y(t,x)dx\Big)dt + \int_{\mathbb{R}^n} k(x,z(T))y(T,x)dx\Big]. \qquad (3.16)
\end{aligned}$$

Since $z(t)$ is a Brownian motion with respect to Q, we see that the partial observation control problem (3.1)–(3.4) is equivalent to the following complete observation SPDE control problem:

$$\sup_u E_Q\Big[\int_0^T \Big(\int_{\mathbb{R}^n} \ell(x,z(t),u(t))y(t,x)dx\Big)dt + \int_{\mathbb{R}^n} k(x,z(T))y(T,x)dx\Big] \qquad (3.17)$$

where $y(t,x)$ is the solution of the SPDE (3.13) and the sup is taken over all controls $u(t) \in U$ which are adapted to the filtration $\mathcal{Z}_t$ of the Q-Brownian motion $z(t)$.

This is a problem of the type discussed in Section 2, Theorem 2.3.

Therefore, combining the above transformation with Theorem 2.3 we obtain the following:

Consider the partial observation problem (3.1)–(3.4). Assume that (3.14) holds and that

$$a(x,z,u) = a(x) \quad \text{does not depend on } z \text{ and } u\ . \qquad (3.18)$$

Let $\mathcal{A}_1$ denote the set of $u \in \mathcal{A}$ such that (3.13) has a unique solution $y(t,x) = y^{(u)}(t,x)$. Define the *partial observation control Hamiltonian*

$$H_1 : V \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times U \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

by

$$H_1(x,y,\gamma,z,u,p,q) = \ell(x,z,u)y + b(x,y,\gamma,z,u)p_1 + h(x)y\,q_1 + q_2\ , \qquad (3.19)$$

where

$$\begin{aligned}
b(x,y,\gamma,z,u) &= \Big[\sum_{i,j=1}^n \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}(x) - \sum_{i=1}^n \frac{\partial \alpha_i}{\partial x_i}(x,z,u)\Big]y \\
&\quad + \sum_{i=1}^n \Big\{\sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_i}(x) - \alpha_i(x,z,u)\Big\}\gamma_i \qquad (3.20)
\end{aligned}$$

with $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$, $p = (p_1, p_2) \in \mathbb{R}^2$ and $q = (q_1, q_2) \in \mathbb{R}^2$.

For each $u \in \mathcal{A}_1$ consider the following adjoint backward SPDE in the unknown $\mathcal{Z}_t$-adapted processes $p(t,x) = (p_1(t,x), p_2((t,x)) \in \mathbb{R}^2$ and $q(t,x) = (q_1(t,x), q_2(t,x)) \in \mathbb{R}^2$:

$$\tag{3.21} \begin{cases} dp_1(t,x) = -\big\{\big(\frac{\partial H_1}{\partial y}\big)(x, y(t,x), \nabla y(t,x), z(t), u(t), p(t,x), q(t,x)) \\ \qquad\qquad + Lp_1(t,x)\big\}dt + q_1(t,x)dz(t)\ ; \qquad (t,x) \in (0,T) \times V \\ dp_2(t,x) = -\frac{\partial H_1}{\partial z}(x, y(t,x), \nabla y(t,x), z(t), u(t), p(t,x), q(t,x))dt \\ \qquad\qquad + q_2(t,x)dz(t)\ ; \qquad (t,x) \in (0,T) \times V \\ p_1(T,x) = k(x, z(T))\ ; \qquad x \in V \\ p_2(T,x) = \frac{\partial k}{\partial z}(x, z(T))y(T,x)\ ; \qquad x \in V \\ p_1(t,x) = p_2(t,x) = 0 \qquad \text{for } (t,x) \in (0,T) \times \partial V\ , \end{cases}$$

where L is given by (1.4), with $a(x)$ as in (3.10).

Theorem 3.1 (Sufficient maximum principle for partial observation control) *Let $\hat{u} \in \mathcal{A}_1$ with corresponding solution $\hat{y}(t,x)$ of (3.13) and let $\hat{p}(t,x), \hat{q}(t,x)$ be a solution of the associated adjoint backward SPDE (3.20). Suppose that (3.14), (3.18) holds and that the following, (3.22) and (3.23) hold:*

(3.22) *The functions*

$$(y,z,u) \to H_1(y,z,u) := H_1(x, y, \nabla\hat{y}(t,x), z, u, \hat{p}(t,x), \hat{q}(t,x));$$
$$y \in \mathbb{R},\ z \in \mathbb{R}^m,\ u \in U$$

and

$$(y,z) \to k(x,z)y\ ;\ y \in \mathbb{R},\ z \in \mathbb{R}^m \ \textit{are } \text{concave}\textit{, for all } (t,x) \in (0,T) \times V$$

$$\tag{3.23} \int_V H_1(x, \hat{y}(t,x), \nabla\hat{y}(t,x), z(t), \hat{u}(t), \hat{p}(t,x), \hat{q}(t,x))dx = \sup_{u \in U} \int_V H_1(x, \hat{y}(t,x), \nabla\hat{y}(t,x), z(t), u, \hat{p}(t,x), \hat{q}(t,x))dx\ .$$

Then $\hat{u}(t)$ is an optimal control for the partial observation control problem (3.1)–(3.4).

Proof. This follows by the transformation (3.16) of the partial observation control problem (3.1)–(3.4) to a complete observation SPDE control of the type discussed in Theorem 2.3, applied to the 2-dimensional system $(y(t,x), z(t))$ given by

$$\begin{cases} dy(t,x) = A^*_{z(t),u(t)}y(t,x)dt + y(t,x)h(x)dz(t) \\ dz(t) = dz(t) \end{cases}$$

For this a slight extension of Theorem 2.1 to the case with 2-dimensional state $Y(t,x) = (y(t,x), z(t))$ is needed. We omit the details. □

If we in addition assume that

$$\tag{3.24} \alpha(x,z,u) = \alpha(x,u),\ \ell(x,z,u) = \ell(x,u) \quad \text{and} \quad k(x,z) = k(x) \ \text{ do not depend on } z$$

then we can choose $p_2(t,x) = q_2(t,x) = 0$ in the adjoint equation (3.21), so that it simplifies to, writing $p_1(t,x) = p(t,x)$ and $q_1(t,x) = q(t,x)$,

$$\tag{3.25} \begin{cases} dp(t,x) = -\big\{\big(\frac{\partial H_1}{\partial y}\big)(x, y(t,x), \nabla y(t,x), u(t), p(t,x), q(t,x)) + Lp(t,x)\big\}dt \\ \qquad\qquad +q(t,x)dz(t)\ ; \qquad (t,x) \in (0,T) \times V \\ p(T,x) = k(x)\ ; \qquad x \in V \\ p(t,x) = 0\ ; \qquad (t,x) \in (0,T) \times \partial V \end{cases}$$

where

$$\tag{3.26} H_1(x,y,\gamma,u,p,q) = \ell(x,u)y + b(x,y,\gamma,u)p + h(x)y\,q$$

with

$$\tag{3.27} \begin{aligned} b(x,y,\gamma,u) &= \Big[\sum_{i,j=1}^{n} \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}(x) - \sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial x_i}(x,u)\Big]y \\ &\quad + \sum_{i=1}^{n} \Big\{\sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_i}(x) - \alpha_i(x,u)\Big\}\gamma_i\ , \\ \gamma &= (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n\ . \end{aligned}$$

Hence Theorem 3.1 simplifies to

Corollary 3.2 *Let $\hat{u} \in \mathcal{A}_1$ with corresponding solution $\hat{y}(t,x)$ of (3.13) and let $\hat{p}(t,x), \hat{q}(t,x)$ be a solution of the associated adjoint backward SPDE (3.25), with H_1 as in (3.26)–(3.27). Suppose that (3.14), (3.18) and (3.24) hold, in addition to the following, (3.28) and (3.29):*

(3.28) *The function*

$$(y,u) \to H_1(y,u) := H_1(x,y,\nabla\hat{y}(t,x),u,\hat{p}(t,x),\hat{q}(t,x)); \quad (y,u) \in \mathbb{R} \times U$$

is concave for all $(t,x) \in (0,T) \times V$

$$\tag{3.29} \begin{aligned} &\int_V H_1(x,\hat{y}(t,x), \nabla\hat{y}(t,x), \hat{u}(t), \hat{p}(t,x), \hat{q}(t,x))dx \\ &\qquad = \sup_{u \in U} \int_V H_1(x,\hat{y}(t,x), \nabla\hat{y}(t,x), u, \hat{p}(t,x), \hat{q}(t,x))dx\ . \end{aligned}$$

Then $\hat{u}(t)$ is an optimal control for the partial observation control problem (3.1)–(3.4).

Example 3.3 (Optimal consumption with partial observation)
Suppose the wealth $x(t)$ at time t is given by

$$\tag{3.30} dx(t) = [\mu x(t) - u(t)]dt + \sigma x(t)dv(t)\ ; \qquad t > 0$$

$$\tag{3.31} x(0) \quad \text{has density} \quad F(x)\ .$$

Here $u(t) \geq 0$ is our consumption rate. We require $x(t) \geq 0$ for all $t \in [0;T]$ for $u(\cdot)$ to be admissible. Moreover, we require that $u(\cdot)$ be adapted to the filtration $\mathcal{Z}_t$ of the observation process $z(t)$ given by

$$\tag{3.32} dz(z) = x(t)dt + dw(t)\ ; \qquad t > 0$$

$$\tag{3.33} z(0) = 0\ .$$

Suppose we want to maximize the total expected utility of the consumption and the terminal wealth, represented by

$$J(u) = E\Big[\int_0^T \frac{u^r(t)}{r}dt + \theta x^r(T)\Big], \tag{3.34}$$

where $r \in (0,1)$ and $\theta > 0$ are given constants. Here the operator $A_{z,u}$ in (3.9) has the form

$$A\phi(x) = \tfrac{1}{2}\sigma^2 x^2 \phi''(x) + (\mu x - u)\phi' = \tfrac{1}{2}\sigma^2 (x^2\phi'(x))' + [x(\mu - \sigma^2) - u]\phi'(x) \tag{3.35}$$

(where $(\ )'$ denotes differentiation with respect to x) and hence

$$A^*_{z,u}\phi(x) = \tfrac{1}{2}\sigma^2(x^2\phi'(x))' - [(x(\mu-\sigma^2) - u)\phi(x)]'$$

$$= \tfrac{1}{2}\sigma^2(x^2\phi'(x))' + (x(\sigma^2-\mu) + u)\phi'(x) + (\sigma^2-\mu)\phi(x) \tag{3.36}$$

$$= \tfrac{1}{2}\sigma^2 x^2\phi''(x) + [x(2\sigma^2-\mu) + u]\phi'(x) + (\sigma^2-\mu)\phi(x)\ . \tag{3.37}$$

Therefore the equation (3.13) for the unnormalized conditional density $y(t,x)$ is

$$dy(t,x) = \big[\tfrac{1}{2}\sigma^2x^2y''(t,x) + [x(2\sigma^2-\mu) + u(t)]y'(t,x) + (\sigma^2-\mu)y(t,x)\big]dt \\ + xy(t,x)dz(t)\ ; \qquad (t,x) \in (0,T)\times(0,\infty) \tag{3.38}$$

$$y(t,x) = F(x)\ ; \qquad x > 0 \tag{3.39}$$

$$y(t,0) = 0\ ; \qquad t \in (0,T)\ . \tag{3.40}$$

The Hamiltonian H_1 of (3.26) gets the form

$$H_1(x,y,\gamma,u,p,q) = \frac{u^r}{r}y + \big[(\sigma^2-\mu)(y+x\gamma) + u\gamma\big]p + xyq\ . \tag{3.41}$$

Hence

$$\frac{\partial H_1}{\partial y} = \frac{u^r}{r} + (\sigma^2-\mu)p + xq$$

and the adjoint equation (3.25) becomes

$$dp(t,x) = -\Big[\frac{u^r(t)}{r} + (\sigma^2-\mu)p(t,x) + xq(t,x) + \tfrac{1}{2}\sigma^2(x^2y'(t,x))'\Big]dt \\ + q(t,x)dz(t)\ ; \qquad (t,x)\in(0,T)\times(0,\infty) \tag{3.42}$$

$$p(T,x) = \theta x^r\ ; \qquad x > 0 \tag{3.43}$$

$$p(t,0) = 0\ ; \qquad t \in (0,T)\ . \tag{3.44}$$

We now use (3.29) to find a candidate for $\hat{u}(t)$:

$$\int_V H_1(x,y(t,x),y'(t,x),u,p(t,x),q(t,x))dx$$

$$= \int_0^\infty \Big\{\frac{u^r(t)}{r}y(t,x) + u\cdot y'(t,x)p(t,x)\Big\}dx + \quad \text{terms not containing } u.$$

This is maximal when

$$u = u^*(t) = \left[\frac{-\int_0^\infty y'(t,x)p(t,x)dx}{\int_0^\infty y(t,x)dx}\right]^{\frac{1}{r-1}}\ . \tag{3.45}$$

Using integration by parts and (3.15) we can rewrite this as

$$\begin{aligned} u = u^*(t) &= \left[\frac{-\int_0^\infty p'(t,x)y(t,x)dx}{\int_0^\infty y(t,x)dx}\right]^{\frac{1}{r-1}} = \left[\frac{E_Q[p'(t,x(t))N_t|\mathcal{Z}_t]}{E_Q[N_t|\mathcal{Z}_t]}\right]^{\frac{1}{r-1}} \\ &= E\big[p'(t,x(t))|\mathcal{Z}_t\big]\ . \end{aligned} \tag{3.46}$$

This is our candidate for the optimal consumption rate $u^*(t)$. However, it remains to find $p(t,x)$ and to verify that $\hat{u}(t) := u^*(t)$ satisfies all the conditions of Corollary 3.2.

For general existence and uniqueness results for backward stochastic partial differential equations see [**?**]

Acknowledgments I am grateful to Mark Davis and Agnès Sulem for helpful comments.

References

[BEK] J. S. Baras, R. J. Elliott and M. Kohlmann: The partialy observed stochastic minimum principle. SIAM J. Control & Optim. **27** (1989), 1279–1292.

[B1] A. Bensoussan: Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions. Stochastics **9** (1983), 169–222.

[B2] A. Bensoussan: Stochastic maximum principle for systems with partial information and application to the separation principle. In M. Davis and R. Elliott (editors): Applied Stochastic Analysis. Gordon & Breach 1991, pp. 157–172.

[B3] A. Bensoussan: Stochastic Control of Partially Observable Systems. Cambridge University Press 1992.

[CE] C. D. Charalambous and R. E. Elliott: Classes of nonlinear partially observable stochastic optimal control problems with explicit optimal control laws. SIAM J. Control Optim. **36** (1998), 542–578.

[D1] M. H. A. Davis: Linear Estimation and Stochastic Control. Chapman & Hall 1977.

[D2] M. H. A. Davis: Lectures on Stochastic Control and Nonlinear Filtering. Tata Institute of Fundamental Research, Bombay 1984.

[DM] M. H. A. Davis and I. Markus: An introduction to nonlinear filtering. In M. Hazewinkel and J.C. Willems (editors): Stochastic Systems: The Mathematics of Filtering and Identification and Applications. D. Reidel Publ. Co 1981, pp. 53–75.

[H] U. G. Haussmann: The maximum principle for optimal control of diffusions with partial information. SIAM J. Control & Optim. **25** (1987), 341–361.

[K] G. Kallianpur: Stochastic Filtering Theory. Springer-Verlag 1980.

[KS] I. Karatzas and S.E. Shreve: Brownian Motion and Stochastic Calculus. Second Edition. Springer-Verlag 1991.

[La] B. Larssen: The partially observed stochastic linear quadratic regulator: A direct approach. Manuscript 2000.

[Le] D. Lefèvre: An introduction to utility maximization with partial observation. Rapport de Recherche 4183, INRIA, May 2001.

[LL] P.-L. Lions and J.-M. Lasry: Stochastic control under partial information and applications to finance. CEREMADE (UMR 7534), Number 9912, 10/03/99.

[M] R. E. Mortensen: Stochastic optimal control with noisy observations. Int. J. Control **4** (1966), 455–464.

[Ø] B. Øksendal: Stochastic Differential Equations. Fifth Edition. Springer-Verlag 2000.

[ØZ] B. Øksendal and T. Zhang: On backward stochastic partial differential equations. Manuscript 2001.

[P1] E. Pardoux: Stochastic partial differential equations and filtering of diffusion processes. Stochastics 3 (1979), 127–167.

[P2] E. Pardoux: Filtrage non lineaire et équations aux derivées partielles stochastiques associées. Ecole d'Été de Probabilités de Saint-Flour 1989.

[R] R. T. Rockafellar: Convex Analysis. Princeton University Press 1970.

[SS] A. Seierstad and K. Sydsæter: Optimal Control Theory with Economic Applications. North-Holland 1987.

[YZ] J. Yong and X. Y. Zhou: Stochastic Controls. Springer-Verlag 1999.