AFFINE TERM STRUCTURE MODELS

C. GOURIEROUX [(1)], MONFORT A. [(2)] and V., POLIMENIS [(3)]

(March 2002)

Acknowledgement : we thank D. Duffie, P. Gagliardini and L. Hansen for helpful comments.

[1]CREST, CEPREMAP and University of Toronto.
[2]CNAM and CREST.
[3]University of California, Riverside.

Affine Term Structure Models

Abstract

This paper gives a general presentation of affine term structure in discrete time. By assuming a compound autoregressive (affine) dynamics of the state variables and an exponential affine stochastic discount factor, it is possible to derive the risk neutral distribution and to check that the term structure is affine. We discuss several examples of one or multifactor models of this type, which can be with discrete or continuous state spaces, parametric or nonparametric. We provide the derivative pricing formulas and discuss the implementation.

Keywords : Interest Rate, Term Structure, Compound Autoregressive Process, Affine Process, Laplace Transform, Nonparametric Pricing.

Modèles Affines de Structure par Terme

Résumé

Ce papier fournit une présentation générale en temps discret des modèles affines de structure par terme. Sous l'hypothèse de variables d'états suivant un processus autorégressif composé et en supposant un facteur d'escompte stochastique exponentiel affine, il est possible de dériver la distribution risque-neutre et de vérifier que la structure par terme est affine. Nous discutons plusieurs modèles de ce type à un ou plusieurs facteurs, à espace d'états discret ou continu, paramétriques ou non paramétriques.

Mots clés : Taux d'intérêt, structure par terme, processus autorégressif composé, processus affine, transformée de Laplace, valorisation non paramétrique.

JEL Number : G13, C51

1. Introduction

Since the publication of the paper by Vasicek (1977), there is a large literature linking the term structure of interest rates to the historical dynamics of the short rate, and more generally pricing derivatives written on this rate. These models differ by the set of admissible values of the prices of zero-coupon bonds, which can be discrete as in Ho and Lee model [Ho, Lee (1986)] or continuous as in the major part of the literature. They also differ by the interpretation of the factors which have been introduced, and by their numbers. These factors can be bond prices [Ball, Torous (1983), Schaefer, Schwartz (1987)], forward rates [Ho, Lee (1986), Heath, Jarrow, Morton (1990), (1992)], external state variables, especially for the so-called equilibrium models [4] or stochastic parameters [5]. They can include one factor, [6] two factors, or more. But the major part of all these approches leads to an affine term structure, in which the interest rates with different maturities satisfy affine relationship [Duffie, Kan (1996)].

The aim of this paper is to develop a general approach of affine term structure in discrete time (see Duffie, Filipovic, Schachermayer (2001) for continuous time and Duffie, Pan, Singleton (2000) for the inclusion of jump processes). Discrete time implies an incomplete market framework and a multiplicity of admissible pricing formulas. The multiplicity problem is solved by imposing a special structure of the stochastic discount factor (sdf), which summarizes both the time discounting and the risk correction. The sdf is assumed to be an exponential-affine function of the future short term interest rate and of other factors. In section 2, we assume a compound autoregressive specification (CAR)(or affine process) for the state variables[Darolles, Gourieroux, Jasiak (2001), Polimenis (2001)], and derive the exponential-affine sdf satisfying the arbitrage free restrictions. Then we explain how to compute recursively the price of zero-coupon bonds and we derive an affine term structure. In section 3 we discuss the multiplicity of state variables

[4] see e.g. [Merton (1974), Long (1974), Vasicek (1977), Dothan (1978), Brennan, Schwartz (1979), Langetieg (1980), Cox, Ingersoll, Ross (1985), Longstaff, Schwartz (1992)].

[5] Fong, Vasicek (1991), Chen (1996), Balduzzi et alii (1996)

[6] see [Vasicek (1977), Cox, Ingersoll, Ross (1985), Longstaff, Schwartz (1992)] for one factor models, [Brennan, Schwartz (1979), Longstaff, Schwartz (1992), Balduzzi et alii (1998)] for two factor models and [Langetieg (1980), Heath, Jarrow, Morton (1992), Duffie, Kan (1996) Dai, Singleton (2000)] for multifactor models.

which can be introduced, and in particular the partition of the set of state variables into components which can be interpreted as interest rates and components without this interpretation. This discussion shows that these different state variables all follow compound autoregressive processes, but the dynamics of the process is generally more constrained when the number of rate type components is larger. We finally focus on the extreme case of mimicking factors, where all components are interest rates and show that the dynamics of interest rate is compound autoregressive both in the historical and risk neutral worlds. Derivative pricing formulas are derived in section 4 following the approach of Duffie, Pan, Singleton (2000). The aim of section 5 it to show that the set of discrete time CAR (affine) processes is much larger than the set of discretely sampled continuous affine processes. We present a variety of examples, compute the associated term structure, discuss the behaviour of the long term interest rate and compare the stationarity conditions in the historical and risk neutral world. Section 6 is concerned by statistical inference. Beside the standard parametric methods developed by Singleton (2001), we focus on the direct modelling approach. In this approach the CAR dynamics is directly introduced on the interest rates themselves. We explain how to incorporate the arbitrage free restrictions in the estimation approach. Moreover the method is extended to the nonparametric framework.

2 The model

The model is written in discrete time and defined in three steps. First we introduce the historical distribution of the process of state variables. Then we specify a class of stochastic discount factors (s.d.f.) [see Harrison, Kreps (1979), Hansen, Richard (1987)], which explain how to pass from the historical dynamics to the state prices. Finally we impose on the s.d.f. the arbitrage free constraints. This s.d.f. approach is now standard in discrete time framework [see e.g. Cochrane (2001), Gourieroux, Jasiak (2001)a for general presentation].

2.1 The historical distribution

As usual in the literature we first introduce a n-dimensional state variable X_t. Some of its components can be interpreted as returns (they are said endogenous), whereas the other ones do not admit a priori this interpretation (they are said exogenous). The distinction between these types of components is important due to arbitrage free restrictions. Indeed with each return state variable is associated an arbitrage free (equilibrium) constraint, which concerns jointly the historical dynamics and the sdf. This feature will be discussed in detail in section 3. In the section below we assume that the first component is the (short term) riskfree rate r_{t+1} predetermined at time t and that the other component f_t is exogenous : $X_t = (r_{t+1}, f_t')'$, where denotes transpose.

The historical dynamics of the state variable is fully described by means of the Laplace transform (also called moment generating function) of its conditional distribution. The conditional Laplace transform can be written either in terms of complex, or real arguments. It is important to first discuss the relations between these functions, since different restrictions of the Laplace transform are used for pricing and estimation purposes. Let us denote by w a real vector and by z a vector with complex components. The conditional Laplace transform [resp. real Laplace transform, Fourier transform [7]] associates to a complex argument z [resp. a real argument w, a pure imaginary argument iw] the value $\psi(z, \underline{X_t}) = E_t(\exp z'X_{t+1})$, where the conditional expectation is taken with respect to the current and lagged values of the factors $\underline{X_t} = (X_t, X_{t-1}, \ldots)$ (resp. $\psi(w, \underline{X_t}) = E_t(\exp w'X_{t+1}), \psi(iw, \underline{X_t}) = E_t(\exp iw'X_{t+1})$). The historical dynamics is restricted by means of the conditional Laplace transform.

Definition 1 : The state process is a compound autoregressive (CAR) process or an affine process if

i) the conditional Laplace transform is analytical in a neighbourhood of $z = 0$;[8]

ii) the log of the conditional Laplace transform is an affine function of the lagged value of the state variable :

$$E[\exp z'X_{t+1}|\underline{X_t}] = \exp[a'(z)X_t + b(z)], \text{ for all } z \text{ (say)}.$$

The following standard properties are useful to avoid the use of complex arguments and to get interpretations in terms of moments.

Proposition 1 : The state process is a compound autoregressive (affine)

[7] or characteristic function.

[8] Note that the conditional Laplace transform depends on the value of the conditioning variable and that the neighbourhood is uniform with respect to this value.

process iff :

i) the conditional real Laplace transform admits series expansion in a neighbourhood of $w = 0$;

ii) $E(\exp w'\underline{X}_{t+1}|X_t) = \exp[a'(w)X_t + b(w)]$, for all w,

Proposition 2 : For a CAR (affine) process the conditional real Laplace transform characterizes the conditional distribution. This distribution admits conditional moments at any order and the conditional cumulants are affine functions of X_t. [9]

This dynamic specification corresponds to a compound autoregressive model, as studied in [Darolles, Gourieroux, Jasiak (2001)] and extends the standard gaussian vector autoregressive model. This is a discrete time analogue of the continuous time affine process introduced in Duffie, Filipovic, Schachermayer (2001). The process $X_t = (r_{t+1}, f_t')'$ is Markovian ; it is stationary if $\lim_{h\to\infty} a^{oh} = 0$, where a^{oh} denotes function a compounded h times [Darolles, Gourieroux, Jasiak (2001)]. [10]

The functions a and b cannot be chosen arbitrarily, but are constrained by the interpretation of the Laplace transform [see e.g. Joe (1997) p. 373 for a discussion of complete monotonicity]. They are also constrained by the set of values that take the variables r_{t+1}, f_t. For instance let us assume that the short term interest rate is nonnegative with an admissible value zero[11] and denote the conditional real Laplace transform by

[9] An affine process has often been defined by restricting the conditional mean and variance to be affine (see e.g. Piazzesi (2001)). The two definitions coincide for diffusion models, but in more general framework the drift-volatility definition has to be extended to moments of larger order.

[10] $a^{oh}(z) = \overbrace{a(a(...(a(z))}^{h \text{ times}} = \underbrace{aoa\dots oa(z)}_{h \text{ times}}.$

[11] In the rest of paper, we impose the nonnegativity of the short term interest rate which is especially important for nominal interest rates. If this assumption is relaxed, the gaussian vector autoregressive model can be used and, since it is compound autoregressive, the whole applies. Thus the results can be used to derive the term structure exhibited by Vasicek (1977) in the one factor case, and Langetieg (1980) in the multifactor case. They can also be used to study models mixing Ornstein-Uhlenbeck and CIR factor processes, for instance.

$$\begin{aligned} & E_t(\exp w'X_{t+1}) \\ = \; & E_t \exp\left[u r_{t+2} + v' f_{t+1}\right] \\ = \; & \exp\left[a(u,v)' \begin{pmatrix} r_{t+1} \\ f_t \end{pmatrix} + b(u,v)\right], \forall u, v. \end{aligned} \tag{2.1}$$

Then the conditional real Laplace transform is nondecreasing with respect to the argument u. This implies that:

$$u \to a_1(u,v) r_{t+1} + a_2(u,v)' f_t + b(u,v),$$

is nondecreasing for any values of v, r_{t+1}, f_t. This condition depends on the signs of the factors:

Condition $C1$: If r_{t+1} and the exogenous components are nonnegative with admissible values zero, the functions a_1, a_2, b are nondecreasing with respect to u.

Condition $C2$: If r_{t+1} is nonnegative with admissible value zero and the exogenous components can take any value, then a_1 and b are nondecreasing with respect to u, whereas a_2 does not depend on u.

When the exogenous components can take any value the conditional distribution of r_{t+2} given r_{t+1}, f_t admits the real Laplace transform:

$$\exp[a_1(u,0) r_{t+1} + a_2(0) f_t + b(u,0)] = \exp[a_1(u,0) r_{t+1} + b(u,0)],$$

since $a_2(0) = 0$. Therefore the exogenous factor process (f_t) does not cause the interest rate process (r_{t+1}) [Granger (1969)] and intuitively contains no useful information on this process. On the contrary, when the factor process is a priori nonnegative, there is a possible causality from (f_t) to (r_{t+1}).

Finally note that, depending on the pattern of functions a and b, the variables can have discrete or continuous support. This allows to include as special case the discretely sampled Cox, Ingersoll, Ross model with continuous state space [Cox, Ingersoll, Ross (1985)], as well as models based on binomial tree [Ho, Lee (1986)].

2.2 Stochastic discount factor

The pricing model is completed by specifying the stochastic discount factor $M_{t,t+1}$ for the period $(t, t+1)$. The sdf is the basis for pricing any derivative written on the spot short term interest rate.[12] For instance the price at t of a european derivative paying $g(r_{t+h+1})$ at $t+h$ is :

$$\begin{aligned} C_t(g,h) &= E_t\left[M_{t,t+1}\ldots M_{t+h-1,t+h}\,g(r_{t+h+1})\right] \\ &= E_t\left[M_{t,t+h}\,g(r_{t+h+1})\right] \text{ (say)}, \end{aligned} \tag{2.2}$$

where E_t denotes the historical expectation conditional on the information including the current and lagged values of the state variables. In discrete time, we are generally in an incomplete market framework with a multiplicity of admissible sdf. To restrict the set of risk neutral distributions, we select a sdf, which is exponential-affine in the state variables :

$$M_{t,t+1} = \exp(\alpha r_{t+2} + \delta' f_{t+1} + \beta_t), \tag{2.3}$$

where α, δ are path independent risk correction factors and the intercept β_t can depend on the past. This specification of the sdf corresponds to the Esscher transform introduced in insurance [Esscher (1932)] and used in finance by [Buhlman et alii (1996), Stutzer (1996), Shyraev (1999), Gourieroux, Monfort (2001)a,b, Darolles, Gourieroux, Jasiak (2001)]. The exponential affine specification of the sdf can be justified in different ways. First it corresponds to a choice of a risk neutral distribution, which satisfies the arbitrage free restrictions and is the closest to the historical distribution for an information criterion [Stutzer (1996)]. Second it is obtained in a general equilibrium framework, when the representative agent has a time separable power utility function and the endowment process depends in an appropriate way of exogenous factors [see e.g. Polimenis (2001)]. Finally it corresponds to the choice of sdf suggested by Hansen, Scheinkman (2002), where the sdf is an exponential affine function of the eigenfunction of the conditional expectation operator associated with the largest eigenvalue. Indeed for CAR processes this eigenfunction is affine [Darolles, Gourieroux, Jasiak (2001)].

[12] and more generally a derivative written on a sequence of yields of different maturities $r_{t+1,t+k}$, say. Then formula (2.2) is valid with a payoff $g(\underline{r_{t+h,t+h+1}}, \underline{r_{t+h,t+h+2}}, \ldots)$, where the bar is introduced for current and lagged values.

2.3 Arbitrage free restrictions

The parameters α, δ, β_t of the sdf cannot be chosen arbitrarily. Indeed, under the assumption of no arbitrage opportunity, the pricing formula (2.2) has to be valid for the zero-coupon bond with unitary residual maturity. The arbitrage free condition is :

$$\exp(-r_{t+1}) = E_t(M_{t,t+1})$$

$$\begin{aligned}
\iff \quad & E_t\left[M_{t,t+1} \exp r_{t+1}\right] = 1 \\
\iff \quad & E_t \exp\left(\alpha r_{t+2} + \delta' f_{t+1} + \beta_t + r_{t+1}\right) = 1 \\
\iff \quad & a(\alpha,\delta)' \begin{pmatrix} r_{t+1} \\ f_t \end{pmatrix} + b(\alpha,\delta) + \beta_t + r_{t+1} = 0, \forall r_{t+1}, f_t. \qquad (2.4)
\end{aligned}$$

The arbitrage free restriction fixes β_t as a linear function of current state variables. By replacing in the expression of $M_{t,t+1}$, the constrained sdf becomes :

$$M_{t,t+1} = \exp\left[\alpha r_{t+2} + \delta' f_{t+1} - a(\alpha,\delta)' \begin{pmatrix} r_{t+1} \\ f_t \end{pmatrix} - b(\alpha,\delta) - r_{t+1}\right]. \qquad (2.5)$$

It is an exponential function of X_t, X_{t+1} :

$$M_{t,t+1} = \exp[\gamma_0 + \gamma_1' X_t + \gamma_2' X_{t+1}], \qquad (2.6)$$

$$\text{where : } \gamma_0 = -b(\alpha,\delta), \gamma_1 = -a(\alpha,\delta) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \gamma_2 = (\alpha, \delta')'. \qquad (2.7)$$

It is important to remark that the incompleteness due to the discretized framework has not been completely suppressed by the exponential affine restriction imposed on the sdf. Indeed the parameters α, δ can still be chosen independently of the historical distributions of the state variables. We say that the dimension of residual incompleteness is equal to the number n of state variables. Finally note that the s.d.f constrained by the arbitrage free restrictions is the same as (2.6), (2.7) if the a priori specifications of $M_{t,t+1}$ is exponential affine in X_t, X_{t+1} : $M_{t,t+1} = \exp[\gamma_o + \gamma_1' X_t + \gamma_2' X_{t+1}]$ with unconstrained γ parameters.

2.4 Factor identifiability

As usual the factors are defined up to an invertible affine function. More precisely, let us consider :

$$f_t^* = Bf_t + br_{t+1}, +b_o,$$

where B is an invertible matrix. It is easily checked that the initial sdf is an exponential-affine function of $(r_{t+2}, f_{t+1}^{*'})'$, and that the process $(r_{t+1}, f_t^{*'})'$ is still a compound autoregressive process. [13]

2.5 Affine term structure

We can easily derive the prices of the zero-coupon bonds from formula (2.2) since :

$$B(t,h) = E_t \left[M_{t,t+1} \dots M_{t+h-1,t+h} \right],$$

where $B(t,h)$ denotes the price at t for residual maturity h.

Proposition 3 : The price at date t of the zero-coupon bond with residual maturity h is :

$$B(t,h) = \exp(c_h' X_t + d_h), h \geq 1,$$

where c_h and d_h satisfy the recursive equations :

$$c_h = a \left[c_{h-1} + \begin{pmatrix} \alpha \\ \delta \end{pmatrix} \right] - a \begin{pmatrix} \alpha \\ \delta \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$d_h = d_{h-1} - b \begin{pmatrix} \alpha \\ \delta \end{pmatrix} + b \left[c_{h-1} + \begin{pmatrix} \alpha \\ \delta \end{pmatrix} \right],$$

[13] In the three factor model this lack of identifiability explains why it is not possible to interpret a priori factors as affecting the level, slope and curvature of the term structure. If such factors with these interpretations exist, they have to be searched among the infinite number of admissible factors.

for $h \geq 2$, with initial conditions : $c_1' = (-1, 0), d_1 = 0$.

Proof : see Appendix 1.

Thus the yields :

$$r_{t,t+h} = -\frac{1}{h} \log B(t, h) = -\frac{c_h'}{h} X_t - \frac{d_h}{h}, \ h \text{ varying}, \tag{2.8}$$

are affine functions of the short term yield and the factors. They generate an affine space [14]. Thus we get an affine term structure of interest rates [see e.g. Duffie, Kan (1996) and the list of references for examples of such models].

Corollary 1 : The term structure of yields is affine.

The dimension of this affine space is equal to the rank of the system $[c_1, c_2, ..., c_h, ...]$ and is always between 1 and the total number n of state variables. To discuss the rank, let us first consider exogenous factor components with any admissible real values. By condition $C2$, the function a_2 depends on v only and by the recursive equation for c_h (see Proposition 3) we find that $c_{2,h} = 0, \forall h$. The dimension of the affine space is equal to one. Since the exogenous factor does not Granger cause the interest rate process, it has no effect on the term structure.

In contrast, factors with nonnegative values generally influence the term structure. To illustrate this point let us consider the case of one additional exogenous factor $n = 2$. We get :

$$\begin{aligned} c_{1,2} &= a_1(\alpha - 1, \delta) - a_1(\alpha, \delta) - 1, \\ c_{2,2} &= a_2(\alpha - 1, \delta) - a_2(\alpha, \delta). \end{aligned} \tag{2.9}$$

The affine space has dimension 2, whenever $a_2(u, v)$ is strictly increasing with respect to u.

Similarly the forward short term interest rates are :

$$f_{t,t+h} \quad = \quad - \log \frac{B(t, h+1)}{B(t, h)}$$

[14] The assumption of path independent risk corrections α, δ is crucial to get this result. Otherwise c_h and d_h may depend on r_{t+1}, f_t and the relation is no longer affine.

$$
\begin{aligned}
&= -(c_{h+1} - c_h)' X_t - (d_{h+1} - d_h) \\
&= -(c_{h+1} - c_h)' X_t + b \begin{pmatrix} \alpha \\ \delta \end{pmatrix} - b \left[c_h + \begin{pmatrix} \alpha \\ \delta \end{pmatrix} \right]. \qquad (2.10)
\end{aligned}
$$

The forward short term interest rates generate generally the same affine space as the yields. Moreover they depend on the residual maturity through c_h only.

Corollary 2 : The term structure of forward rates is affine.

Finally note that the sequences of coefficients c_h, d_h is very easy to compute numerically.

3. Endogenous and exogenous state variables

In the section above we considered a special partition of the state variable into an interest rate r_{t+1} and exogenous factors f_t. However other partitions can be considered a priori. For instance some authors have introduced models in which all state variables correspond to yields of various maturities. The advantage is that the yields are observable and the derivatives are generally written on yields. The drawback is that the dynamics is constrained due to arbitrage free restrictions. On the contrary other authors considered only exogenous state variables. This allows for an unconstrained dynamic model, but requires additional transformations to solve the estimation and pricing problems. The aim of this section is to show that more financial return state components in the partition imply more nonlinear restrictions on the dynamics of the state variable. We will also discuss carefully the dimension of residual incompleteness.

In a second step we explain how the exogenous factors can be replaced by endogenous ones, called mimicking factors.

3.1 Free arbitrage restrictions and constraints on parameters

Let us consider a n-dimensional state variable, which can be partitioned into a H-dimensional subvector of yields and $n - H$ additional exogenous factors : $X_t = \left(\tilde{r}'_{t+1}, f'_t\right)'$. For expository purpose, we assume that $\tilde{r}_{t+1}$ includes the H first yields, and introduce a common a priori specification of the sdf which is exponential affine in X_t, X_{t+1} (see (2.6)).

i) The structural parameters

The state variable follows a compound autoregressive model :

$$E_t[\exp z'X_{t+1}] = \exp[a(z,\theta)'X_t + b(z;\theta)], \tag{3.1}$$

where θ is a p-dimensional vector of parameters.

The sdf is given by :

$$M_{t,t+1} = \exp\left(\gamma_0 + \gamma_1' X_t + \gamma_2' X_{t+1}\right). \tag{3.2}$$

Thus the total number of parameters is $p + 2n + 1$. The parameter θ characterizes the historical distribution of the state variables, whereas the γ parameter concerns the pricing kernel. However these parameters are not necessarily independent.

ii) The arbitrage free restrictions

They correspond to the pricing of zero-coupon bonds :

$$B(t,h) = E_t[M_{t,t+1}B(t+1,h-1)], h \text{ varying}, \tag{3.3}$$

or equivalently in term of yields :

$$\exp(-hr_{t,t+h}) = E_t\left\{M_{t,t+1}\exp[-(h-1)r_{t+1,t+h}]\right\}, h \text{ varying}. \tag{3.4}$$

The first H restrictions involve only components of X_t and will imply constraints on the structural parameters. On the contrary the other arbitrage free restrictions do not imply constraints on parameters, but define the other yields as functions of the state components (as for equation (2.8)). In fact the h^{th} restriction corresponds to an equilibrium condition on the bond market, which justifies the terminology endogenous state variable for $r_{t,t+h}, h = 1, \ldots, H$.

More precisely the structural parameters are constrained by :

$$\exp(-hr_{t,t+h}) = E_t(M_{t,t+1}\exp[-(h-1)r_{t+1,t+h}]), h = 1, \ldots, H.$$

These restrictions are of the type :

$$\exp(\lambda_h' X_t) = E_t[M_{t,t+1} \exp(\mu_h' X_{t+1})], h = 1, \dots H, \text{ (say)},$$

where λ_h, μ_h are given. We deduce :

$$\exp(\lambda_h' X_t) = \exp(\gamma_0 + \gamma_1' X_t) E_t\{\exp[(\mu_h + \gamma_2)' X_{t+1}]\}, h = 1, \dots, H$$

$$\text{or : } \quad (\gamma_1 + a(\mu_h + \gamma_2) - \lambda_h)' X_t + \gamma_0 + b(\mu_h + \gamma_2) = 0, h = 1, \dots, H.$$

These equalities have to be satisfied for any admissible values of X_t, which implies :

$$\gamma_1 + a(\mu_h + \gamma_2) - \lambda_h = 0, \gamma_0 + b(\mu_h + \gamma_2) = 0, h = 1, \dots, H. \qquad (3.5)$$

We get $H(n+1)$ constraints on the structural parameters. Comparing with the number of parameters, we get the following order conditions.

Proposition 4 :

i) If $H = 0$, that is if any state variable is exogenous, the θ and γ parameters are unconstrained. There is a multiplicity of admissible derivative prices, with a dimension of residual incompleteness equal to $2n + 1$. [15]

ii) If $H = 1$, which corresponds to the situation of section 2 [see also Duffie, Pan, Singleton (2000), p 1355], the θ parameters are unconstrained, but the γ parameters are partly dependent. There is a multiplicity of admissible derivative prices, with a dimension of residual incompleteness equal to n.

iii) If $H \geq 2$ and $p \geq H(n+1) - 2n - 1$, the γ parameters are functions of the θ-parameters and the θ parameters are partly constrained. There is a unique pricing formula and a restricted historical dynamics.

iv) If $H \geq 2$ and $p < H(n+1) - 2n - 1$, the model defined by (3.1), (3.2) is not compatible with arbitrage free restrictions.

Proposition 4 shows that different modelling approaches can be followed.

[15] Note that the residual incompleteness can be diminished by introducing additional restrictions on the γ parameters, such as $\gamma_1 = 0$ (see Polimenis (2001)]. In this special case endogenizing the riskfree rate is equivalent to specifying the stochastic discount factor.

i) When all state variables are exogenous, the parameters are unconstrained, but the conditional distribution of the short term interest rate is not directly specified.

ii) When one state variable corresponds to the short rate and the other ones are exogenous, the historical dynamics is still unconstrained and in particular, we can easily select directly the form of the conditional distribution of the short rate process.

iii) With more endogenous state variables, the historical dynamics has to be restricted in an appropriate way. [16]

3.2 Mimicking factors

Let us now come back to the framework of section 2 and explain how the initial exogenous factors can be replaced by endogenous ones. As noted in section 2.4 the exogenous factors are defined up to an invertible affine transformation. We directly deduce from (2.8) [resp. (2.10)] that a finite number of interest rates [resp. forward rates] may constitute an admissible set of factors. Therefore we can replace the initial specification by a model whose influencing factors are interest rates with given maturities, or forward rates [as in Heath, Jarrow, Morton (1990), (1992), Jamshidian (1989)], called mimicking factors. For expository purpose we assume that the affine space has dimension n and that the mimicking state variables are the rates with the smallest maturities. The new state variable is denoted by $X_t^* = (r_{t+1}, R_t')'$, where $R_t = (r_{t,t+2}, \ldots, r_{t,t+n})'$.

The historical dynamics of X_t^* corresponds to a compound autoregressive model :

$$E_t(\exp w' X_{t+1}^*) = \exp[a^*(w)' X_t^* + b^*(w)], \tag{3.6}$$

and the sdf admits an exponential affine expression :

[16] For instance the Ball, Torous' model is inconsistent with the absence of arbitrage (see Cheng (1987)). The same remark applies for the Nelson, Schaefer model [see Martellini, Priaulet (2001)].

$$M_{t,t+1} = \exp\left(\gamma_0^* + \gamma_1^{*'} X_t^* + \gamma_2^{*'} X_{t+1}^*\right). \quad (3.7)$$

The interest rates at different maturities can often be observed, in contrast to initial factors [17]. Thus the compound autoregressive form (3.6) is a testable restriction induced by the initial model. For instance we can check if the conditional mean and variance of the rates are affine functions of the previous rates [see Ghysels, Ng (1998)].

i) Constrained historical dynamics

The functions a^*, b^* (and the pricing parameters γ^*) are functions of the initial parameters $a, b, (\alpha, \delta)$, and deduced by means of relations (2.8).

We have already noted that a^*, b^*, γ^* were strongly constrained by the arbitrage free restrictions (see the discussion of section 3.1). In our framework these restrictions can be obtained in the following way.

Let us denote by c_h^*, d_h^* the solution of the recursive equation of Proposition 3, when functions a and b are replaced by functions a^* and b^* and (α, δ) are replaced by (α^*, δ^*). By applying condition (2.8), we get :

$$\begin{aligned} r_{t,t+h} &= -\frac{c_h^{*'}}{h}\begin{pmatrix} r_{t+1} \\ R_t \end{pmatrix} - \frac{d^*}{h}, h = 2, \ldots, n \\ \Longleftrightarrow \quad R_t &= -\begin{bmatrix} c_2^{*'}/2 \\ \vdots \\ c_n^{*'}/n \end{bmatrix}\begin{pmatrix} r_{t+1} \\ R_t \end{pmatrix} - \begin{bmatrix} d_2^*/2 \\ \vdots \\ d_n^*/n \end{bmatrix}. \end{aligned}$$

Since this equality has to be satisfied for any admissible values of the rates, we deduce that functions a^*, b^* are constrained by :

$$d_h^* = 0, h = 2, \ldots, n, \quad (3.8)$$

$$\begin{bmatrix} c_2^{*'}/2 \\ \vdots \\ c_n^{*'}/n \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \quad (3.9)$$

[17] This remark is important for statistical inference (see section 6).

$$Id + \begin{bmatrix} c_2^{*'}/2 \\ \vdots \\ c_n^{*'}/n \end{bmatrix} \begin{pmatrix} 0 \\ Id \end{pmatrix} = 0. \tag{3.10}$$

These restrictions on functions a^*, b^* are automatically satisfied when the dynamics of the rates is deduced from (2.1), (2.3), (2.8) [see Appendix 2]. Indeed the dynamics of the rates depend not only on the underlying factors, but also on the value of the endogenous risk corrections α, δ : "The spot rate and bond price processes parameters are not independent of the market prices for risk" [Heath, Jarrow, Morton (1992)] .

ii) Risk neutral dynamics

The dynamics of the interest rate process under the risk neutral probability[18] is characterized by its conditional Laplace transform at horizon 1, which is given by :

$$\begin{aligned}
& \overset{Q}{E_t} \left[\exp(w' X_{t+1}^*)\right] \\
= \;& \frac{E_t\left[M_{t,t+1}\exp(w' X_{t+1}^*)\right]}{E_t(M_{t,t+1})} \\
= \;& \exp(r_{t+1}) E_t \left[\exp(\gamma_0^* + \gamma_1^{*'} X_t^* + \gamma_2^{*'} X_{t+1}) \exp(w' X_{t+1}^*)\right] \\
= \;& \exp\left\{[a^*(\gamma_2^* + w) - a^*(\gamma_2^*)] X_t^* + b^*(\gamma_2^* + w) - b^*(\gamma_2^*)\right\},
\end{aligned}$$

with $\gamma_0^* = -b^*(\gamma_2^*), \gamma_1^* = -a^*(\gamma_2^*) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We deduce the proposition below.

Proposition 5 : The term structure of interest rate $X_t^* = (r_{t+1}, R_t')'$ is a compound autoregressive (affine) process under the risk neutral probability. It corresponds to shifted functions :

[18] In a similar way it is possible to derive the risk neutral dynamics of the initial state variables X_t. However the derivatives proposed on the market are written directly in terms of the rates X_t^* (not of X_t).

$$A^*(w) = a^*(\gamma_2^* + w) - a^*(\gamma_2^*),$$

$$B^*(w) = b^*(\gamma_2^* + w) - b^*(\gamma_2^*).$$

The fact that a compound autoregressive representation is valid both under the historical and risk neutral probabilities allows for a direct analysis of the stationarity properties. Let us recall that the term structure is stationary under the historical (resp. risk neutral) probability if $\lim_{h\to\infty}(a^*)^{oh} = 0$ [resp. $\lim_{h\to\infty}(A^*)^{oh} = 0$].

4. Derivative Pricing

When the state process follows CAR (affine) dynamics and the sdf is log-linear, derivatives' prices also get closed form. To simplify the presentation we consider european derivatives at maturity h. The discrete time approach was developed in Polimenis (2001), and is analogous to the method presented in Duffie, Pan, Singleton (2000) for continuous time models. We first explain how to price the derivatives with exponential payoff of the type $\exp(z' X^*_{t+h})$, directly written on interest rates X^*. When $z = w$ is real the payoff is simply exponential, but the computation has to be extended to complex argument z. Indeed with complex argument, we can invert complex Laplace transform to deduce the price of derivatives with other types of payoff such as european call written on bonds.

4.1 Exponential payoff

Let us recall that the process X^*_t is compound autoregressive :

$$E_t[\exp z' X^*_{t+1}] = \exp[a^*(z)' X^*_t + b^*(z)], \tag{4.1}$$

and that the sdf can be written as :

$$M_{t,t+1} = \exp(\gamma_0^* + \gamma_1^{*'} X^*_t + \gamma_2^{*'} X^*_{t+1}), \tag{4.2}$$

where : $\gamma_0^* = -b^*(\gamma_2^*), \gamma_1^* = -a(\gamma_2^*) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The exponential payoffs $\exp(z' X^*_{t+h}), z$ varying, constitute a generating system of the set of all payoffs. Let us denote by $C^*_t(z, h)$ the price at t of this derivative :

$$C_t^*(z,h) = E_t[M_{t,t+h} \exp(z' X_{t+h}^*)]. \qquad (4.3)$$

Proposition 6 : The price $C_t^*(z,h)$ of the contingent payoff $\exp(z' X_{t+h}^*)$ is given by :

$$C_t^*(z,h) = \exp[c^*(h,z)' X_t^* + d^*(h,z)],$$

where c^*, d^* satisfy the recursive equations :

$$\begin{aligned} c^*(h,z) &= A^*[c^*(h-1,z)] - \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ d^*(h,z) &= d^*[h-1,z] + B^*[c^*(h-1,z)], \end{aligned}$$

and the initial conditions :

$$\begin{aligned} c^*(1,z) &= A^*(z) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ d^*(1,z) &= B^*(z). \end{aligned}$$

Proof : See Appendix 3.

Proposition 6 is similar to the standard property for continuous time models, saying that the price of a derivative satisfies a partial differential equation, which is independent of the payoff (see also the complex valued ordinary differential equations derived by Duffie, Pan, Singleton (2000), p1351, or the generalized Ricatti equations in Duffie et alii (2001) equation (11.10)). However in the discrete framework the equations are easily solved recursively.

The general formula can be easily particularized to derivatives with exponential payoff on the short term interest rate only and written with respect to the initial factors. We get the following corollary.

Corollary 2 : The price $C_t(u,h)$ of the derivative paying $\exp(u r_{t+h})$ at date $t+h$ is :

$$C_t(u,h) = \exp[c(h,u)' X_t + d(h,u)],$$

where c and d satisfy the same nonlinear recursive equations as in Proposition 3, with initial conditions :

$$
\begin{aligned}
c(1,u) &= a(\alpha+u,\delta) - a(\alpha,\delta) - (1,0)', \\
d(1,u) &= b(\alpha+u,\delta) - b(\alpha,\delta).
\end{aligned}
$$

4.2 Inversion formula

The pricing formula (4.3) can be considered as a Fourier transform, and used to derive the price of other derivatives by inversion. For instance it is possible to use the inversion formula described in Duffie, Pan, Singleton (2000) [Proposition 2] and Polimenis (2001) for continuous and discrete time economies, respectively. Let us denote :

$$
G_t(w_0, w_1, K; h) = E_t[M_{t,t+h} \exp w_0' X^*_{t+h} \mathbb{1}_{w_1' X^*_{t+h} < K}], \tag{4.4}
$$

where $\mathbb{1}$ denotes the indicator function. The function G_t is a truncated real Laplace transform. From Duffie, Pan, Singleton (2000) the truncated real Laplace transform can be deduced from the (untruncated) complex Laplace transform :

$$
G_t(w_0, w_1, K; h) = \frac{C^*_t(w_0, h)}{2} - \frac{1}{\pi} \int_0^\infty \frac{Im[C^*_t(w_0 + i w_1 y, h) \exp -iyK]}{y} dy, \tag{4.5}
$$

where $Im(z)$ denotes the imaginary part of the complex z.

Then the truncated real Laplace transform can be used in the usual way to derive the price of a european call option on a coupon bond. It is first noted that holding a call option with maturity $t+h$ on a coupon bond paying coupons c_i at date $t+h+k_i$ (say) is equivalent to holding a portfolio of call options with same maturity $t+h$ on discount bonds $B(t+h, k_i)$ [Jamshidian (1989)]. Thus it is sufficient to price a european call on discount bond $B(t+h, H-h)$, say. The price of this call is :

$$
\begin{aligned}
C^*_t(h; H, K) &= E_t\left[M_{t,t+h}(B(t+h, H-h) - K)^+\right] \\
&= E_t\left\{M_{t,t+h}(\exp[-(H-h) r_{t+h,t+H}] - K)^+\right\}.
\end{aligned}
$$

By the arbitrage free restrictions the rate $r_{t+h,t+H}$ can be written as an affine function of the state variables X^*_{t+h} :

$$\exp[-(H-h)r_{t+h,t+H}] = \exp[c^*(h,H)'X^*_{t+h} + d^*(h,H)], \text{ (say)}. \tag{4.6}$$

Therefore :

$$\begin{aligned} C_t^*(h;H,k) &= E_t\left\{M_{t,t+h}(\exp[c^*(h,H)'X^*_{t+h} + d^*(h,H)] - K]^+\right\} \\ &= E_t\{M_{t,t+h}(\exp[c^*(h,H)'X^*_{t+h} + d^*(h,H)] - K) \\ &\quad \mathbf{1}_{c^*(h,H)'X^*_{t+h}+d^*(h,H)>\log K}\Big\} \\ &= \exp d^*(h,H)E_t\left\{M_{t,t+h}\exp[c^*(h,H)'X^*_{t+h}]\mathbf{1}_{-c^*(h,H)'X^*_{t+h}<-\log K+d^*(h,H)}\right\} \\ &\quad -KE_t\left\{M_{t,t+h}\mathbf{1}_{-c^*(h,H)'X^*_{t+h}<-\log K+d^*(h,H)}\right\}. \end{aligned}$$

Proposition 7 : The price of the european call on discount bond is :

$$\begin{aligned} C_t^*(h,H,K) &= \exp(d^*(h,H))G_t[c^*(h,H), -c^*(h,H), -\log K + d^*(h,H)] \\ &- KG_t[0, -c^*(h,H), -\log K + d^*(h,H)], \end{aligned}$$

where G_t denotes the truncated Laplace transform (4.4) and $c^*(h,H), d^*(h,H)$ are the coefficients in the decomposition of $-(H-h)r_{t,h,t+H}$ on the state variables.

The pricing formula only requires one-dimensional integration to derive the values of the truncated Laplace transform [see (4.5)]. This integration is equivalent to the computation of the cumulative distribution function of the gaussian (resp. gamma, bivariate chi-square) distribution in the Black-Scholes (resp. CIR, Longstaff-Schwartz) model.

5. The pattern of the term structure

Various patterns can be derived for the term structure according to the number of factors and to the selected functions a and b that capture the dynamics of the state variables X_t (or a^* and b^* that capture the dynamics of the rates X_t^*). We first consider one factor models and give examples when the short term interest rate is discrete, or continuous. Then we propose a general approach for building multifactor models.

The aim of this section is threefold. First we present a large variety of models to show that the set of discrete CAR (affine) processes is much larger than the set of discretely sampled continuous time affine processes [see Appendix 5]. Second we want to compare the stationarity properties of the rate dynamics under the historical and risk neutral probabilities. Finally we analyse the dynamics of the long term interest rate. Recall that in the absence of arbitrage the long term zero-coupon rate can never decrease (Dybvig, Ingersoll, Ross (1996), El-Karoui, Frachot, Geman (1998)). Thus the following situations are possible.

i) The long term interest rate does not exist;

ii) The long term interest rate exists and its historical (or risk neutral) distribution is stationary. Then it is necessarily constant.

iii) The long term interest rate exists and both its historical and risk neutral distributions are nonstationary.

5.1 One factor model

If the short term interest rate is the single state variable, the dynamics is characterized by $E_t(\exp u r_{t+2}) = \exp[a(u) r_{t+1} + b(u)]$, and the sdf is : $M_{t,t+1} = \exp(\alpha r_{t+2} + \beta_t)$. When, in an equilibrium framework, higher rates coincide with lower economic growth, the risk correction α becomes positive. Cash flows that arrive in high interest rate states allow agents to diversify their exposure to the fundamental uncertainty and thus have higher value. In this case, we get :

$$\begin{aligned} r_{t,t+h} &= -\frac{c_h}{h} r_{t+1} - \frac{d_h}{h}, \\ \text{where} \ : & \quad c_h = a[c_{h-1} + \alpha] - a(\alpha) - 1, c_1 = -1, \\ & \quad d_h = d_{h-1} - b(\alpha) + b[c_{h-1} + \alpha], d_1 = 0. \end{aligned}$$

The term structure is driven by r_{t+1}. The long term interest rate exists if and only if c_h/h and d_h/h tend to finite limits, when h tends to infinity. Moreover it is stochastic if $\lim_{h\to\infty} c_h/h \neq 0$.

The historical and risk neutral dynamics are characterized by the Laplace transforms :

$$E_t(\exp u r_{t+2}) = \exp[a(u) r_{t+1} + b(u)], \quad (5.1)$$

$$\overset{Q}{E_t} (\exp u r_{t+2}) = \exp[A(u) r_{t+1} + B(u)], \quad (5.2)$$

$$\text{where : } A(u) = a(\alpha+u) - a(\alpha), B(u) = b(\alpha+u) - b(\alpha). \quad (5.3)$$

We give below examples of term structure patterns.

Example 1 : The compound Poisson process

We assume that the interest rate can only take discrete nonnegative values $j\gamma, j \in I\!R$, where $\gamma \in I\!R^+$ is given. Its conditional distribution is such that :

$$r_{t+2}/\gamma = Z_{t+1} + \varepsilon_{t+1},$$

where Z_{t+1} and ε_{t+1} are conditionally independent, with conditional distributions :

$$Z_{t+1} \sim \mathcal{B}(r_{t+1}/\gamma; \pi), \pi \in (0,1), \text{ (Binomial)},$$

$$\varepsilon_{t+1} \sim \mathcal{P}(\lambda), \lambda > 0, \text{ (Poisson)}.$$

The conditional Laplace transform is :

$$\begin{aligned} E_t(\exp u r_{t+2}) &= E_t[\exp u\gamma(Z_{t+1} + \varepsilon_{t+1})] \\ &= E_t \exp(u\gamma Z_{t+1}) E_t \exp(u\gamma\varepsilon_{t+1}) \\ &= [\pi \exp(u\gamma) + 1 - \pi]^{r_{t+1}/\gamma} \exp[-\lambda(1 - \exp u\gamma)]. \end{aligned}$$

Thus : $a(u) = \dfrac{1}{\gamma} \log[\pi \exp(u\gamma) + 1 - \pi], b(u) = -\lambda(1 - \exp u\gamma).$

A simple computation given in Appendix 4 provides the term structure :

$$r_{t,t+h} = -\frac{r_{t+1}}{\gamma h} \log \left\{ A \left(\frac{\pi \exp[\gamma(\alpha-1)]}{\pi \exp(\gamma\alpha) + 1 - \pi} \right)^h + B \right\}$$

$$- \frac{C}{h} - D - \frac{E}{h} \left(\frac{\pi \exp[\gamma(\alpha-1)]}{\pi \exp(\gamma\alpha) + 1 - \pi} \right)^h,$$

where A,B,C,D,E are constant coefficients. For large maturities the interest rate tends to a constant at an hyperbolic rate. Examples of term structures are provided in Figure 1 for different values of the parameters. These values are : $\gamma = 0.0015, \lambda = 0.5, \pi$ varying between 0.9 and 0.99, whereas the risk adjustment coefficient α is $1, 100, -100, 500, -500$. We observe monotonic convex or concave patterns.

[Insert Figures 1 : Term structure for compound Poisson]

It is easily checked that the risk neutral dynamics of the short term interest rate corresponds also to a compound Poisson process, with parameters :

$$\gamma^* = \gamma, \pi^* = \exp(\alpha\gamma)[\pi \exp(\alpha\gamma) + 1 - \pi]^{-1},$$

$$\lambda^* = \lambda \exp(\alpha\gamma).$$

There exists an infinite number of risk neutral probabilities indexed by the risk adjustment parameter α.

When $\pi < 1$, the process is both stationary under the historical and risk neutral probabilities. The process has a longer memory under the risk neutral probability ($\pi^* > \pi$), if the correction term α is positive. When $\pi = 1$, the process is a random walk with drift under both probabilities. The drift is larger under the risk neutral probability ($\lambda^* > \lambda$), if the correction term α is positive.

Example 2 : The autoregressive gamma process

The autoregressive gamma process is the discrete time counterpart of the Cox-Ingersoll-Ross process [Cox-Ingersoll-Ross (1985)]. Conditionally to r_{t+1}, the future interest rate r_{t+2} is such that r_{t+2}/c follows a gamma distribution $\gamma(\nu + Z_t)$, where Z_t is drawn in the Poisson distribution $\mathcal{P}[\rho \, r_{t+1}/c]$.

[see e.g. Gourieroux, Jasiak (2001)b]. The parameters are constrained by $c > 0, \rho > 0$. Its conditional log-Laplace transform is given by :

$$\log E_t[\exp u r_{t+2}] = -\nu \log(1 - uc) + \frac{\rho u}{1 - uc}\, r_{t+1}.$$

We get :

$$a(u) = \frac{\rho u}{1 - uc}, b(u) = -\nu \log(1 - uc).$$

The coefficient c_h satisfies the recursion :

$$c_h = \frac{\rho(c_{h-1} + \alpha)}{1 - c(c_{h-1} + \alpha)} - \frac{\rho\alpha}{1 - c\alpha} - 1.$$

It is possible to get the explicit expression of the coefficient c_h. Indeed the series c_h satisfies a rational recursive equation, which is equivalent to :

$$\frac{c_h - \gamma_1}{c_h - \gamma_2} = \frac{1 + \gamma_1}{\gamma_1} \frac{\gamma_2}{1 + \gamma_2} \frac{c_{h-1} - \gamma_1}{c_{h-1} - \gamma_2},$$

where γ_1, γ_2 are distinct real roots of the second degree polynomial : $c^*\gamma^2 + \gamma[\rho^* + c^* - 1] - 1 = 0$, and $c^* = c[1 - \alpha c]^{-1}, \rho^* = \rho[1 - \alpha c]^{-2}$.

Thus we get :

$$\frac{c_h - \gamma_1}{c_h - \gamma_2} = \left(\frac{1 + \gamma_1}{\gamma_1} \frac{\gamma_2}{1 + \gamma_2}\right)^{h-1} \frac{1 + \gamma_1}{1 + \gamma_2},$$

$$c_h = \frac{\gamma_1 - \gamma_2 \left(\frac{1 + \gamma_1}{\gamma_1} \frac{\gamma_2}{1 + \gamma_2}\right)^{h-1} \frac{1 + \gamma_1}{1 + \gamma_2}}{1 - \left(\frac{1 + \gamma_1}{\gamma_1} \frac{\gamma_2}{1 + \gamma_2}\right)^{h-1} \frac{1 + \gamma_1}{1 + \gamma_2}}.$$

It can be checked that the short term interest rate follows a gamma autoregressive process under both historical and risk neutral probability. Under the risk neutral probability the parameters are $\nu^* = \nu, c^*, \rho^*$. The parameters c^* and ρ^* have to be positive, which implies the restriction $\alpha < 1/c$ on the risk correction parameter.

Moreover if $\rho < 1$, the interest rate process is stationary under the historical probability. Under the risk neutral probability the process is stationary if $\rho^* < 1$, that is if $\alpha < \frac{1 - \sqrt{\rho}}{c}$; it is nonstationary, otherwise.

As illustration Figure 2 reports the term structures corresponding to different parameter values : $c = .1, \nu = .1, .2, \alpha = -4, 0, 4$, whereas ρ varies between 0 and 0.9.

[Insert Figures 2 : Terms Structure for Autoregressive Gamma Process]

Note the presence of patterns with bumps.

Example 3 : Infinitely divisible distribution

Let us consider an infinitely divisible distribution on $\mathbb{R}^+$ (see e.g. Sato (1999)) with Laplace transform $\exp a(u)$ and another distribution on $\mathbb{R}^+$ with Laplace transform $\exp b(u)$. Then function a satisfies a complete monotonicity condition, that is function a is infinitely differentiable with non-negative derivatives [Joe (1997)]. Then we can consider the process with conditional Laplace transform :

$$E_t[\exp ur_{t+2}] = \exp[a(u)(\gamma_o r_{t+1} + \gamma_1) + b(u)],$$

where $\gamma_o \geq, \gamma_1 \geq 0$.

Since function a is increasing, convex, with value 0 for $u = 0$, it is easily checked that the stationarity condition : $\lim_{h\to\infty} \gamma_0^h a^{0h} = 0, \forall u < 0$ is equivalent to the Lipschitz condition $\gamma_0 \dfrac{da(0)}{du} < 1$.

Under the risk neutral probability, we get :

$$A(u) = \gamma_0[a(u + \alpha) - a(\alpha)],$$

$$B(u) = \gamma_1[a(u + \alpha) - a(\alpha)] + b(u + \alpha) - b(u).$$

The process is also based on an infinitely divisible distribution with log-Laplace transform A. Since : $\dfrac{dA}{du}(0) = \gamma_0 \dfrac{da}{du}(\alpha)$, the process is stationary under the risk neutral condition if $\alpha < \alpha_0$, where α_0 satisfies $\dfrac{da}{du}(\alpha_0) = 1/\gamma_0$.

In this special case the recursive equations become :

$$
\begin{aligned}
c_h &= \gamma_0 a(c_{h-1}+\alpha) - \gamma_0 a(\alpha) - 1, c_1 = -1, \\
d_h &= d_{h-1} - \gamma_1 a(\alpha) - b(\alpha) + \gamma_1 a(c_{h-1}+\alpha) + b(c_{h-1}+\alpha), \\
& \quad d_1 = 0.
\end{aligned}
$$

Then the coefficient c_h satisfies the recursive equation : $c_h = A(c_{h-1}) - 1, c_1 = -1$, where A is increasing convex. Two cases can be distinguished.

(i) If $\lim_{u\to-\infty} \frac{dA(u)}{du} = \lim_{u\to-\infty} \gamma_0 \frac{da(u)}{du} < 1$, there is a fixed point c_∞, such that $c_\infty = A(c_\infty) - 1 < -1$, and the sequence c_h tends to c_∞. Then $d_h = d_{h-1} + \frac{\gamma_1}{\gamma_0}(c_h + 1)$ is asymptotically equivalent to $d_h \sim h\frac{\gamma_1}{\gamma_0}(c_\infty + 1)$ and the long term interest rate exists and is constant $r_{t,t+\infty} = -\frac{\gamma_1}{\gamma_0}(c_\infty + 1)$.

(ii) If $\lim_{u\to-\infty} \gamma_0 \frac{da(u)}{du} > 1$, the sequence c_h tends to $-\infty$, and the long term interest rate can depend on r_{t+1} according to the rate of divergence.

This example shows the large variety of dynamics, which correspond to a compound autoregressive model. For instance the baseline distribution with log-Laplace transform a can be discrete with integer value and probabilities $p_i, i \in \mathbb{N}$, such that the ratio p_{i+1}/p_i is increasing in i (a sufficient condition for infinite divisibility [Warde, Katti (1971)]). It includes as special cases the power series with Laplace transform $1 - (1 - \exp s)^{1/\theta}, \theta \geq 1$, and the logarithmic series with Laplace transform $-\theta \log[1 - (1 - \exp -\theta) \exp s], \theta > 0$. The baseline distribution can also be continuous as for stable distribution with Laplace transform $\exp(-|s|^{1/\theta}), \theta \geq 1$.

5.2 Multifactor model

The extension of the above examples to a multifactor framework is important, if we want to introduce independent evolutions for the level, slope and curvature of the term structure. In this section we present general approaches to build multifactor models.

i) A transformation approach.

It is well known that gaussian vector autoregressive models admit an affine log-Laplace transform. Let us consider such a model :

$$Y_t = \Phi Y_{t-1} + u_t, \tag{5.4}$$

where $u_t \sim N(0, \Sigma)$. Then we have :

$$\Sigma^{-1/2}\ Y_t = (\Sigma^{-1/2}\Phi\Sigma^{1/2})\Sigma^{-1/2}\ Y_{t-1} + \Sigma^{-1/2}\ u_t,$$

$$\text{or :}\quad Z_t = \Phi^* Z_{t-1} + v_t,\ \text{(say)},$$

where $v_t \sim N(0, Id)$. Thus (Y_t) is a linear transformation of a process (Z_t), whose components are instantaneously independent and which admits a CAR representation.

A similar approach can be followed for defining CAR distributions of nonnegative variables.

In a first step we define a nonnegative process with conditionally independent components. The conditional Laplace transform is defined by :

$$\begin{aligned} & E_t[\exp v' Z_{t+1}] \\ = \ & E_t[\exp(\sum_{j=1}^n v_j Z_{j,t+1})] \\ = \ & \exp[\sum_{j=1}^n \{a_j(v_j)(\gamma_j' Z_t + \gamma_{jo}) + b_j(v_j)\}], \end{aligned} \tag{5.5}$$

where a_j, j varying, is the log-Laplace transform of an infinitely divisible distribution [19] on $I\!R^+$, $\gamma_j = (\gamma_{j1}, \ldots, \gamma_{jn})'$, and γ_{ji}, γ_{jo} are nonnegative coefficients.

Then in a second step the process of interest is defined by :

$$X_t = \begin{pmatrix} r_{t+1} \\ f_t \end{pmatrix} = PZ_t, \tag{5.6}$$

where P has nonnegative elements and is invertible (P can be chosen triangular). Then the process of interest admits the conditional Laplace transform :

[19] The choice of an infinitely divisible distribution ensures that $\exp(a_j(v_j)y)$ is a Laplace transform for any nonnegative y.

$$
\begin{aligned}
& E_t[\exp w' X_{t+1}] \\
= \; & E_t[\exp w' P Z_{t+1}] \\
= \; & \exp\left\{\sum_{j=1}^{n} [a_j[(w'P)_j](\gamma_j' \, P^{-1} X_t + \gamma_{jo}, + b_j[(w'P)_j]])\right\}.
\end{aligned} \tag{5.7}
$$

We get a CAR representation :

$$
\begin{aligned}
& E_t\left[\exp w' X_{t+1}\right] \\
& = \exp[A(w)' X_t + B(w)], \\
\text{where : } \quad A(w)' \; & = \sum_{j=1}^{n} a_j[(w'P)_j]\gamma_j' P^{-1} \\
B(w) \; & = \sum_{j=1}^{n} \left\{ a_j[(w'P)_j]\gamma_{jo} + b_j[(w'P)_j] \right\}.
\end{aligned}
$$

The specification depends on the basic distribution by means of functions $a_1, \ldots, a_n$, on the vectors $\gamma_o, \gamma_j, j = 1, \ldots, n$, which capture the time dependence and on the matrix P defining the linear transformation. It involves $2n(n+1)$ parameters in the definition of the conditional affine mean and covariance matrix, and induces strict restrictions on these affine moments.

Example 4 : Let us consider for a moment the continuous time framework and assume that the process (X_t) satisfies a multidimensional stochastic differential equation with drift $\mu(X_t)$ and volatility matrix $\Sigma(X_t)$. Since the instantaneous Laplace transform of the process is :

$$
\exp[w'\mu(X_t) + \frac{1}{2} w' \Sigma(X_t) w],
$$

the affine condition is satisfied if and only if μ and Σ are affine functions of (X_t). [20] It is well-known that this condition plus the nonnegativity of the

[20] The affine restriction on the two first conditional moments is often used as definition of affine models [see e.g. Piazzesi (2001) section 2.4].

variance matrix restrict the admissible processes. For instance the multivariate continuous time extensions of the CIR process are very constrained and can require independent factors (Duffie-Kan (1996), Dai, Singleton (2000)]. The restrictions are weaker in discrete time and for instance, the previous approach can be applied to get a multivariate extension of the autoregressive gamma process.

ii) Recursive multifactor models

Let us discuss again the gaussian vector autoregressive model. For convenience, the approach is described in dimension two. Instead of autoregressive equation (4.4), we can first define the conditional distribution of $Y_{1,t}$ given $Y_{2,t}, Y_{1,t-1}, Y_{2,t-1}$:

$$Y_{1,t} = b_0 + b_1 Y_{2,t} + b_2 Y_{1,t-1} + b_3 Y_{2,t-1} + v_{1,t}, \text{ (say)},$$

then the conditional distribution of $Y_{2,t}$ given $Y_{1,t-1}, Y_{2,t-1}$:

$$Y_{2,t} = c_0 + c_1 Y_{1,t-1} + c_2 Y_{2,t-1} + v_{2,t} \text{ (say)}.$$

The first equation can be seen as a gaussian autoregressive model, in which a stochastic factor $Y_{2,t}$ has been introduced in the drift. This approach is typically used in the Ornstein-Uhlenbeck process with stochastic long run equilibrium.

We can now apply this recursive scheme to the general compound autoregressive process. Let us introduce the log-Laplace transforms of two infinitely divisible distributions a_1, a_2 (say). We can define the conditional distribution of r_{t+2} given f_{t+1}, r_{t+1}, f_t by :

$$\begin{aligned} & E_t \left[\exp(u r_{t+2}) | f_{t+1}, r_{t+1}, f_t \right] \\ = \; & \exp \left[a_1(u) [\alpha_1 f_{t+1} + \beta_1 r_{t+1} + \beta_2 f_t] + b_1(u) \right], \end{aligned}$$

then the conditional distribution of f_{t+1} given r_{t+1}, f_t by :

$$E_t \left[\exp(u f_{t+1}) | r_{t+1}, f_t \right] = \exp \left[a_2(u) (\gamma_1 r_{t+1} + \gamma_2 f_t) \right].$$

We deduce by iterated expectations that :

$$\begin{aligned} & E_t \left[\exp(u r_{t+2} + v f_{t+1}) | r_{t+1}, f_t \right] \\ = \; & \exp \left[a_2 \left[v + a_1(u)\alpha_1 \right] (\gamma_1 r_{t+1} + \gamma_2 f_t) + a_1(u)\beta_1 r_{t+1} + a_1(u)\beta_2 f_t + b_1(u) + b_2[v + a_1(u)\alpha_1] \right] \end{aligned}$$

which is a compound autoregressive process.

This approach can also be used to extend the autoregressive gamma process to the multivariate framework and does not provide the same dynamics as in the example of the section above.

iii) Recursive specification and stochastic parameter models

Finally it is interesting to relate the recursive modelling approach with the practice of considering models with stochastic parameters, such as stochastic volatility or stochastic long run parameter. Let us consider a parametric CAR process :

$$E[\exp u'Y_{t+1}|Y_t;\delta] = \exp[a_o(u)'y_t + b_o(u)'\delta],$$

where parameter δ affects linearly the b function. The model can be extended by introducing a stochastic parameter δ satisfying also a CAR specification. The model becomes :

$$\begin{aligned} E[\exp u'Y_{t+1}|Y_t,\delta_{t+1}] &= \exp[a_0(u)'Y_t + b_0(u)'\delta_{t+1}], \\ E[\exp v'\delta_{t+1}|Y_t,\delta_t] &= \exp[a_1(v)'\delta_t + b_1(v)]. \end{aligned}$$

We get a recursive model of the type discussed in the section above. For instance for the compound Poisson process [section 5.1, Example 1], the intensity λ can be modified into a stochastic intensity λ_t. In the autoregressive gamma case [see 5.1, Example 2], the degree of freedom ν can be made stochastic. [21]

6. Implementation

The implementation of the model for pricing requires a preliminary estimation of the parameters involved in the historical dynamics of the interest rates, that are functions a^*, b^* defined in equation (3.6) characterizing the dynamics of X_t^*.

Two types of approaches can be followed. In the latent modelling approach the dynamics concerns a vector of state variables including exogenous

[21] If we do not impose the positivity of the short term interest rate, we can consider the gaussian AR(1) model, where both the volatility and mean are stochastic [see Chen (1996)].

components (as in section 2). The functions a^*, b^* depend on a set of parameters $\theta^* = (\theta, \alpha, \delta)$, where θ parametrizes the initial functions a and b. In the direct modelling approach, a parametric specification of a^*, b^* is assumed, and we explain how to take account of the arbitrage free restrictions in the statistical step.

6.1 Latent modelling approach

i) Parametric specification with known transition

When functions a^*, b^* are known up to a finite dimensional parameter θ^*, and the conditional density of $X_t^* = [r_{t+1}, R'_{t+1}]'$ admits a closed form expression, the parameter is estimated by maximum likelihood. Moreover the risk neutral distribution generally belongs to the same family as the historical distribution, and it is possible to get analytical expressions of the prices of european calls written on bonds. This is the case of the standard CIR model.

ii) Parametric specification easy to simulate

For some parametric specifications the closed form expression of the conditional pdf is not available, but simulated paths are easily constructed. The parameter can be estimated by simulated maximum likelihood [Pedersen (1995), Brandt, Santa-Clara (2001)], or indirect inference [Gourieroux, Monfort, Renault (1993), Gallant, Tauchen (1996)a]. In this framework, it is generally easy to simulate under the risk neutral probability. Thus the derivative prices can be computed by Monte-Carlo to avoid the integral formula based on the complex Laplace transform.

iii) Parametric specification difficult to simulate

When a closed form expression of the transition is not available and simulated path are not easily constructed, the parameter can be estimated by a method of moments based on the Laplace transform [22]. Two approaches have yet been developed in the literature according to the type of argu-

[22] It is also possible to use the first and second order conditional moments for calibration, since they have simple affine expression [Darolles et alii (2001), Singleton (2001)]. The method is simple since it involves seemingly unrelated regressions. However it may lack of efficiency in finite sample.

ment introduced in the Laplace transform either real, or pure imaginary (see Darolles, Gourieroux, Jasiak (2001), Singleton (2001), respectively). For real arguments, the estimator is defined as a solution of :

$$\hat{\theta} = \arg\min_{\theta^*} \sum_{j=1}^{J} \left(\sum_{t=1}^{T} \omega_{jt} [\exp(u_j r_{t+1} + v_j' R_{t+1}) - \exp\left[a^*(w_j, \theta^*)' \begin{pmatrix} r_{t+1} \\ R_t \end{pmatrix} + b^*(w_j, \theta^*) \right] \right)^2 , \tag{6.1}$$

where $w_j = (u_j, v_j)', j = 1, \ldots, J$ are given values, $\omega_j = (\omega_{j1}, \ldots, \omega_{jt}), j$ varying, are instrumental variables. [23]

For imaginary arguments, the estimator is defined as a solution of :

$$\tilde{\theta} = \arg\min_{\theta^*} \sum_{j=1}^{J} \| \sum_{t=1}^{T} \omega_{jt} [\exp(iu_j r_{t+1} + iv_j' R_{t+1}) - \exp[a^*(iw_j, \theta^*)' \begin{pmatrix} r_{t+1} \\ R_t \end{pmatrix} + b^*(iw_j, \theta^*)] \|^2 . \tag{6.2}$$

Since both the real Laplace transform and the Fourier transform characterize the distribution for nonnegative variables, accurate estimations can be expected if the number of moments J, the instruments ω_j and the grid $w_j = (u_j, v_j), j = 1, \ldots, J$ are well-chosen [see Feuerverger (1990), Feuerverger, Mc Dunnough (1991) a,b, Singleton (2001)]. This is a consequence of general results on GMM with a continuum of moment conditions established by Carrasco, Florens (2000), (2002).

In fact the approaches above do not use the same basic moments for calibration. The first approach focuses on exponential transformations, whereas the second one considers sine and cosine functions. Intuitively the first approach is more appropriate for getting accurate results on the tails of the distribution in finite sample, since some of the selected moments overweight the tails. On the other hand the second approach involves transformations with the same order of magnitude and can be more easily applied with constant instruments $\omega_{jt} = 1, \forall j, t$, independent of j.

6.2 Direct modelling approach

[23] To improve the asymptotic efficiency it is possible to introduce a matrix of weights, which is not diagonal. We retained an objective function of type (6.1) for expository purpose.

We first explain how this approach can be implemented in a parametric framework. Then we consider its extension to the nonparametric case.

i) Parametric specification

Let us denote by θ^* the parameter introduced in the direct specification of a^*, b^*. We assume that the dimension of θ^* is sufficiently large to avoid the unconsistency discussed in Proposition 4, that is dim $\theta^* = p \geq n^2 - n - 1$.[24] By applying one of the methods described in section 6.1, it is possible to get an unconstrained estimator $\hat{\theta}^*$ of the parameter and an estimator $\hat{\Omega}$ of its variance-covariance matrix.

However the estimated functions $\hat{a}^*(.) = a^*(., \hat{\theta}^*), \hat{b}^* = b^*(., \hat{\theta}^*)$ are not necessarily compatible with the arbitrage free restrictions. Indeed from (3.5), there exist constants $\gamma_0^*, \gamma_1^*, \gamma_2^*$ such that :

$$\gamma_1^* + a^*(\mu_h + \gamma_2^*; \theta^*) - \lambda_h = 0, \gamma_0^* + b(\mu_h + \gamma_2^*; \theta^*) = 0, h = 1, \dots, n, \quad (6.3)$$

where θ^* denotes the true value of parameter.

This is a standard constrained estimation problem (Gourieroux, Monfort (1989), (1995), Vol 2), in which the constraint (6.3) has a mixed form :

$$G(\gamma^*, \theta^*) = 0, \text{ say.}$$

Then the estimator $\hat{\theta}^*$ can be corrected to incorporate the constraint. The constrained asymptotic least squares estimators of both type of parameters θ^*, γ^* are solutions of :

$$(\tilde{\theta}^*, \tilde{\gamma}^*) = \arg\min_{\theta^*, \gamma^*} (\hat{\theta}^* - \theta^*)' \hat{\Omega}^{-1} (\hat{\theta}^* - \theta^*)$$

$$\text{s.t. } G(\gamma^*, \theta^*) = 0.$$

Moreover it is known that the optimal value of the criterion is a natural test statistic for the hypothesis $G(\gamma^*, \theta^*) = 0$ (see Gourieroux, Monfort (1995) Vol 2].

In summary this approach provides arbitrage free constrained estimators of θ^*, γ^*, but also a specification test, which is easy to implement. When

[24] In multifactor models a large number of parameters is easily introduced by following one of the approaches proposed in section 5.2.

the arbitrage free restrictions are rejected, the initial model can be modified either by changing the parametric form of the distribution, or by increasing the number of factors. This specification test is similar to the approach developped by De Munnik, Schotman (1994), Bams, Schotman (1997), Bams (1998), De Jong (1997) in special cases.

ii) Nonparametric specification

More important, the CAR (affine) processes can also be used in a nonparametric framework. Let us recall how functions a^*, b^* restricted to real arguments can be estimated in a nonparametric setup [see Darolles, Gourieroux, Jasiak (2001)]. For any real argument w the values $a^*(w), b^*(w)$ can be approximated by :

$$[\hat{a}(w), \hat{b}(w)] = \arg\min_{a^*,b^*} \sum_{t=1}^{T} \{\exp(w' X^*_{t+1}) - \exp(a^{*'} X^*_t + b^*)\}^2. \tag{6.4}$$

These functional estimators are consistent and converge at rate $\sqrt{T}$. They can be used to estimate the underlying prices of risk by considering :

$$\begin{aligned}(\hat{\gamma}^*_0, \hat{\gamma}^*_1, \hat{\gamma}^*_2) &= \arg\min_{\gamma^*} \sum_{h=1}^{n} \| \gamma^*_1 + \hat{a}(\mu_h + \gamma^*_2) - \lambda_h \|^2 \\ &+ \sum_{h=1}^{n} \| \gamma^*_0 + \hat{b}(\mu_h + \gamma^*_2) \|^2,\end{aligned}$$

and as above the optimal value of the criterion function can be used to test the arbitrage free restrictions.

The estimated a^* and b^* functions restricted to real arguments can be used to determine the term structure of interest rates and to price derivatives with real exponential payoff. But the expression of the Laplace transform should be estimated for complex arguments also in order to apply a pricing formula based on the inversion of a Fourier transform [see section 4.2]. The nonparametric estimation of the Laplace transform for complex arguments is less tractable. Indeed two approaches can be followed. The first one consists in applying a calibration criterion (6.4) for complex arguments $z = w_0 + iw_1$. Thus the dimension of the optimization problem is $2n$, since it concerns both w_0 and w_1. Likely the curse of dimensionality will be encountered. Moreover the results will not be very accurate. Indeed we now that the Laplace transform defined for complex numbers is characterized by the real Laplace transform. Thus the fact that $z \to a(z), z \to b(z)$ depend on the real functions $w \to a(w), w \to b(w)$ of smaller dimension is not taken into account. A second approach consists in computing the complex Laplace transform from the real one. The idea is to derive the series expansion of the [complex] Laplace transform by estimating the derivatives at $w = 0$ of the real Laplace transform $\exp[\hat{a}(w)'X_t + \hat{b}(w)]$.

7. Affine versus quadratic term structure models

This paper develops a general approach for affine term structure in discrete time. The specification assumes that the underlying factors satisfy a compound autoregressive (affine) process and that the stochastic discount factor is an exponential affine function of the lagged values of the factors. Under these assumptions, the term structure is affine and it is possible to derive the admissible risk neutral probabilities. Several examples have been considered and the implementation has been discussed in a parametric or nonparametric framework.

Several authors reported strong evidence against the set of restrictions implied by the general class of affine models. But either they have considered specific parametrized specifications often deduced from affine continuous time models, implying a constant long term interest rate [Ait-Sahalia (1996), Anderson, Lund (1996), Chan et al. (1992), Conley et alii (1996), Gallant, Tauchen (1996), Stanton (1997)], or introduced a priori observable economic factors [Ghysels, Ng (1998)].

These stylized facts are against the affine specifications written in continuous time. At this point two strategies can be followed.

1) It is possible to still consider continuous time model and to relax the affine assumption. This is done in the quadratic term structure models [see e.g. Leippold, Wu (2002)]. But as for affine models, the admissible dynamics are very restricted. Typically [25], the short term interest rate has to be a quadratic function of factors, following a differential stochastic system with linear drift and constant volatility [see e.g. Leippold, Wu (2002), Proposition 1]. Thus the quadratic model involves only $K(K+1)/2$ more parameters

[25] Under the risk neutral probability.

than the affine model in continuous time where K is the number of factors. Moreover this quadratic specification is not easy to implement in practice since the unobservable factors have to be recovered from the yields by solving multivariate quadratic system.

2) The alternative is to stay in the affine class, but to assume a minimal time unit which is not infinitesimal. Since the class of discrete time compound autoregressive (affine) processes is quite large in a nonparametric setup, a reasonable fit can be expected (see the application on absolute value of returns in Darolles et alii (2001))].

Of course it would be possible to mix the two extensions and consider quadratic models in discrete time, but it is out of the scope of the present paper.

Finally as in Duffie et alii (2001) the approach points out the importance of the conditional Laplace transform instead of the standard conditional drift and volatility functions.

REFERENCES

Ait-Sahalia, Y. (1996) : "Nonparametric Pricing of Interest Rate Derivative Securities", Econometrica, 64, 527-560.

Anderson, T., and J., Lund (1998) : "Stochastic Volatility and Mean Drift in the Short Term Interest Rate Diffusion : Sources of Steepness, Level and Curvature in the Yield Curve", DP. Northwestern University.

Babbs, S., and K., Nowman (1999) : "Kalman Filtering of Generalized Vasicek Term Structure Models", Journal of Financial and Quantitative Analysis, 34, 115-130.

Backus, D., Foresi, S., and C., Telmer (1998) : "Discrete Time Models of Bond Pricing", NBER.

Backus, D., Foresi, S. and C., Telmer (2001) : "Affine Term Structure Models and the Forward Premium Anomaly", Journal of Finance, 56, 279-304.

Backus, D., Telmer, C., and L., Wu (1999) : "Design and Estimation of Affine Yield Models", Manuscript, New-York University.

Balduzzi, P., Das, S., and S., Foresi (1998) : "The Central Tendency : A Second Factor in Bond Yields", The Review of Economics and Statistics, 80, 62-72.

Balduzzi, P., Das, S., Foresi, S., and R., Sundaran (1996) : "A Simple Approach to Three Factor Affine Term Structure Models", Journal of Fixed Income, 6, 43-53.

Ball, C., and W., Torous (1983) : "Bond Price Dynamics and Options", Journal of Financial and Quantitative Analysis, 18, 517-531.

Bams, D. (1998) : "An Empirical Comparison of Time Series and Cross-Sectional Information in the Longstaff-Schwartz Term Structure Model", Maastricht, Univ. DP.

Bams, D., and P., Schotman (1997) : "A Panel Data Analysis of Affine

Term Structure Models", Maastricht Univ. DP.

Bansal, R., and H., Zhou (2000) : "Term Structure of Interest Rates with Regime Shifts", Working Paper, Duke University.

Berardi, A., and M., Esposito (1999) : "A Base Model for Multifactor Specifications of the Term Structure", Economic Notes, 28, 145-170.

Bjork, T., Kabanov, Y., and W., Runggaldier (1997) : "Bond Market Structure in the Presence of Marked Point Processes", Mathematical Finance, 7, 211-239.

Brandt, M., and P., Santa-Clara (2001) : "Simulated Likelihood Estimation of Diffusions with an Application to Exchange Rates Dynamics in Incomplete Markets", Journal of Financial Economics, 63, 161-210.

Brennan, M., and E., Schwartz (1979) : "A Continuous Time Approach to the Pricing of Bonds", Journal of Banking and Finance, 3, 135-155.

Brown, R. and S., Schaefer (1994) : "Interest Rate Volatility and the Shape of the Term Structure", Philosophical Transactions of the Royal Society : Physical Sciences and Engineering, 347, 449-598.

Buhlman, N., Delbean, F., Embrechts, P., and A., Shyraev (1996) : "No Arbitrage, Change of Measure and Conditional Esscher Transforms in a Semi-Martingale Model of Stock Processes", CWI Quarterly, 9, 291-317.

Carrasco, M., and J.P., Florens (2000) : "Generalization of a GMM to a Continuum of Moment Conditions", Econometric Theory, 16, 797-834.

Carrasco, M., and J.P., Florens (2002) : "Efficient GMM Estimation Using the Empirical Characteristic Function", Univ. Toulouse, DP.

Chacko, G., and L., Viceira (2001) : "Spectral GMM Estimation of Continuous Time Processes", Mimeo Harvard Univ.

Chan, K., Karolyi, A., Longstaff, F., and A., Sanders (1992) : "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate", Journal of Finance, 47, 1209-1227.

Chen, L., (1996) : "Stochastic Mean and Stochastic Volatility : A Three-Factor Model of the Term Structure of Interest Rates and its Application to the Pricing of Interest Rate Derivatives", Oxford, Blackwell Publishers.

Chen, R., and L., Scott (1993) : "Maximum Likelihood Estimation for a Multifactor Equilibrium Model of the Term Structure of Interest Rates", Journal of Fixed Income, 3, 14-31.

Chen, R., and L., Scott (1995) : "Multifactor Cox-Ingersoll-Ross Models of the Term Structure : Estimate and Tests from a Kalman Filter Model", Working Paper, University of Georgia.

Cheng, S. (1987) : "On the Feasibility of Arbitrage Based Option Pricing when Stochastic Bond Prices are Involved", DP Columbia University.

Cochrane, J. (2001) : "Asset Pricing", Princeton University Press.

Conley, T., Hansen, L., Luttmer, E., and J., Scheinkman (1996) : "Estimating Subordinated Diffusions from Discrete Data", DP University of Chicago.

Courtadon, G. (1982) : "The Pricing of Options on Default Free Bonds", Journal of Financial and Quantitative Analysis, 17, 75-100.

Cox, J., Ingersoll, J., and S., Ross (1985) : "A Theory of the Term Structure of Interest Rates", Econometrica, 53, 385-408.

Dai, Q., and K., Singleton (2000) :"Specification Analysis of Affine Term Structure Models", Journal of Finance, 55, 1943-1978 .

Dai, Q., and K., Singleton (2002) : "Expectation Puzzles, Time-Varying Risk Premia, and Dynamic Models of the Term Structure", Journal of Financial Economics, 63, 415-441.

Darolles, S., Gourieroux, C., and J., Jasiak (2001) : "Compound Autoregressive Models", CREST DP, 2001-20, revided version entitled "Structural Laplace Transform and Compound Autoregressive Models".

De-Jong, F. (1997) : "Time Series and Cross Section Information in Affine Term Structure Models", Working paper, University of Amsterdam.

De Munnik, J., and P., Schotman (1994) :"Cross Sectional Versus Time Series Estimation of Term Structure Models : Empirical Results for the Dutch Bond Market", Journal of Banking and Finance, 18, 997-1025.

Dothan, L. (1978) : "On the Term Structure of Interest Rates", Journal of Financial Economics, 6, 59-69.

Duarte, J. (1999) : "The Relevance of the Price of the Risk in Affine Term Structure Models", Working paper, Chicago GSB.

Duffee, G. (1999) : "Forescasting Future Interest Rates : An Affine Models Failures ?", Working paper, UC Berkeley.

Duffee, G. (2001) : "Term Premia and Interest Rate Forecasts in Affine Models", Journal of Finance, 57,

Duffie, D., Filipovic, D., and W., Schachermayer (2001) : "Affine Processes and Applications in Finance", Working Paper, Stanford University.

Duffie, D., and R., Kan (1996) :"A Yield Factor Model of Interest Rates", Mathematical Finance, 379-406.

Duffie, D., Pan, J., and K., Singleton (2000) : "Transform Analysis and Asset Pricing for Affine Jump Diffusions", Econometrica, 68, 1343-1376.

Dybvig, P., Ingersoll, J., and S., Ross (1996) : "Long Forward and Zero Coupon Rates can Never Fall", Journal of Business, 69, 1-25.

El Karoui, N., Frachot, A., and H., Geman (1998) : "On the Behaviour of Long Zero Coupon Rates in a No Arbitrage Framework", Review of Derivatives Research, 1, 351-369.

Esscher, F. (1932) : "On the Probability Function in the Collective Theory of Risk", Skandinavisk Aktuariedskrift, 15, 165-195.

Feuerverger, A. (1990) : "An Efficiency Result for the Empirical Characteristic Function in Stationary Time Series Models", The Canadian Journal of Statistics, 18, 2, 155-161.

Feuerverger, A., and P., Mc Dunnough (1981)a : "On Some Fourier Methods for Inference", Journal of the American Statistical Association, 76, 319-387.

Feuerverger, A., and P., Mc Dunnough (1981)b : "On the Efficiency of the Empirical Characteristic Function Procedures", Journal of the Royal Statistical Society, B43, 20-77.

Filipovic, D. (2000) : "Exponential Polynomial Families and the Term Structure of Interest Rates", Bernoulli, 6, 1-27.

Filipovic, D. (2001) : "A General Characterization of one Factor Affine Term Structure Models", Finance Stoch., 5, 389-412.

Fong, H. and O., Vasicek (1991) : "Fixed Income Volatility Management", Journal of Portfolio Management, 17, 41-46.

Gallant, R., and G., Tauchen (1996)a : "Which Moments to Match", Econometric Theory, 12, 657-681.

Gallant, R., and G., Tauchen (1996)b : "Reprojecting Partially Observed Systems with Applications to Interest Rate Diffusions", DP Duke University.

Geyer, A., and S., Pichler (1999) : "A State Space Approach to Estimate and Test Multifactor Cox-Ingersoll-Ross Models of the Term Structure", Journal of Financial Research, 22, 107-130.

Ghysels, E., and S., Ng (1998) : "A Semi-Parametric Factor Model of Interest Rates and Tests of the Affine Term Structure", Review of Economic and Statistics, 80, 535-548.

Gibbons, M., and K., Ramaswamy (1993) : "A Test of the Cox-Ingersoll-Ross Model of the Term Structure", Review of Financial Studies, 6, 619-658.

Goldstein, R. (2000) : "The Term Structure of Interest Rate as a Random Field", Review of Financial Studies, 13, 365-38?.

Gourieroux, C., and J., Jasiak (2001)a : " Financial Econometrics", Princeton University Press.

Gourieroux, C., and J., Jasiak (2001)b : "Autoregressive Gamma Processes", CREST DP.

Gourieroux, C., and A., Monfort (1989) : "A General Framework for Testing a Null Hypothesis in a Mixed Form", Econometric Theory, 5, 63-82.

Gourieroux, C., and A., Monfort (1995) : "Statistics and Econometric Models", Vol 2, Cambridge Univ. Press.

Gourieroux, C., and A., Monfort (2001)a : "Pricing with Splines", CREST, DP.

Gourieroux, C., and A., Monfort (2001)b : "Econometric Specifications of Stochastic Discount Factor Models", CREST, DP.

Gourieroux, C., Monfort, A., and E., Renault (1993) : "Indirect Inference", Journal of Applied Econometrics, 8, 85-118.

Granger, C. (1969) : "Investigating Causal Relations by Econometric Models and Cross Spectral Methods", Econometrica, 37, 424-439.

Hansen, L. and S., Richard (1987) : "The Role of Conditioning Information in Deducing Testable Restrictions Implied by Dynamic Asset Pricing Models", Econometrica, 55, 587-613.

Hansen, L., and J., Scheinkman (2002) : "Semi-Group Pricing", Univ. of Chicago, DP.

Harrison, M., and D., Kreps (1979) : "Martingales and Arbitrage in Multiperiod Securities Markets", Journal of Economic Theory, 20, 381-408.

Harrison, J., and S., Pliska (1981) : "Martingales and Stochastic Integrals in the Theory of Continuous Trading", Stochastic Processes and Their Applications, 11, 215-260.

Heath, D., Jarrow, R., and A., Morton (1990) : "Bond Pricing and the Term Structure of Interest Rates : A Discrete Time Approximation", Journal of Financial and Quantitative Analysis, 25, 419-440.

Heath, D., Jarrow, R., and A., Morton (1992) : "Bond Pricing and the Term Structure of Interest Rates : A New Methodology", Econometrica, 60, 77-105.

Ho, T., and S., Lee (1986) : "Term Structure Movements and Pricing Interest Rate Contingent Claims", Journal of Finance, 41, 1011-1028.

Jamshidian, F. (1996) : "Bond, Futures and Option Valuation in the Quadratic Interest Rate Model", Applied Mathematical Finance, 3, 93-115.

Jamshidian, F. (1989) : "An Exact Bond Option Formula", Journal of Finance, 44, 205-209.

Joe, H. (1997) : "Multivariate Models and Dependence Concepts", Monographs on Statistics and Applied Probability, 73, Chapman and Hall.

Langetieg, T. (1980) : "A Multivariate Model of the Term Structure", Journal of Finance, 35, 71-97.

Leipold, M., and L., Wu (2002) : "Asset Pricing Under the Quadratic Class", JFQA, forthcoming.

Long, J. (1974) : "Stock Prices, Inflation and the Term Structure of Interest Rates", Journal of Financial Economics.

Longstaff, F., and E., Schwartz (1992) : "Interest Rate Volatility and the Term Structure : A Two Factor General Equilibrium Model", Journal of Finance, 47, 1259-1282.

Martellini, L., and P., Priaulet (2001) : "Fixed Income Securities", Wiley.

Merton, R. (1974) : "On the Pricing of Corporate Debt : The Risk Structure of Interest Rates", Journal of Finance, 29, 449-470.

Nelson, J., and S., Schaefer (1983) : "The Dynamics of the Term Structure and Alternative Portfolio Immunization Strategies", in Innovations in Bond Portfolio Management : Duration Analysis and Immunization, Greenwich, JAI Press.

Pearson, N., and T., Sun (1994) : "Exploiting the Conditional Density in Estimating the Term Structure : An Application to the Cox-Ingersoll-Ross Model", Journal of Finance, 49, 1279-1329.

Pedersen, A. (1995) : "A New Approach to Maximum Likelihood Estimation for Stochastic Differential Equations Based on Discrete Observations", Scandinavian Journal of Statistics,??;55-71.

Piazzesi, M. (2001) : "Affine Term Structure Models", in Handbook of Financial Econometrics, ed. Ait-Sahalia, Y. and L., Hansen, Elsevier.

Polimenis, V. (2001) : "Essays in Discrete Time Asset Pricing", Ph. D Thesis, Wharton School, University of Pennsylvania.

Sato, K. (1999) : "Levy Processes and Infinitely Divisible Distributions", Cambridge University Press.

Schaefer, S., and E., Schwartz (1987) : "Time Dependent Variance and the Pricing of Bond Options", Journal of Finance, 423, 1113-1128.

Shyraev, A. (1999) : "Essentials of Stochastic Finance : Facts, Models, Theory", World Scientific Publishing, London.

Singleton, K. (2001) : "Estimation of Affine Diffusion Models Based on the Empirical Characteristic Function", Journal of Econometrics, 102, 111-141.

Stanton, R. (1997) : "A Nonparametric Model of the Term Structure Dynamics and the Market Price of Interest Rate Risk", Journal of Finance, 52,

Stutzer, M. (1996) : "A Simple Nonparametric Approach to Derivative Security Valuation", Journal of Finance, 51, 1633-1652.

Vasicek, O. (1977) : "An Equilibrium Characterization of the Term Structure", Journal of Financial Economics, 5, 177-188.

Warde, W., and S., Katti (1971) : "Infinite Divisibility of Discrete Distributions, II", Annals of Mathematical Statistics, 42, 1088-1090.

Appendix 1
Proof of Proposition 3

$$
\begin{aligned}
B(t,h) &= E_t\left[M_{t,t+1}\dots M_{t+h-1,t+h}\right] \\
&= E_t\left[M_{t,t+1}B(t+1,h-1)\right] \\
&= E_t \exp\left[\alpha r_{t+2} + \delta' f_{t+1} - a(\alpha,\delta)' X_t - b(\alpha,\delta) - r_{t+1}\right. \\
&+ \left. c'_{h-1} X_{t+1} + d_{h-1}\right] \\
&= \exp\left[-a(\alpha,\delta)' X_t - b(\alpha,\delta) - r_{t+1} + d_{h-1}\right] \\
& \quad E_t\left[\exp\left[c_{h-1} + \begin{pmatrix} \alpha \\ \delta \end{pmatrix}\right]' X_{t+1}\right] \\
&= \exp\left[-a(\alpha,\delta)' X_t - b(\alpha,\delta) - r_{t+1} + d_{h-1}\right. \\
& \quad \left. + a\left(c_{h-1} + \begin{pmatrix} \alpha \\ \delta \end{pmatrix}\right)' X_t + b\left(c_{h-1} + \begin{pmatrix} \alpha \\ \delta \end{pmatrix}\right)\right].
\end{aligned}
$$

The result follows by identifying the coefficients.

QED

Appendix 2

Constraints in the two factor case

As an illustration let us consider a model with two state variables, that are the riskfree rate r_{t+1} and a nonnegative factor f_t. From (2.7), (2.8) and Proposition 3 we get:

$$
\begin{aligned}
r_{t,t+2} &= -\frac{c_{1,2}}{2} r_{t+1} - \frac{c_{2,2}}{2} f_t - \frac{d_2}{2} \\
&= \frac{a_1(\alpha,\delta) - a_1(\alpha-1,\delta)+1}{2} r_{t+1} + \frac{a_2(\alpha,\delta) - a_2(\alpha-1,\delta)}{2} f_t + \frac{b(\alpha,\delta)-b(\alpha-1,\delta)}{2} \\
&= \gamma_1 r_{t+1} + \gamma_2 f_t + \gamma_3, \text{ say.}
\end{aligned}
$$

Whenever functions a_1, a_2, b are strictly increasing with respect to u [see Condition C1], the affine space can be generated by r_{t+1} and $r_{t,t+2}$. Moreover $r_{t,t+2}$ automatically takes nonnegative values.

Let us now consider the joint dynamics of the two rates. The conditional Laplace transform of $r_{t+2}, r_{t+1,t+3} = R_{t+1}$ is:

$$
\begin{aligned}
& E_t \exp(u r_{t+2} + v R_{t+1}) \\
&= E_t \exp[u r_{t+2} + v(\gamma_1 r_{t+2} + \gamma_2 f_{t+1} + \gamma_3)] \\
&= \exp[a_1(u+\gamma_1 v, \gamma_2 v) r_{t+1} + a_2(u+\gamma_1 v, \gamma_2 v) f_t + b(u+\gamma_1 v, \gamma_2 v) + \gamma_3 v] \\
&= \exp[a_1(u+\gamma_1 v, \gamma_2 v) r_{t+1} + a_2(u+\gamma_1 v, \gamma_2 v)[R_t/\gamma_2 - (\gamma_1/\gamma_2) r_{t+1} - \gamma_3/\gamma_2] \\
&\quad + b(u+\gamma_1 v, \gamma_2 v) + \gamma_3 v] \\
&= \exp[a_1^*(u,v) r_{t+1} + a_2^*(u,v) R_t + b^*(u,v)],
\end{aligned}
$$

where

$$
\begin{aligned}
a_1^*(u,v) &= a_1(u+\gamma_1 v, \gamma_2 v) - (\gamma_1/\gamma_2) a_2(u+\gamma_1 v, \gamma_2 v), \\
a_2^*(u,v) &= a_2(u+\gamma_1 v, \gamma_2 v)(1/\gamma_2), \\
b^*(u,v) &= b(u+\gamma_1 v, \gamma_2 v) + \gamma_3 v - (\gamma_3/\gamma_2) a_2(u+\gamma_1 v, \gamma_2 v).
\end{aligned}
$$

Similarly the stochastic discount factor becomes:

$$
\begin{aligned}
M_{t,t+1} &= \exp(\alpha r_{t+2} + \delta f_{t+1} + \beta_t) \\
&= \exp(\alpha^* r_{t+2} + \delta^* R_{t+1} + \beta_t^*),
\end{aligned}
$$

where $\alpha^* = \alpha - \delta\gamma_1/\gamma_2, \delta^* = \delta/\gamma_2$.

It follows directly from this example that:

$$\begin{aligned}
a_1^*(\alpha^*, \delta^*) - a_1^*(\alpha^* - 1, \delta^*) &= -1, \\
a_2^*(\alpha^*, \delta^*) - a_2^*(\alpha^* - 1, \delta^*) &= 2, \\
b^*(\alpha^*, \delta^*) - b^*(\alpha^* - 1, \delta^*) &= 0,
\end{aligned}$$

and that the arbitrage free restriction is automatically satisfied, since:

$$\begin{aligned}
& \frac{a_1^*(\alpha^*, \delta^*) - a_1^*(\alpha^* - 1, \delta^*) + 1}{2} r_{t+1} + \frac{a_2^*(\alpha^*, \delta^*) - a_2^*(\alpha^* - 1, \delta^*)}{2} R_t \\
+ & \frac{b^*(\alpha^*, \delta^*) - b^*(\alpha^* - 1, \delta^*)}{2} = R_t.
\end{aligned}$$

Appendix 3
Proof of Proposition 6

The proof of similar to the proof of Appendix 1 after noting that :

$$C_t^*(z,h) = E_t\left[M_{t,t+1}C_{t+1}^*(z,h-1)\right].$$

i) Recursive equation

We get :

$$\begin{aligned}
& E_t[M_{t,t+1}C_{t+1}^*(z,h-1)] \\
= \; & E_t\{\exp(\gamma_0^* + \gamma_1^{*'}X_t + \gamma_2^{*'}X_{t+1})\exp[c^*(h-1,z)'X_{t+1} + d^*(h-1,z)]\} \\
= \; & \exp[\gamma_0^* + \gamma_1^{*'}X_t + d^*(h-1,z)]E_t\{\exp[(c^*(h-1,z)+\gamma_2^*)'X_{t+1}]\} \\
= \; & \exp\{(a^*[c^*(h-1,z)+\gamma_2^*] + \gamma_1^*)'X_t + b^*[c^*(h-1,z)+\gamma_2^*] + \gamma_0^* + d^*(h-1,z)\} \\
= \; & \exp\{(A^*[c^*(h-1,z)] - \begin{pmatrix} 1 \\ 0 \end{pmatrix})'X_t + d^*(h-1,z) + B^*[c^*(h-1,z)]\}.
\end{aligned}$$

Thus we deduce :

$$\begin{aligned}
c^*(h,z) &= A^*[c^*(h-1,z)] - \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
d^*(h,z) &= d^*(h-1,z) + B^*[c^*(h-1,z))].
\end{aligned}$$

ii) Initial conditions

For $h = 1$, we get :

$$
\begin{aligned}
C_t^*(z,1) &= E_t[M_{t,t+1} \exp z' X_{t+1}^*] \\
&= \exp(\gamma_0^* + \gamma_1^{*'} X_t) E_t[\exp(z+\gamma_2^*)' X_{t+1}^*] \\
&= \exp[(a^*(z+\gamma_2^*) + \gamma_1^*)' X_t + b^*(z+\gamma_2^*) + \gamma_0^*] \\
&= \exp\{[A^*(z) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}]' X_t + B^*(z)\}.
\end{aligned}
$$

Appendix 4
Term structure for the compound Poisson process

We have to compute the expressions of the coefficients c_h and d_h. The coefficient c_h satisfies the recursive equation :

$$c_h = \frac{1}{\gamma} \log \left[\frac{\pi \exp(\gamma c_{h-1} + \gamma \alpha) + 1 - \pi}{\pi \exp \gamma \alpha + 1 - \pi} \right] - 1$$

$$\iff \quad \exp \gamma \exp(\gamma c_h) = \frac{\pi \exp \gamma \alpha}{\pi \exp \gamma \alpha + 1 - \pi} \exp \gamma c_{h-1} + \frac{1 - \pi}{\pi \exp \gamma \alpha + 1 - \pi}.$$

We get a linear recursive equation, whose solution is of the type :

$$\exp(\gamma c_h) = A \left(\frac{\pi \exp[\gamma(\alpha - 1)]}{\pi \exp(\gamma \alpha) + 1 - \pi} \right)^h + B$$

$$\iff \quad c_h = \frac{1}{\gamma} \log \left\{ A \left(\frac{\pi \exp[\gamma(\alpha - 1)]}{\pi \exp(\gamma \alpha) + 1 - \pi} \right)^h + B \right\},$$

for some constants A and B.

Since $\frac{\pi \exp[\gamma(\alpha - 1)]}{\pi \exp(\gamma \alpha) + 1 - \pi} < 1$, c_h/h tends asymptotically to zero and features hyperbolic decay.

The coefficient d_h satisfies the equation :

$$d_h = d_{h-1} + \lambda \exp(\alpha \gamma)(\exp(\gamma c_{h-1}) - 1),$$

and can be written as :

$$d_h = C + Dh + E \left(\frac{\pi \exp[\gamma(\alpha - 1)]}{\pi \exp(\gamma \alpha) + 1 - \pi} \right)^h,$$

for some constants C, D, E. Thus d_h/h tends to a limit D asymptotically.

Appendix 5
Discrete and continuous time affine processes

The class of continuous time (c.t.) affine processes has been introduced by Duffie, Filipovic, Schachermayer (2001). Let us recall that a c.t. Markov process is affine if :

$$E_t(\exp z'X_{t+h}) = \exp[a_h(z)'X_t + b_h(z)], \forall z \in C, \forall t \in I\!R, \forall h \in I\!R^+.$$

(see Definition (2.1), Duffie et alii (2001)). In particular this condition is satisfied when t and h are integers. We deduce that any discretely sampled c.t. affine process is a compound autoregressive process.

It is known that the class of c.t. affine processes is rather small. "Roughly speaking the c.t. affine processes with state space $I\!R^n_+$ are branching process with immigration and those with state space $I\!R^n$ are of Ornstein-Uhlenbeck type" (Duffie et alii (2001)). In fact these processes have to be infinitely decomposable. This decomposability condition is not necessary in discrete time, which explains the much larger number of affine dynamics in discrete time. Let us now provide examples of CAR processes without continuous time interpretation.

i) Gaussian Vector Autoregressive Process

The Laplace transform of a process : $X_t = AX_{t-1} + \varepsilon_t, \varepsilon_t \sim N(0, Id)$ is :

$$E_t(\exp z'X_{t+1}) = \exp[z'AX_t + \frac{z'z}{2}],$$

and satisfies the CAR condition for any matrix A

A necessary and sufficient condition for infinite decomposability is the existence of a matrix $\wedge$ such that $A = \exp -\wedge$. In the one dimensional case the gaussian AR (1) process is given by $X_t = \rho X_{t-1} + \varepsilon_t$, with $|\rho| < 1$ to ensure the stationarity. It is decomposable if $\rho > 0$, (and is the discretized Ornstein-Uhlenbeck process), not decomposable if $\rho < 0$.

ii) Markov of order p.

The class of CAR processes is compatible with autoregressive lags larger than 2. Indeed, let us assume that $X_t = (r_{t+1}, r_t)'$ (say). The Laplace transform is :

$$\begin{aligned} E_t(\exp z' X_{t+1}) &= E_t[\exp(z_1 r_{t+2} + z_2 r_{t+1})] \\ &= \exp\{a(z)' X_t + b(z)\} \\ &= \exp[a_1(z) r_{t+1} + a_2(z) r_t + b(z)]. \end{aligned}$$

In particular : $E_t \exp(z_1 r_{t+2}) = \exp[a_1(z_1, 0) r_{t+1} + a_2(z_1, 0) r_t + b(z_1, 0)]$ and the short term interest rate is Markov of order 2. Such a process cannot be the discretized version of a c.t. process of order 1.

iii) Count processes

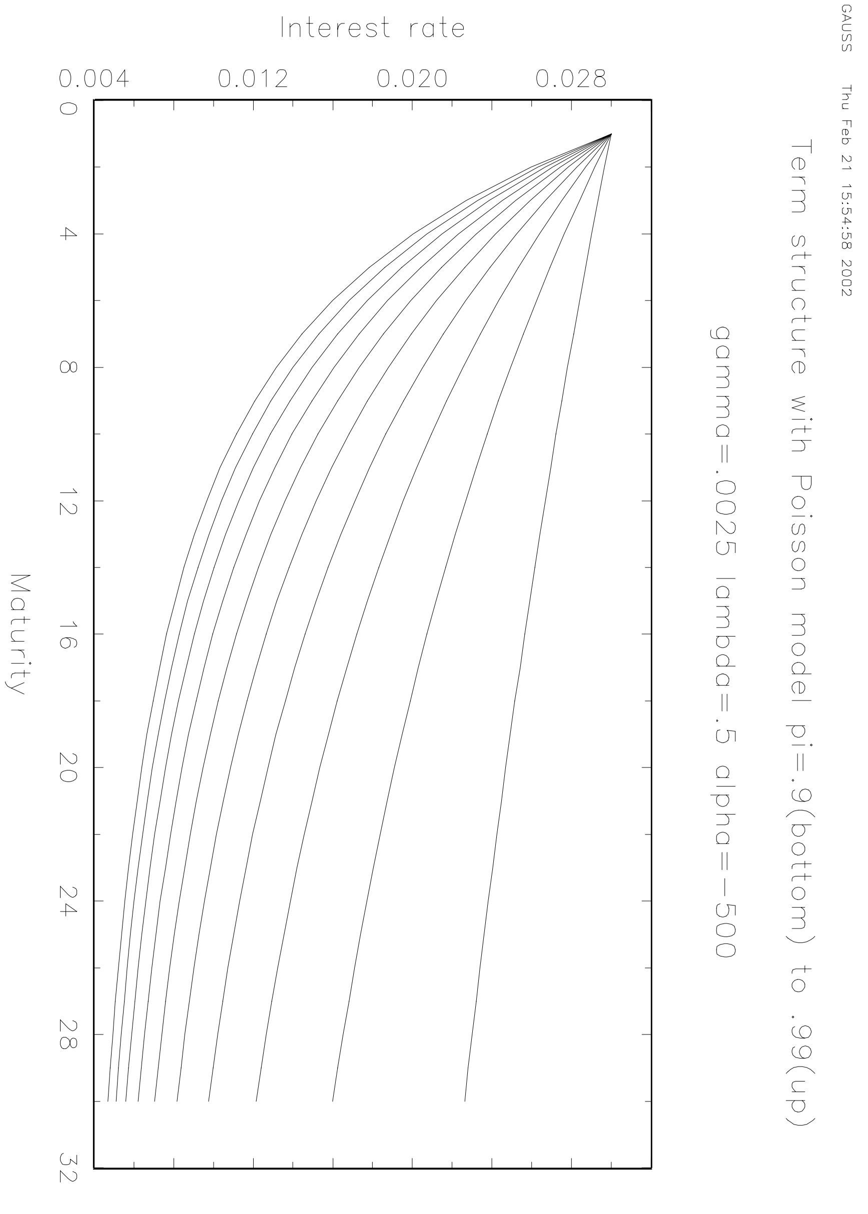
CAR specifications for count data time series are easily derived in the following way. Let us consider independent discrete variables $Z_{i,t}, i, t$ varying (ε_t, t varying) with identical distribution on $\mathbb{N}$ with Laplace transform $\exp a(z)$ [resp. $\exp b(z)$]. We assume that the variables $(Z_{i,t})$ and (ε_t) are independent. The one dimensional process X_t defined by :

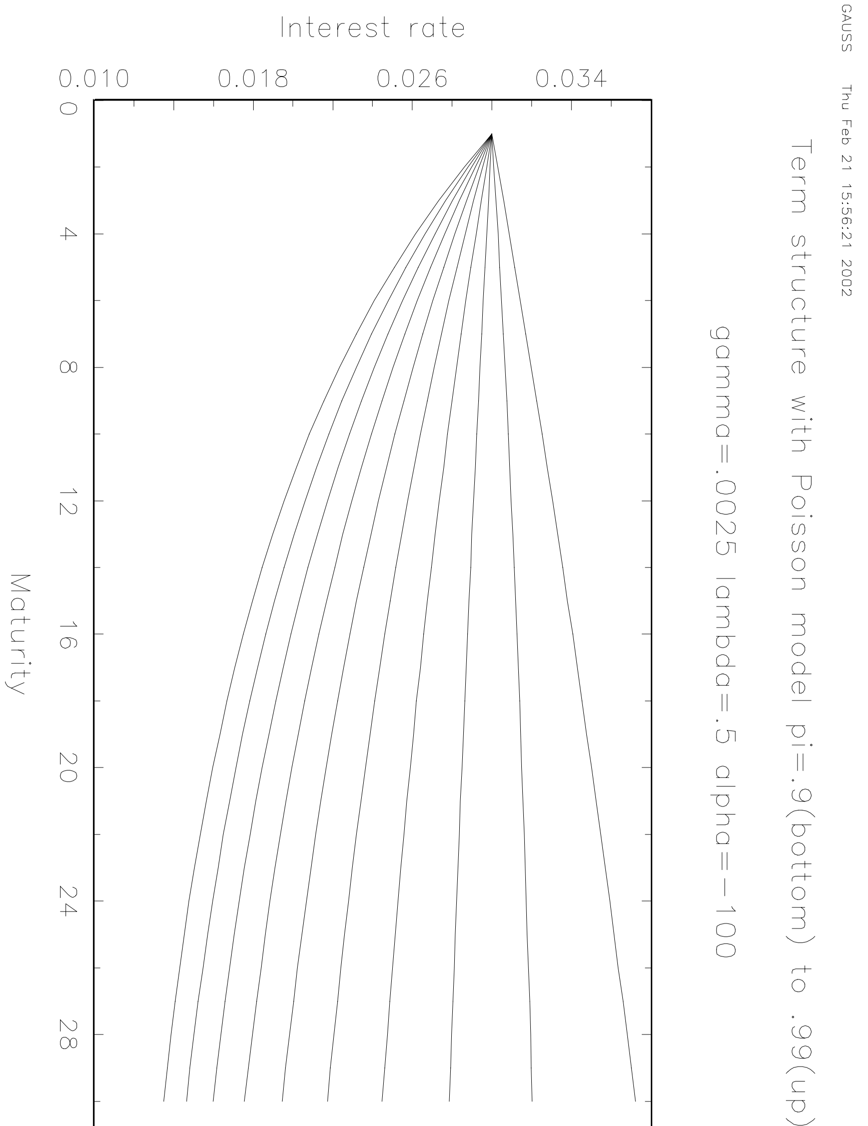
$$X_t = \sum_{i=1}^{X_{t-1}} Z_{i,t} + \varepsilon_t,$$

admits the Laplace transform :

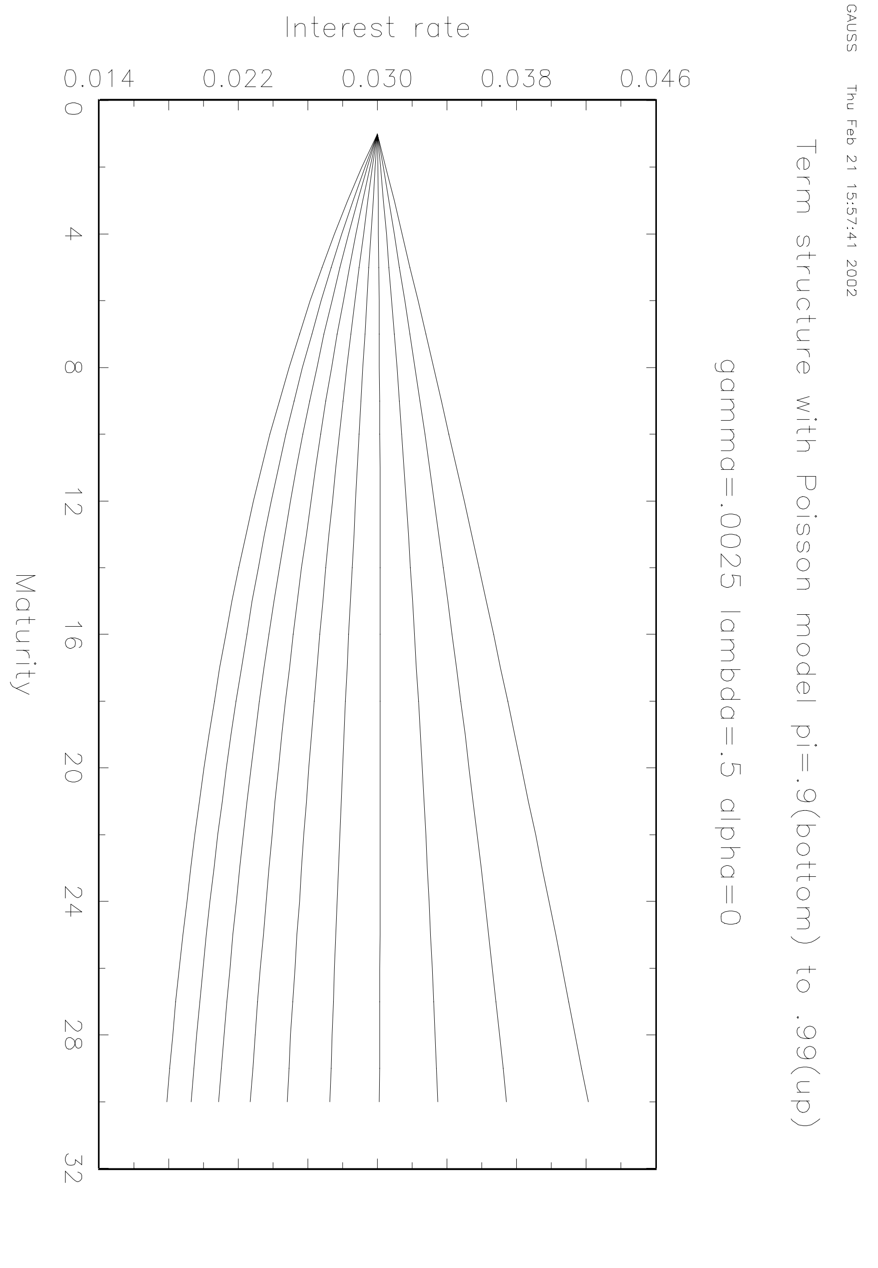
$$E_t[\exp z X_{t+1}] = \exp[a(z) X_t + b(z)].$$

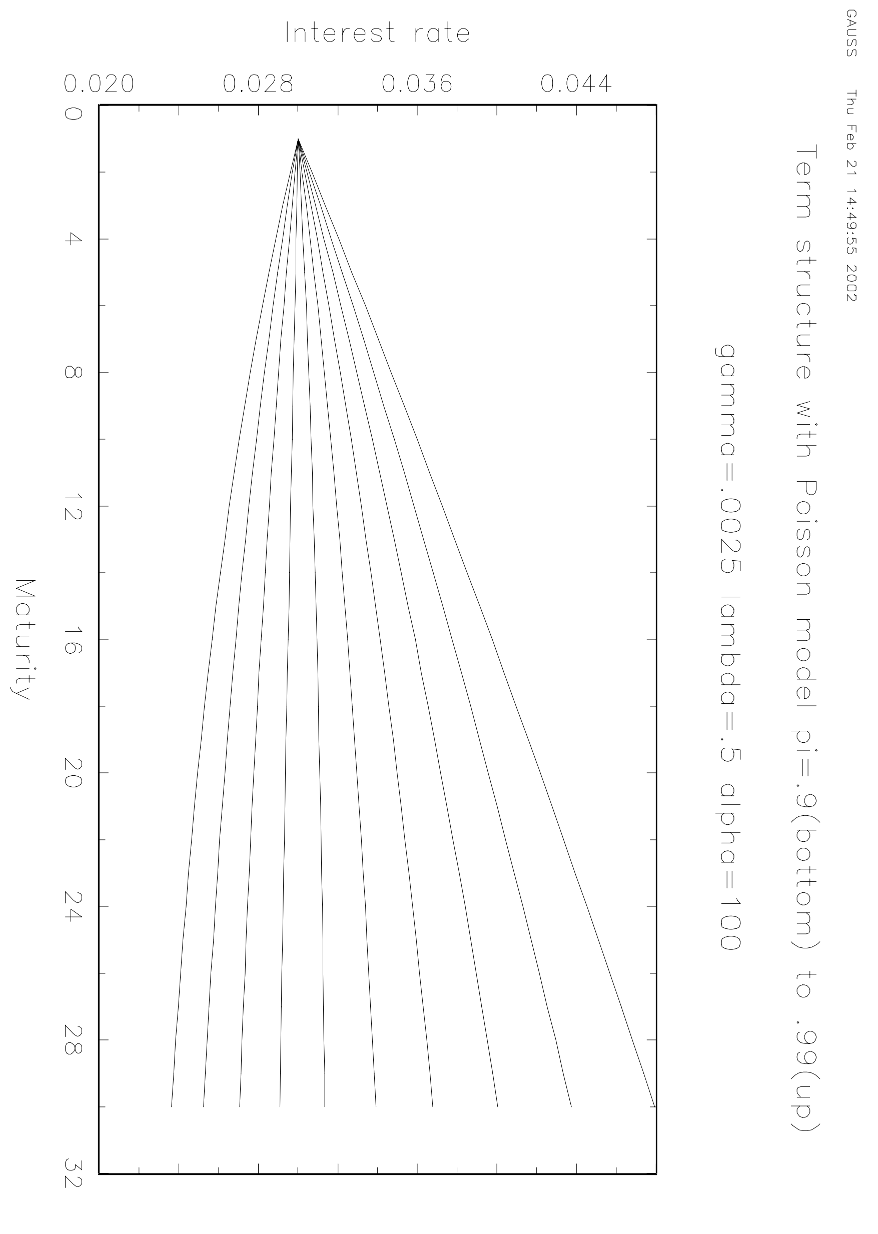
It is important to note that the discrete distributions a, b can be chosen arbitrarily.


GAUSS Thu Feb 21 15:54:58 2002
Term structure with Poisson model pi=.9(bottom) to .99(up)
gamma=.0025 lambda=.5 alpha=-500
Interest rate
0.004
0.012
0.020
0.028
Maturity
0
4
8
12
16
20
24
28
32

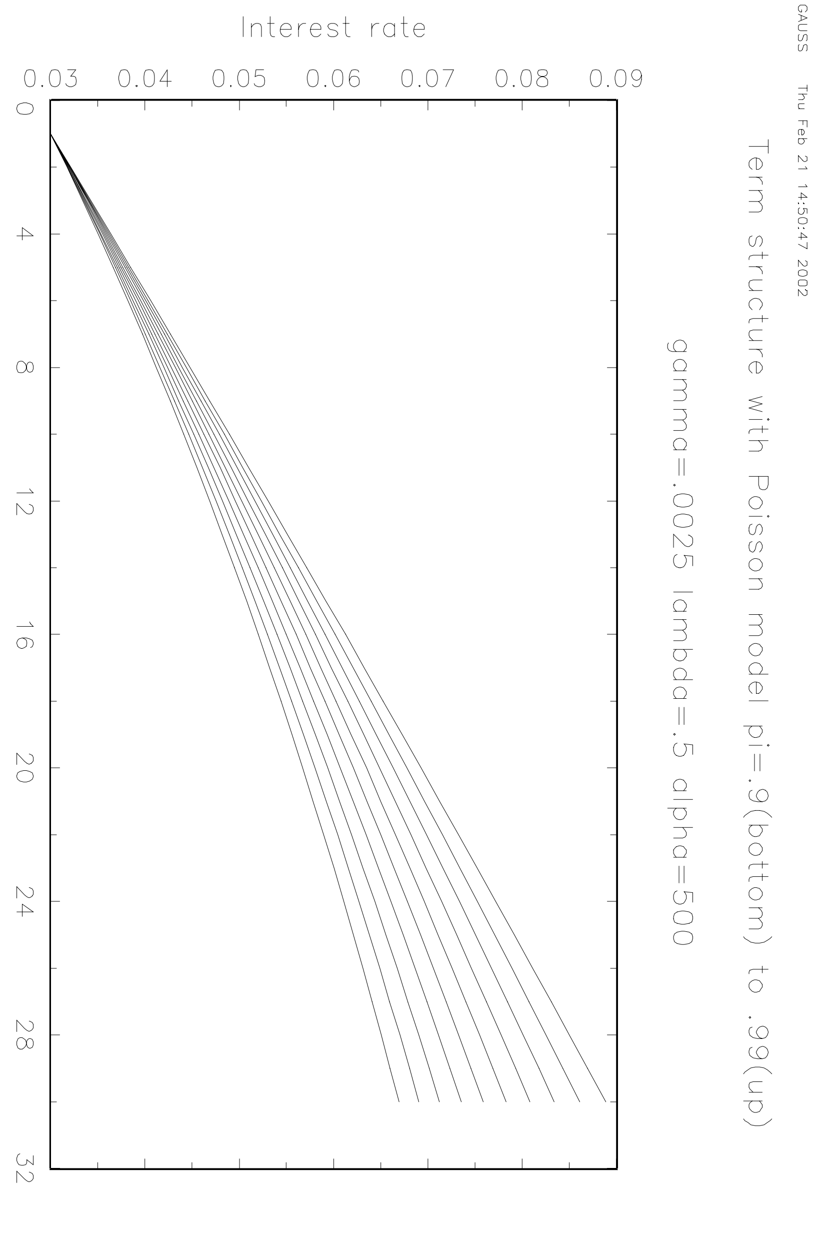


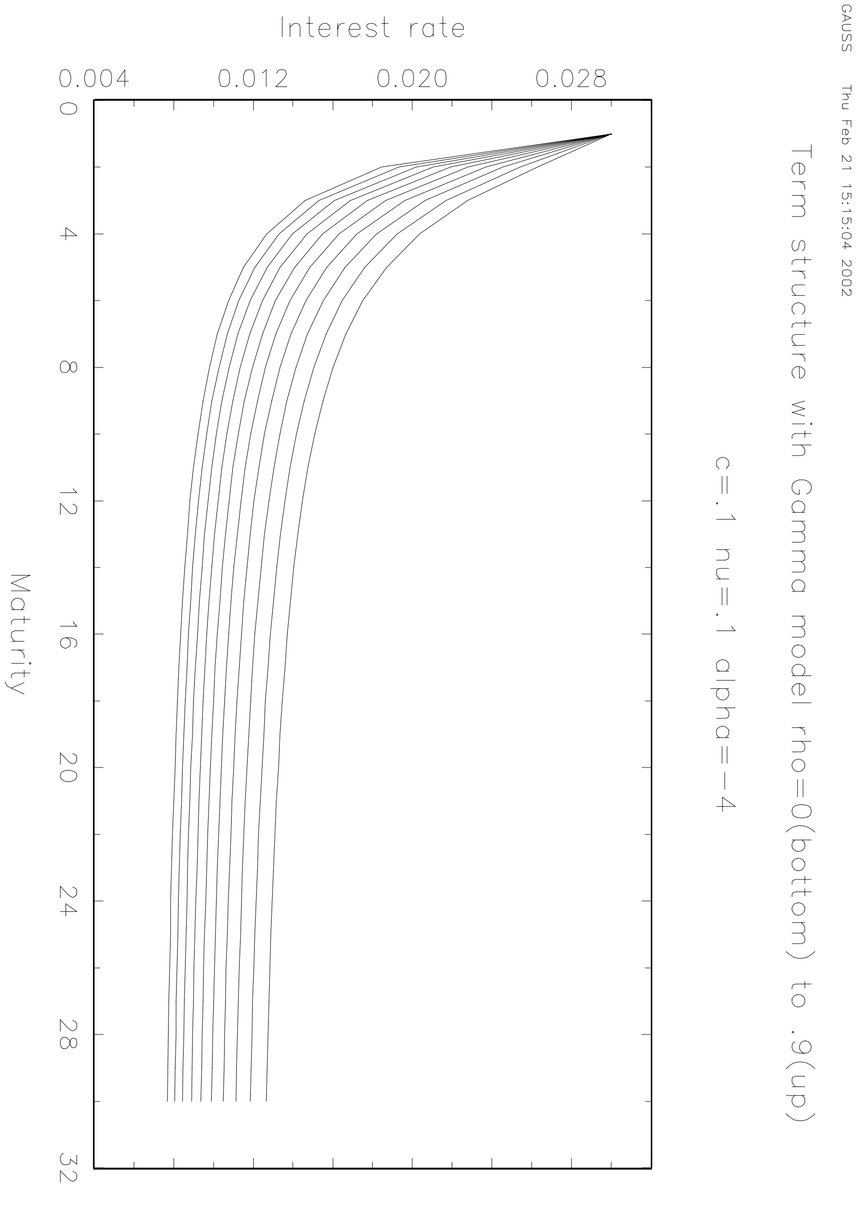
GAUSS Thu Feb 21 15:56:21 2002
Term structure with Poisson model pi=.9(bottom) to .99(up)
gamma=.0025 lambda=.5 alpha=-100
Interest rate
0.010
0.018
0.026
0.034
Maturity
0
4
8
12
16
20
24
28
32

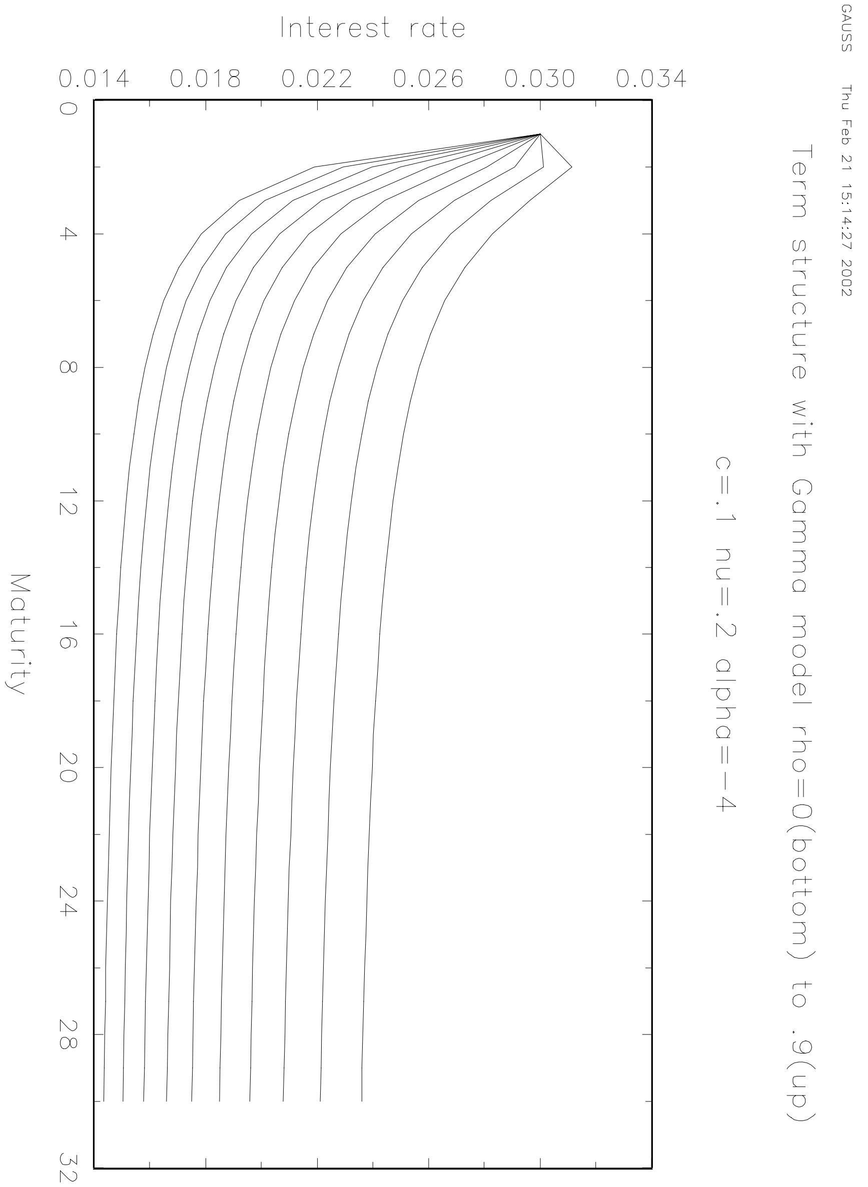

GAUSS Thu Feb 21 15:57:41 2002
Term structure with Poisson model pi=.9(bottom) to .99(up)
gamma=.0025 lambda=.5 alpha=0
Interest rate
0.014
0.022
0.030
0.038
0.046
Maturity
0
4
8
12
16
20
24
28
32

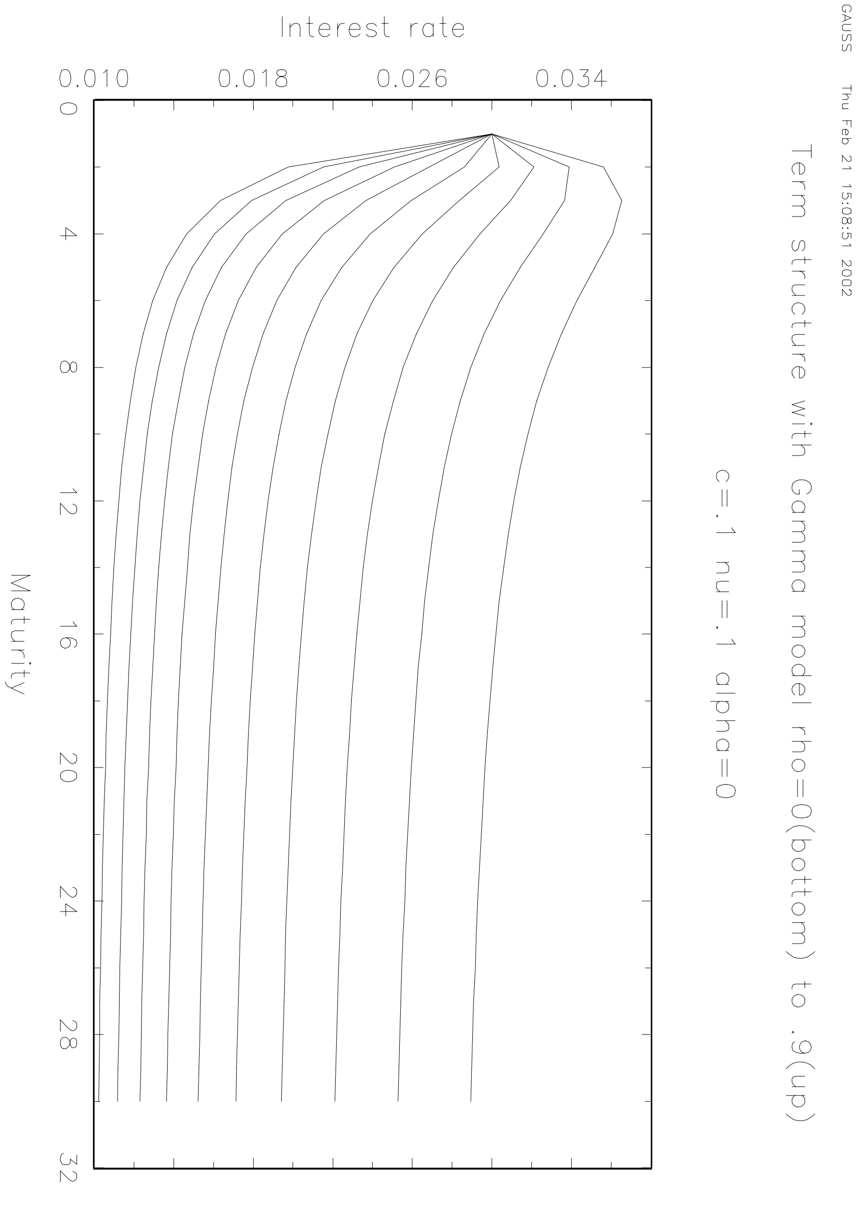


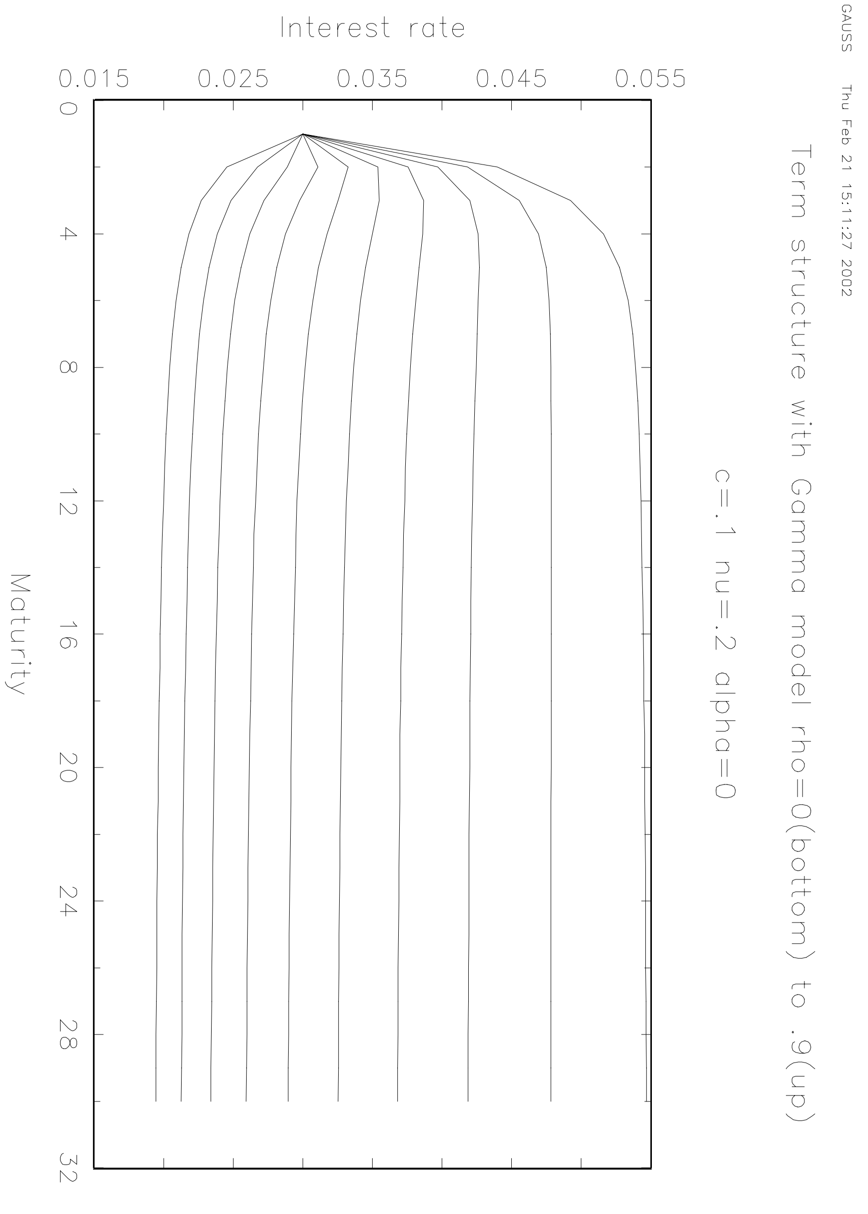
GAUSS Thu Feb 21 14:49:55 2002
Term structure with Poisson model pi=.9(bottom) to .99(up)
gamma=.0025 lambda=.5 alpha=100
Interest rate
0.020
0.028
0.036
0.044
Maturity
0
4
8
12
16
20
24
28
32

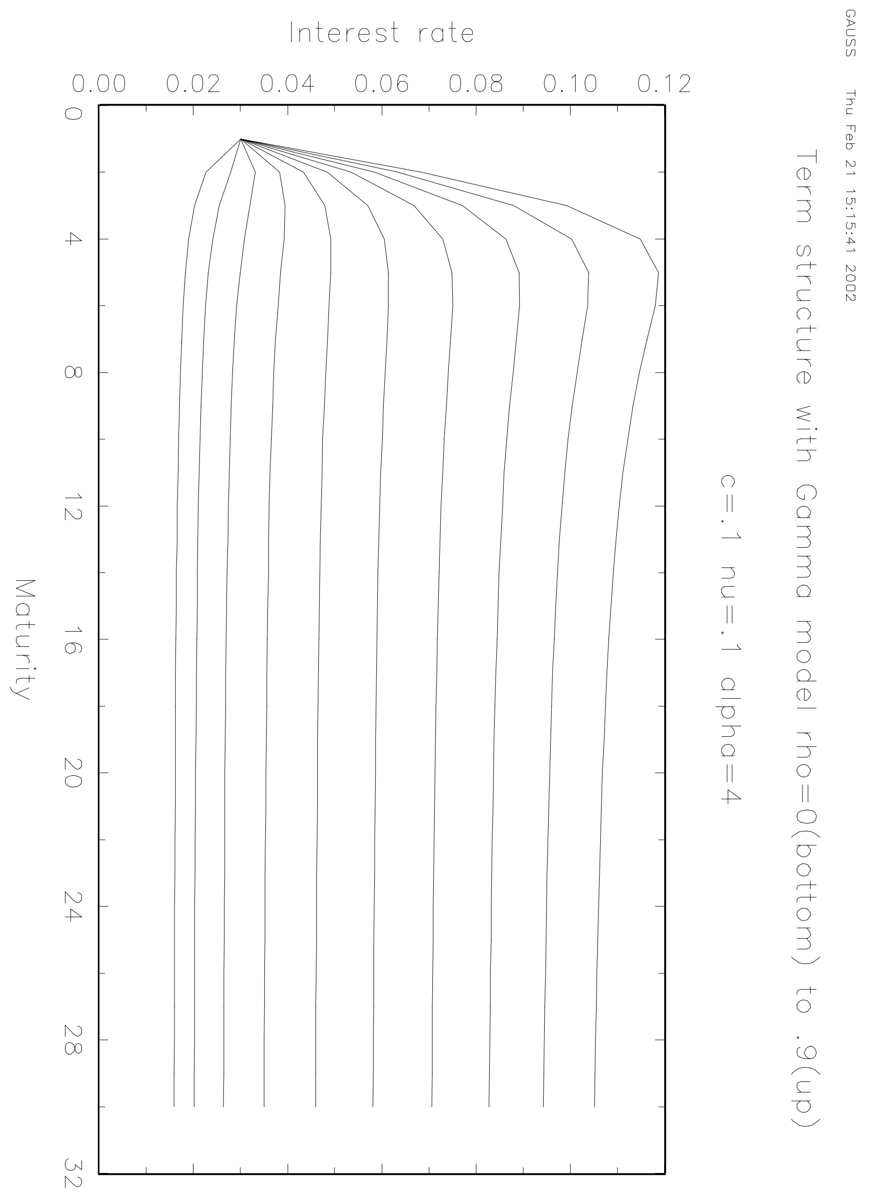

GAUSS Thu Feb 21 14:50:47 2002
Term structure with Poisson model pi=.9(bottom) to .99(up)
gamma=.0025 lambda=.5 alpha=500
Interest rate
0.03
0.04
0.05
0.06
0.07
0.08
0.09
Maturity
0
4
8
12
16
20
24
28
32

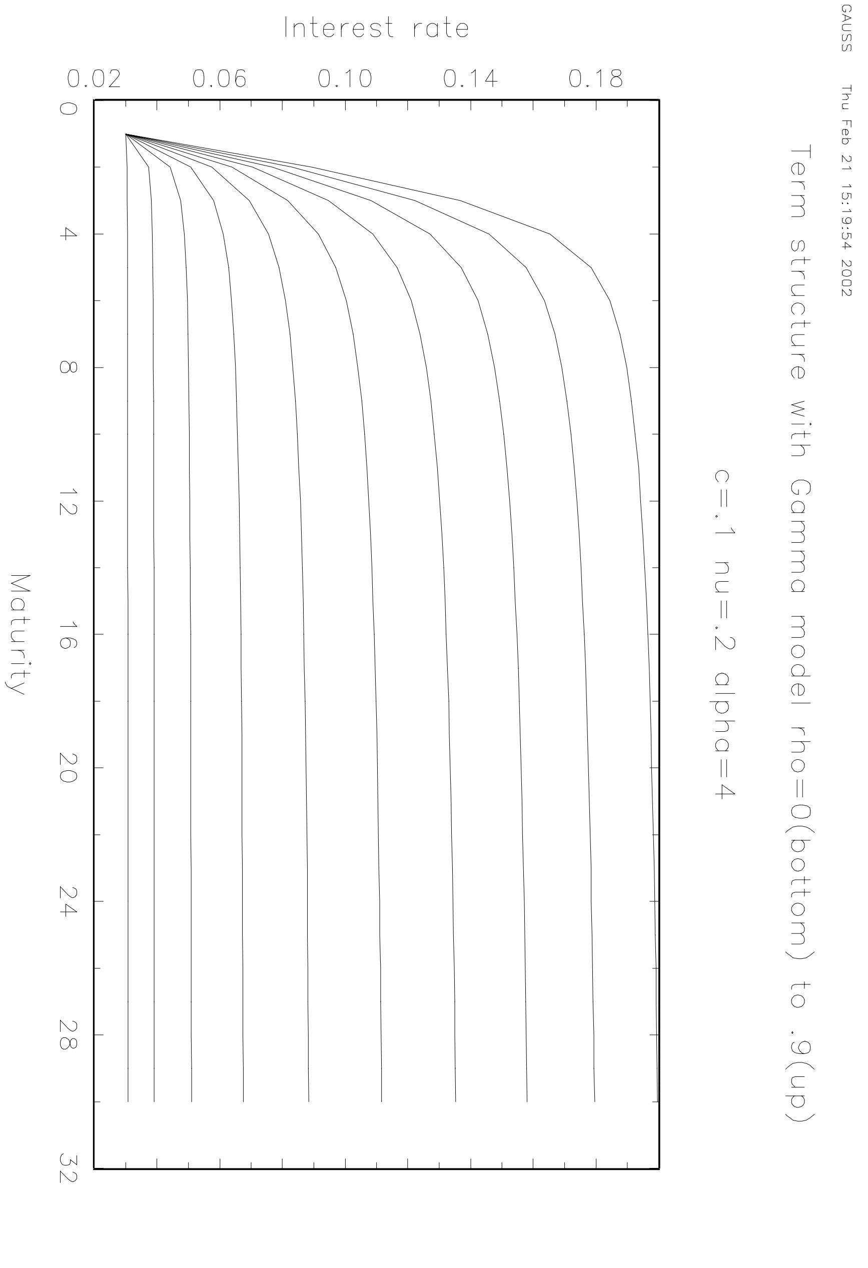

GAUSS Thu Feb 21 15:15:04 2002
Term structure with Gamma model rho=0(bottom) to .9(up)
c=.1 nu=.1 alpha=-4
Interest rate
0.004
0.012
0.020
0.028
Maturity
0
4
8
12
16
20
24
28
32


GAUSS Thu Feb 21 15:14:27 2002
Term structure with Gamma model rho=0(bottom) to .9(up)
c=.1 nu=.2 alpha=-4
Interest rate
0.014
0.018
0.022
0.026
0.030
0.034
Maturity
0
4
8
12
16
20
24
28
32


GAUSS Thu Feb 21 15:08:51 2002
Term structure with Gamma model rho=0(bottom) to .9(up)
c=.1 nu=.1 alpha=0
Interest rate
0.010
0.018
0.026
0.034
Maturity
0
4
8
12
16
20
24
28
32


Term structure with Gamma model rho=0(bottom) to .9(up)
c=.1 nu=.2 alpha=0
GAUSS Thu Feb 21 15:11:27 2002
Interest rate
0.015
0.025
0.035
0.045
0.055
Maturity
0
4
8
12
16
20
24
28
32


GAUSS Thu Feb 21 15:15:41 2002
Term structure with Gamma model rho=0(bottom) to .9(up)
c=.1 nu=.1 alpha=4
Interest rate
0.00 0.02 0.04 0.06 0.08 0.10 0.12
0 4 8 12 16 20 24 28 32
Maturity


GAUSS Thu Feb 21 15:19:54 2002
Term structure with Gamma model rho=0(bottom) to .9(up)
c=.1 nu=.2 alpha=4
Interest rate
0.02
0.06
0.10
0.14
0.18
Maturity
0
4
8
12
16
20
24
28
32