Tutorial lecture 2: System identification

Data driven modeling: Find a good model from noisy data.

- Model class: Set of all a priori feasible candidate systems
- Identification procedure: Attach a system from the model class to time series data y_t, $t = 1, \ldots, T$
  - ⋆ Development of procedures
  - ⋆ Evaluation: Asymptotic properties

Semi-nonparametric approach: Model specification leads to a finite dimensional model (sub)class

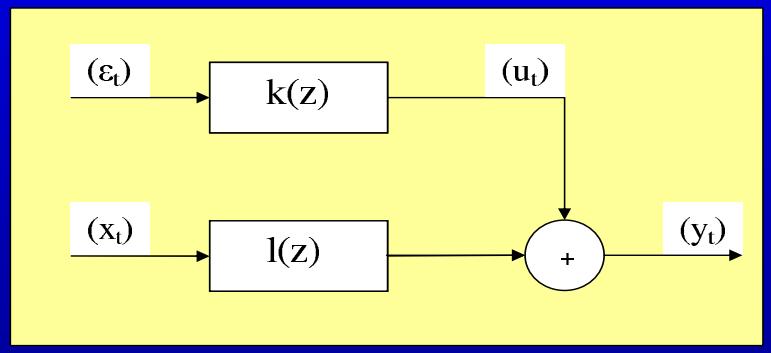
Three modules in semi-nonparametric identification

- Structure theory: Idealized Problem; we commence from the stochastic processes generating the data (or their population moments) rather than from data. Relation between 'external behaviour' and 'internal parameters'.

- Estimation of real valued parameters: Subclass is assumed to be given; parameter space is a subset of an Euclidean space and contains a nonvoid open set. Estimation e.g. by M-estimators.

- Model selection: In general, the orders, the relevant inputs or even the functional forms are not known a priori and have to be determined from data. In many cases, this corresponds to estimating a model subclass within the original model class. This is done, e.g. by estimation of integers, e.g. using information criteria or test sequences.

Linear systems



- (ε_t): white noise (s-dimensional), $\mathbb{E}\varepsilon_t\varepsilon_t' = \Sigma$
- (x_t): (observed) inputs (m-dimensional)
- (u_t): noise to output (s-dimensional), $\mathbb{E}x_t u_s' = 0$
- (y_t): output (s-dimensional)

$l(z)$ is the input-to-output transfer function and $k(z)$ the noise-to-output transfer function.

Relation between second moments and (l, k, Σ)

Relation between second (population) moments of the observations and (l, k, Σ):

$$f_{yx} = l \cdot f_x \quad (1)$$

$$f_y = l \cdot f_x \cdot l^* + \frac{1}{2\pi} k \cdot \Sigma \cdot k^* \quad (2)$$

If $f_x > 0$, then $l = f_{yx} \cdot f_x^{-1}$.

Main model classes for linear systems

AR(X) models: $a(z)y_t = (d(z)x_t) + \varepsilon_t$

- (ε_t): white noise, $\mathbb{E}\varepsilon_t\varepsilon_t' = \Sigma$, $\mathbb{E}x_t\varepsilon_s' = 0$
- $a(z) = \sum_{j=0}^{p} a_j z^j$, $d(z) = \sum_{j=0}^{r} d_j z^j$
- Integer parameters: p, r
- Real valued parameters: $((a_0, \ldots, a_p, d_0, \ldots, d_r), \Sigma)$; free parameters

Model class:
$\{(a_0, \ldots, a_p, d_0, \ldots, d_r) \in \mathbb{R}^{s^2(p+1)+sm(r+1)} | det(a(z)) \neq 0 | z| \leq 1\} \times \underline{\Sigma}$,
$\underline{\Sigma} \subset \mathbb{R}^{s(s+1)/2}$

Relation betw. transfer functions and internal parameters: $l = a^{-1}d$, $k = a^{-1}$

Stability condition: $det(a(z)) \neq 0 | z| \leq 1$

Main model classes for linear systems

ARMA(X) models: $a(z)y_t = (d(z)x_t) + b(z)\varepsilon_t$

- (ε_t): white noise, $\mathbb{E}\varepsilon_t\varepsilon_t' = \Sigma$, $\mathbb{E}x_t\varepsilon_s' = 0$
- $a(z) = \sum_{j=0}^{p} a_j z^j$, $d(z) = \sum_{j=0}^{r} d_j z^j$, $b(z) = \sum_{j=0}^{q} b_j z^j$
- Integer parameters: e.g. p, q, r
- Real valued parameters: $((a_0, \ldots, a_p, b_0, \ldots, b_q, d_0, \ldots, d_r), \Sigma)$; free pars

Relation betw. transfer functions and internal parameters: $l = a^{-1}d$, $k = a^{-1}b$

Stability condition: $det(a(z)) \neq 0 |z| \leq 1$

Miniphase condition: $det(b(z)) \neq 0 |z| \leq (<) 1$

Left coprimeness of (a, b, d) and non-redundancy of dynamics $a_0 = b_0$

Main model classes for linear systems

State Space (StS) models (in innovations form): s_t is the n-dimensional state

$$s_{t+1} = As_t + B\varepsilon_t(+Lx_t) \quad (3)$$

$$y_t = Cs_t + \varepsilon_t(+Dx_t) \quad (4)$$

- Integer parameters: e.g. n
- Real valued parameters: $((A, B, C, L, D), \Sigma)$

Relation betw. transfer functions and internal parameters: $l(z) = D + C(z^{-1}I - A)^{-1}L$, $k(z) = I + C(z^{-1}I - A)^{-1}B$

Stability condition: $|\lambda_{max}(A)| \leq 1$

Miniphase condition: $|\lambda_{max}(A - BC)| < (\leq)1$

Minimality of (A, B, C) (i.e. n is minimal for given $k(z)$) $\Longleftrightarrow$
rk$(B, AB, \ldots, A^{n-1}B)$ =rk$(C', A'C', \ldots, (A')^{n-1}C')' = n$

The state s_t is obtained by projecting the future of (y_t) onto its past

Applications

In applications, AR(X) models still dominate because of their advantages

- no problems with non-identifiability
- maximum likelihood estimates are of least-squares type, asymptotically efficient and easy to calculate

Their disadvantages are:

- less flexible; more parameters may have to be estimated

Comparison ARMA and state-space systems

ARMA and StS models describe the same class of transfer functions.

Theorem:

- Every ARMA system and every StS system has a rational transfer function $k(z)$ that is causal and stable and satisfies $det(k(z)) \neq 0 \; |z| \leq 1$.
- Conversely, for every rational, causal and stable transfer function $k(z)$ satisfying $det(k(z)) \neq 0 \; |z| \leq 1$ there is an ARMA as well as a StS representation.

Relation between second moments and (k, Σ): $f_y = \frac{1}{2\pi} k \cdot \Sigma \cdot k^*$

Note:

- Due to the stability and miniphase condition, k corresponds to the Wold representation.
- For $\Sigma > 0$, $k(0) = I$, k and Σ are unique for given f_y.
- For identifiability it remains to give conditions such that (a, b) or (A, B, C) are unique for given k

2.1 AR(X) identification (1)

This is classical.

Structure theory:

- For $\Sigma > 0$, two AR(X) systems $(\bar{a}, \bar{d})$ and (a, d) are observationally equivalent if and only if there exists a nonsingular matrix t such that $\bar{a} = ta$, $\bar{d} = td$, $\bar{\Sigma} = t\Sigma t'$.

- Thus identifiability is obtained by assuming $a_0 = I$ or by suitable 'structural' restrictions.

- Parameter space in the AR case:
  $\Theta = \underbrace{\{(a_1, \ldots, a_p) \in \mathbb{R}^{s^2 p} | det(a(z)) \neq 0, |z| \leq 1\}}_{\text{open subset of } \mathbb{R}^{s^2 p}} \times \Sigma$

- No 'bad' points in parameter space even for e.g. $a_p = 0$.

2.1 AR(X) identification (2)

Estimation of real valued parameters:

- OLS type estimation for the $a_0 = I$ case (or for the just identifiable case).
- Estimators are consistent and asymptotically efficient.
- Simultaneous equations methods (such as TSLS) for the overidentifiable case.

Model selection:

- Estimation of p and selection of inputs (subset selection); see below.

2.2 Structure theory for ARMA and state-space systems

Relation to internal parameters:

$k = a^{-1}(z)b(z)$ or $k(z) = \sum_{j=0}^{\infty} k_j z^j$ where $k_j = CA^{j-1}B$ for $j \geq 1$, $k_0 = I$.

$U_A = \{k|$ rational, $s \times s$, $k(0) = I$, no poles for $|z| \leq 1$ and no zeros for $|z| < 1\}$

$M(n) \subset U_A$: Set of all transfer functions of order n.

T_A: Set of all A, B, C for fixed s, but n variable, satisfying stability + miniphase assumption.

$S(n) \subset T_A$: Subset of all (A, B, C) for fixed n.

$S_m(n) \subset S(n)$: Subset of all minimal (A, B, C).

2.2 Structure theory for ARMA and state-space systems

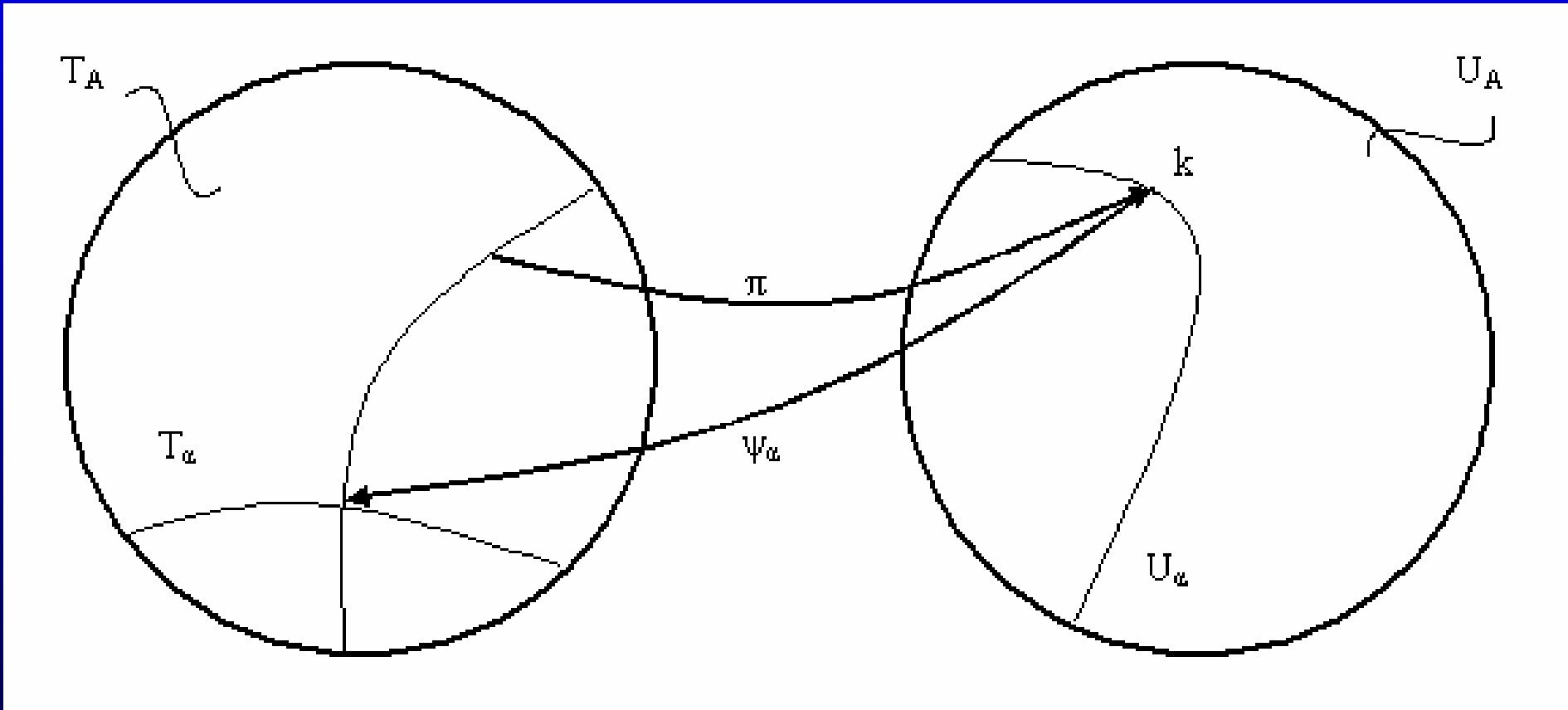
$\pi : T_A \to U_A : \pi(A, B, C) = k = C(Iz^{-1} - A)^{-1}B + I$

π is surjective but not injective

Note: T_A is not a good parameter space because:

- T_A is infinite dimensional
- lack of identifiability
- lack of "well posedness": There exists no continuous selection from the equivalence classes $\pi^{-1}(k)$ for T_α.

2.2 Structure theory for ARMA and state-space systems



2.2 Structure theory for ARMA and state-space systems

Desirable properties of parametrizations:

- U_A and T_A are broken into bits, U_α and T_α, $\alpha \in I$, such that k restricted to T_α: $\pi_{|T_\alpha} : T_\alpha \to U_\alpha$ is bijective. Then there exists a parametrization $\psi_\alpha : U_\alpha \to T_\alpha$ such that $\psi_\alpha(\pi(A, B, C)) = (A, B, C) \quad \forall (A, B, C) \in T_\alpha$.
- U_α is finite dimensional in the sense that $U_\alpha \subset \cup_{i=1}^{n} M(n)$ for some n.
- Well posedness: The parametrization $\psi_\alpha : U_\alpha \to T_\alpha$ is a homeomorphism (pointwise topology T_{pt} for U_A).
- U_α is T_{pt}-open in $\bar{U}_\alpha$.
- $\cup_{\alpha \in I} U_\alpha$ is a cover for U_A.

Examples:

- Canonical forms based on $M(n)$, e.g. echelon forms and balanced realizations. Decomposition of $M(n)$ into sets U_α of different dimension. Nice free parameters vs. nice spaces of free parameters.
- ”Overlapping description” of the manifold $M(n)$ by local coordinates.

2.2 Structure theory for ARMA and state-space systems

- "Full parametrization" for state space systems. Here $S(n) \subset \mathbb{R}^{n^2+2ns}$ or $S_m(n)$ are used as parameter spaces for $\bar{M}(n)$ or $M(n)$, respectively. Lack of identifiability. The equivalence classes are n^2 dimensional manifolds. The likelihood function is constant along these classes.
- Data driven local coordinates (`DDLC`): Orthonormal coordinates for the $2ns$ dimensional ortho-complement of the tangent space to the equivalence class at an initial estimator. Extensions: `slsDDLC` and `orthoDDLC`
- ARMA systems with prescribed column degrees.
- ARMA parametrizations commencing from writing k as $c^{-1}p$ where c is a least common denominator polynomial for k and where the degrees of c and p serve as integer valued parameters.

In general, state space systems have larger equivalence classes compared to ARMA systems: More freedom in selection of optimal representatives.

Main unanswered question: Optimal tradeoff between "number" and dimension of the pieces U_α.

2.2 Structure theory for ARMA and state-space systems

Problem: Numerical properties of parametrizations

Different parametrizations:

$\psi_1 : U_1 \to T_1 \subset T_A, \psi_2 : U_2 \to T_2 \subset T_A$

For the asymptotic analysis, in the case that $U_1 \supset U_2$, U_2 contains a nonvoid open (in U_1) set and $k_0 \in U_2$, we have:

STATISTICAL ANALYSIS (*"real world"*):

- no essential differences: coordinate free consistency
- different asymptotic distributions, but we know the transformation

NUMERICAL ANALYSIS (*"integer world"*):

- The selection from the equivalence class matters
- Dependency on algorithm

2.2 Structure theory for ARMA and state-space systems

Questions:

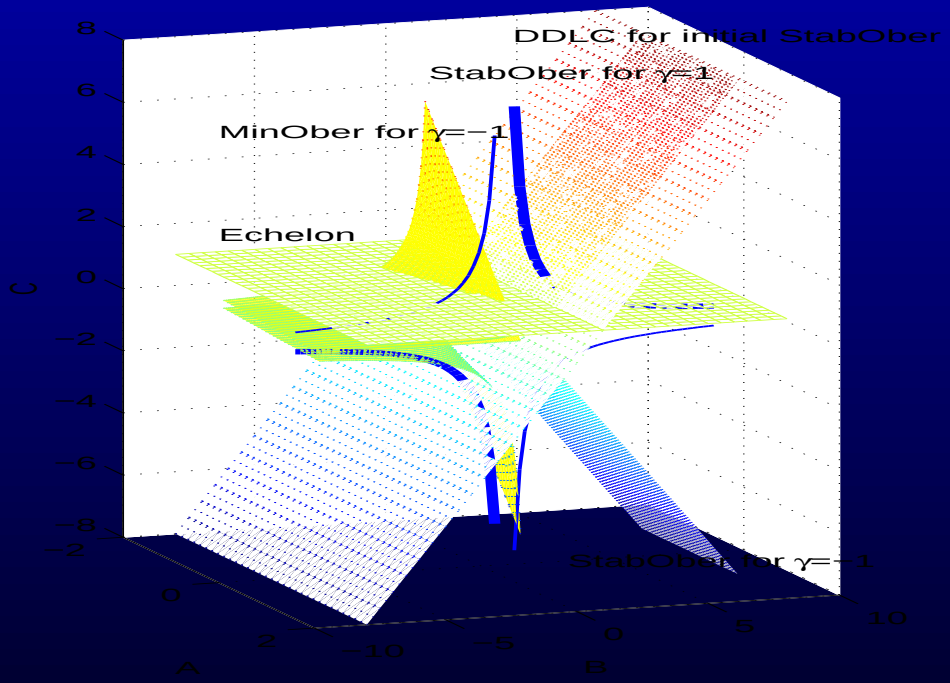
- What are appropriate evaluation criteria for numerical properties?
- Which are the optimal parameter spaces (algorithm specific)?

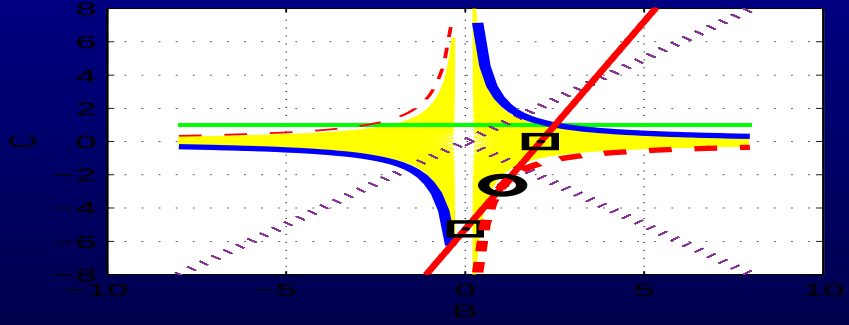
Relation between statistical and numerical precision: curvature of the criterion function:

2.2 Structure theory for ARMA and state-space systems

Consider the case $s = n = 1$ where $(a, b, c) \in \mathbb{R}^3$:

- Minimality: $b \neq 0$ and $c \neq 0$
- Equivalence classes of minimal systems: $\bar{a} = a$, $\bar{b} = tb$, $\bar{c} = ct^{-1}$, $t \in \mathbb{R} \setminus \{0\}$





2.3 Estimation for a given subclass

We here assume that U_α is given.

Identifiable case: $\psi_\alpha : U_\alpha \to T_\alpha$ has the desirable properties.

$\tau \in T_\alpha \subset \mathbb{R}^{d_\alpha}$: vector of free parameters for U_α.

$\sigma \in \underline{\Sigma} \subset \mathbb{R}^{\frac{n(n+1)}{2}}$: free parameters for $\Sigma > 0$.

Overall parameter space: $\Theta = T_\alpha \times \underline{\Sigma}$.

Many procedures, at least asymptotically, commence from sample 2^{nd} moments of the data

GENERAL FEATURES: $\hat{\gamma}(s) = T^{-1} \sum_{t=1}^{T-s} y_{t+s} y_t^{'}, \quad s \geq 0$

Now, $\hat{\gamma}$ can be directly realized as an MA system typically of order Ts; $\hat{\hat{k}}_T$

IDENTIFICATION:

Projection step (model reduction): important for statistical qualities.

Realization step.

2.3 Estimation for a given subclass

- M-estimators:

$$\hat{\theta}_T = \text{argmin} L_T(\theta; y_1, \ldots, y_T)$$

- Direct procedures: Explicit functions.

2.3 Estimation for a given subclass

GAUSSIAN MAXIMUM LIKELIHOOD:

$$\hat{L}_T(\theta) = T^{-1}\text{log det}\Gamma_T(\theta) + T^{-1}y'(T)\Gamma_T(\theta)^{-1}y(T)$$

where $y(T) = (y_1', \ldots, y_T')'$, $\quad \Gamma_T(\theta) = \mathbb{E}y(T;\theta)y'(T;\theta)$, , $\hat{\theta}_T = \text{argmin}_{\theta\in\Theta}\hat{L}_T(\theta)$

- No explicit formula for MLE, in general.
- $\hat{L}_T(k, \Sigma)$ since $\hat{L}_T$ depends on τ only via k: parameter free approach.
- Boundary points are important.

Whittle likelihood:

$$\hat{L}_{W,T}(k,\sigma) \quad = \quad \text{log det}\Sigma + (2\pi)^{-1}\int_{-\pi}^{\pi}\text{tr}\left[\left(k(e^{-i\lambda})\Sigma k^*(e^{-i\lambda})\right)^{-1} I(\lambda)\right] d\lambda$$

where $I(\lambda)$ is the periodogram.

2.3 Estimation for a given subclass

EVALUATION:

- Coordinate free consistency: for $k_0 \in U_\alpha$ and $\lim T^{-1} \sum_{t=1}^{T-s} \varepsilon_{t+s} \varepsilon_t^{'} = \delta_{0,s} \Sigma_0$ a.s. for $s \geq 0$ we have $\hat{k}_T \to k_0$ a.s. and $\hat{\Sigma}_T \to \Sigma_0$ a.s.
  Consistency proof: basic idea Wald (1949) for i.i.d. case.
  Noncompact parameter spaces:

$$\lim_{T \to \infty} \hat{L}_T(k, \sigma) = L(k, \sigma) = \log \det \Sigma + (2\pi)^{-1} \int_{-\pi}^{\pi} \operatorname{tr} \left[\left(k(e^{-i\lambda}) \Sigma k^*(e^{-i\lambda}) \right)^{-1} \left(k_0(e^{-i\lambda}) \Sigma_0 k_0^*(e^{-i\lambda}) \right) \right] d\lambda \text{ a.s.} \tag{5}$$

  - ⋆ L has a unique minimum at k_0, Σ_0.
  - ⋆ $(\hat{k}_T, \hat{\Sigma}_T)$ enters a compact set, uniform convergence in (5).
- Generalized, coordinate free consistency for $k_0 \notin \bar{U}_\alpha$, $(\hat{k}_T, \hat{\Sigma}_T) \to D$ a.s D: Set of all best approximants to k_0, Σ_0 in $\bar{U}_\alpha \times \underline{\Sigma}$.
- Consistency in coordinates: $\psi_\alpha(\hat{k}_T) = \hat{\tau}_T \to \tau_0 = \psi_\alpha(k_0)$ a.s.

2.3 Estimation for a given subclass

- CLT:
  Under $\mathbb{E}(\varepsilon_t|\mathcal{F}_{t-1}) = 0$ and $\mathbb{E}(\varepsilon_t\varepsilon_t'|\mathcal{F}_{t-1}) = \Sigma_0$, we have
  $\sqrt{T}(\hat{\tau}_T - \tau_0) \xrightarrow{d} N(0, V)$
  Idea of proof: Cramer (1946) i.i.d. case: Linearization.

Direct Estimators:

IV Methods, subspace methods: Numerically faster, in many cases not asymptotically efficient.

CALCULATIONS OF ESTIMATES

Usual procedure: consistent initial estimator (e.g. IV or subspace estimator) + one Gauss-Newton step gives an asymptotically efficient procedure (e.g. Hannan-Rissanen)

2.3 Estimation for a given subclass

HOWEVER THERE ARE STILL PROBLEMS

- Problem of local minima: “good” initial estimates are required

- Numerical problems: Optimization over a grid
  Statistical accuracy may be higher than numerical accuracy
  Valleys close to equivalence classes corresponding to lower dimensional systems
  “Intelligent” parametrization may help DDLC’s and extensions:
  Data driven selection of coordinates from an uncountable number of possibilities
  Only locally homeomorphic

- “Curse of dimensionality”
  lower dimensional parametrizations (e.g. reduced rank models)
  concentration of the likelihood function by a least squares step.

2.4 Model selection

Automatic vs. nonautomatic procedures.

Information criteria: Formulate tradeoff between fit and complexity. Based on e.g. Bayesian arguments, coding theory . . .

Order estimation (or more general closure nested case): $n_1 < n_2$ implies $\bar{M}(n_1) \subset \bar{M}(n_2)$ and dim$(M(n_1))$ <dim$(M(n_2))$.

Criteria of the form $A(n) =$ log det$\hat{\Sigma}_T(n) + 2ns \cdot c(T) \cdot T^{-1}$ where $\hat{\Sigma}_T(n)$ is the MLE for Σ_0 over $\bar{M}(n) \times \underline{\Sigma}$; $c(T) = 2$: AIC criterion; $c(T) = c \cdot$log$T$, $c \geq 1$: BIC criterion

Estimator: $\hat{n}_T =$argmin$A(n)$

Statistical evaluation: $\hat{n}_T$ is consistent for $\lim_{T\to\infty} \frac{c(T)}{T} = 0$, $\liminf_{T\to\infty} \frac{c(T)}{\log T} > 0$

Evaluation of uncertainty coming from model selection for estimators of real valued parameters.

2.4 Model selection

Note: Complexity is in the eye of the beholder. Consider e.g. AR models for $s = 1$:
$y_t + a_1 y_{t-1} + a_2 y_{t-2} = \varepsilon_t$

Parameter spaces:

$T = \{(a_1, a_2) \in \mathbb{R}^2 | 1 + a_1 z + a_2 z^2 \neq 0 \text{ for } |z| \leq 1\}$

$T_0 = \{(0, 0)\}$

$T_1 = \{(a_1, 0) | |a_1| < 1, a_1 \neq 0\}$

$T_2 = T - (T_0 \cup T_1)$

2.4 Model selection

Bayesian justification:

- Positive priors for all classes, otherwise MLE is asymptotically normal
- Certain properties of U_α, $\alpha \in I$ are needed, e.g. for BIC to give consistent estimators: closure nestedness, e.g. $n_1 > n_2 \Rightarrow M(n_1) \supset M(n_2)$

Main open question:

- Optimal tradeoff between dimension and "number" of pieces.

2.4 Model selection

Problem: Properties of post model selection estimators

- The statistical analysis of the MLE $\hat{\tau}_T$ traditionally does not take into account the additional uncertainty coming from model selection.
- This may result in very misleading conclusions

Consider AR case (nested): $y_t = a_1 y_{t-1} + \ldots + a_p y_{t-p} + \varepsilon_t$, where
$T_p = \{(a_1, \ldots, a_p)' \in \mathbb{R}^p | \text{stability}\}$

The estimator (LS) for given p is $\hat{\tau}_p = \left(X(p)'X(p)\right)^{-1} X(p)y$

The post model selection estimator is

$$\tilde{\tau} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} 1_{\{\hat{p}=0\}} + \begin{pmatrix} \hat{a}_1(1) \\ \vdots \\ 0 \end{pmatrix} 1_{\{\hat{p}=1\}} + \ldots + \begin{pmatrix} \hat{a}_1(p) \\ \vdots \\ \hat{a}_p(p) \end{pmatrix} 1_{\{\hat{p}=p\}}$$

Main problem: Essential lack of uniformity in convergence of finite sample distributions.