CHARACTERIZATION OF THE TAILS

- Classification of distributions
- Characterisation of fat-tailed distributions
- Empirical estimation of tail behavior
- Importance of the sample size
- How heavy are the tails of financial instruments?
- Aggregation properties

CLASSIFICATION OF DISTRIBUTIONS

- Thin-tailed distributions:
  - Cumulative distribution function declines exponentially in the tails
  - All moments are finite
- Fat-tailed distributions:
  - Cumulative distribution function asymptotically declines as a power with exponent α
- Bounded distributions without tails

MOMENTS OF FAT-TAILED DISTRIBUTIONS

- The density of fat-tailed distributions asymptotically behave as

$$\phi(x) = a\,(x - x_0)^{-\alpha-1} + o\left((x - x_0)^{-\alpha-1}\right)$$

- Moments:

$$\begin{aligned} E[X^k] &= C + a \int_y^\infty (x - x_0)^{-\alpha+k-1} dx \\ &+ o\left((x - x_0)^{k-\alpha}\right) \end{aligned}$$

  only exist for $k < \alpha$.
- Tail index stable under aggregation.

HOW HEAVY ARE TAILS OF FINANCIAL DISTRIBUTIONS?

- Mandelbrot: Returns follow stable distributions. Implications:
  - Tail indices between 0 and 2
  - Variance does not exist.
- Risk has often be associated with variance (e.g. Markowitz portfolio theory)
- Existence of variance central for all areas of finance
- Try to decide this question empirically

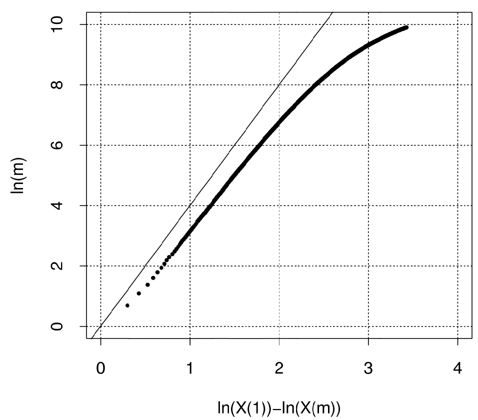
EMPIRICAL ESTIMATION OF TAIL BEHAVIOR

- Let $X_1, X_2, \ldots, X_n$ a sample of n independent n observations with unknown probability distribution function F.
- Let $X_{(1)} \geq X_{(1)} \geq \ldots \geq X_{(n)}$ the descending order statistics.
- The so-called **Hill**-estimator (Hill, 1975)

$$\hat{\gamma}^H_{n,m} = \frac{1}{m-1} \sum_{i=1}^{m-1} \ln X_{(i)} - \ln X_{(m)}$$

with $m > 1$ is a consistent stimator of $\gamma = 1/\alpha$.

TAIL BEHAVIOR OF STUDENT-4 DISTRIBUTION



Sample size: 100000, average computed over 50 simulations.

HILL-ESTIMATOR: PROPERTIES

- $(\gamma_{n,m} - \gamma)\sqrt{m}$ is asymptotically normally distributed.
- $\hat{\gamma}^H_{n,m}$ biased for finite sample size.
- $\hat{\gamma}^H_{n,m}$ depends on m.
- No easy way to determine best value of m

EMPIRICAL RESULTS

- Estimated by a subsample bootstrap method
- Confidence ranges are standard errors times 1.96 (would correspond to 95% confidence for normally-distributed errors)
- Computation of standard error by jackknife method:
  - Data sample modified in 10 different ways by removing one-tenth of the total sample
  - Tail index compuated separately for every modified sample

TAIL INDICES FOR MAIN FX RATES

Rate	30m	1 hr	2 hr	6h	1 day
USD_DEM	3.18 ±0.42	3.24 ±0.57	3.57 ±0.90	4.19 ±1.82	5.70 ±4.39
USD_JPY	3.19 ±0.48	3.65 ±0.79	3.80 ±1.08	4.40 ±2.13	4.42 ±2.98
GBP_USD	3.58 ±0.53	3.55 ±0.65	3.72 ±1.00	4.58 ±2.34	5.23 ±3.77
USD_CHF	3.46 ±0.49	3.67 ±0.77	3.70 ±1.09	4.13 ±1.77	5.65 ±4.21
USD_FRF	3.43 ±0.52	3.67 ±0.84	3.54 ±0.97	4.27 ±1.94	5.60 ±4.25
USD_ITL	3.36 ±0.45	3.08 ±0.49	3.27 ±0.79	3.57 ±1.35	4.18 ±2.44
USD_NLG	3.55 ±0.57	3.43 ±0.62	3.36 ±0.92	4.34 ±1.95	6.29 ±4.96
XAU_USD	4.47 ±1.15	3.96 ±1.13	4.36 ±1.82	4.13 ±2.22	4.40 ±2.98
XAG_USD	5.37 ±1.55	4.73 ±1.93	3.70 ±1.52	3.45 ±1.35	3.46 ±1.97

TAIL INDICES FOR SOME CROSS RATES

Rate	30m	1 hr	2 hr	6h
DEM_JPY	4.17 ±1.13	4.22 ±1.48	5.06 ±1.40	4.73 ±2.19
GBP_DEM	3.63 ±0.46	4.09 ±1.98	4.78 ±1.60	3.22 ±0.72
GBP_JPY	3.93 ±1.16	4.48 ±1.20	4.67 ±1.94	5.60 ±2.56
DEM_CHF	3.76 ±0.79	3.64 ±0.71	4.02 ±1.52	6.02 ±2.91
GBP_FRF	3.30 ±0.41	3.42 ±0.97	3.80 ±1.34	3.48 ±1.75
FRF_DEM	2.56 ±0.34	2.41 ±0.14	2.36 ±0.27	3.66 ±1.17
DEM_ITL	2.93 ±1.01	2.60 ±0.66	3.17 ±1.28	2.76 ±1.49
DEM_NLG	2.45 ±0.20	2.19 ±0.13	3.14 ±0.95	3.24 ±0.87
FRF_ITL	2.89 ±0.34	2.73 ±0.49	2.56 ±0.41	2.34 ±0.66

RESULTS: SUMMARY

- For 30 min returns all rates against USD as wells as nearly all cross rates outside the EMS have tail indices around 3.5 (between 3.1 and 3.9).
- Gold and silver have tail indices above 4.
  These markets differ from the FX market, with very high volatilities in the 80s and much lower volatilities in the 90s (structural change).
- EMS rates show lower tail indices around 2.7:
  → Reduced volatility induced by the EMS rules is at the cost of a larger probability of extreme events for realigning the system.
  → Argument against the credibility of systems like the EMS.

RESULTS: SUMMARY (CONT.)

- Tail index reflects:
  - Institutional setup
  - The way different agents interact
- They are an empirical measure of market regulation and efficiency
- Tail index large:
  - Free interactions between agents with different time horizons
  - Low degree of regulation
  - Smooth adjustment to external shocks
- Tail index small:
  - High degree of regulation
  - Abrupt adjustment to external shocks

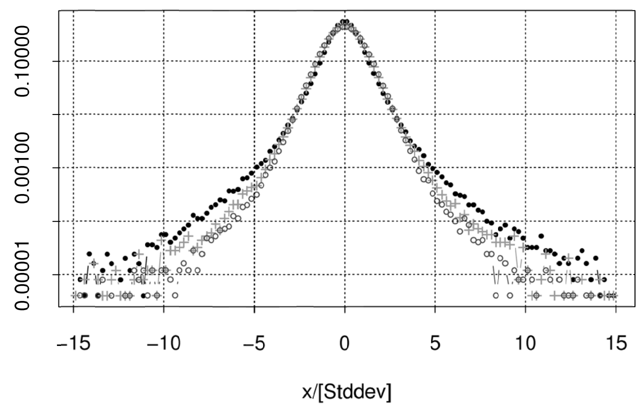
RESULTS: SUMMARY (CONT.)

- Tail indices between 2 and 4
- 2nd moments of return distributions finite
- 4th moments of return distributions usually diverge
- → Absolute returns are preferred for computation of volatility autocorrelation.

RESULTS: SUMMARY (CONT.)

- Return distributions are **fat-tailed** with tail index > 2 and therefore **non-stable**.
- Invariance under aggregation satisfied up to $2 \ldots 6$ hours.
- For larger time horizons tail behavior can no longer be observed because:
  - Transition to tail behavior occurs at increasinly larger (relative) returns.
  - Sample size decreases.
- Detailed studies have shown that 18 years of daily data are not enough to estimated the tail index reliably.

AGGREGATION OF t-3.5 DISTRIBUTION



Density of t-3.5 distribution (black squares), sum of 4 t-3.5 distributed random variables (+) and sum of 10 t-3.5 distributed random variables. Note the transition from normal to tail behavior.

EXTREME RISKS IN FINANCIAL MARKETS

- What are the extreme movements to be expected?
- Are there theoretical processes to model them?
- What is the best hedging strategy?

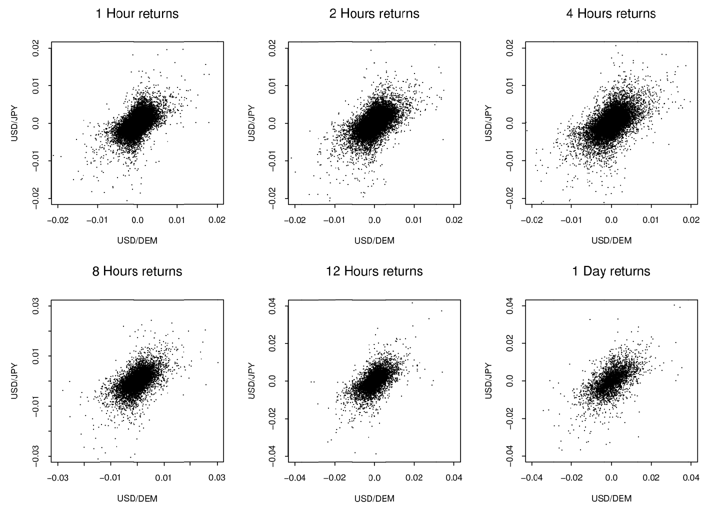
EXTREME EVENTS IN MODELS AND REALITY

	Probabilities [1/years]					
	1/1	1/5	1/10	1/15	1/20	1/25
Normal	0.4%	0.5%	0.6%	0.6%	0.7%	0.7%
Student-3	0.5%	0.8%	1.0%	1.1%	1.2%	1.2%
GARCH(1,1)	1.5%	2.1%	2.4%	2.6%	2.7%	2.9%
USD_DEM	1.7%	2.5%	3.0%	3.3%	3.5%	3.7%
USD_JPY	1.7%	2.4%	2.9%	3.2%	3.4%	3.6%
GBP_USD	1.6%	2.3%	2.6%	2.9%	3.1%	3.2%
USD_CHF	1.8%	2.7%	3.1%	3.5%	3.7%	4.0%
DEM_JPY	1.3%	1.9%	2.2%	2.5%	2.6%	2.8%
GBP_DEM	1.1%	1.7%	2.1%	2.3%	2.5%	2.6%
GBP_JPY	1.6%	2.3%	2.7%	3.0%	3.2%	3.4%
DEM_CHF	0.7%	1.0%	1.2%	1.3%	1.4%	1.5%

SF: DISTRIBUTIONAL PROPERTIES OF MULTIVARIATE RETURNS: DEPENDENCE STRUCTURE

- Covariance matrix describes only linear part of the dependence

RETURN SCATTER PLOTS



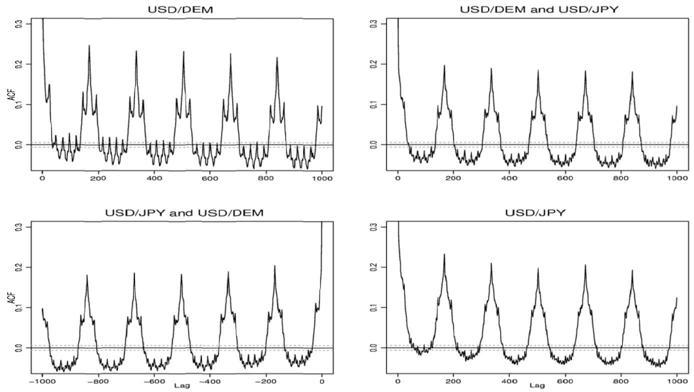
SF: DISTRIBUTIONAL PROPERTIES OF MULTIVARIATE RETURNS: DEPENDENCE STRUCTURE

- Covariance matrix describes only linear part of the dependence
- Before description of the full dependences structure, deseasonalization necessary

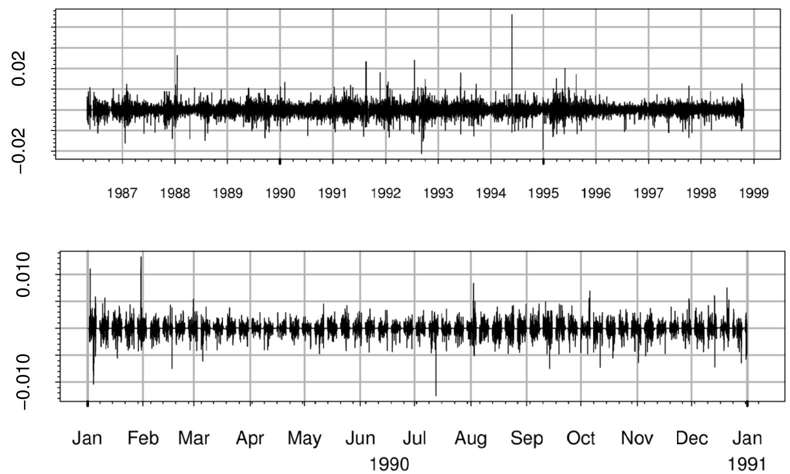
S.F.: VOLATILITY AUTOCORRELATION AND SEASONALITIES

- Pronounced daily and weekly seasonalities in autocorrelation function of absolute returns
- Slowly decaying autocorrelation function of absolute returns (after deseasonalization)

AUTOCORRELATION FUNCTIONS OF ABSOLUTE RETURNS



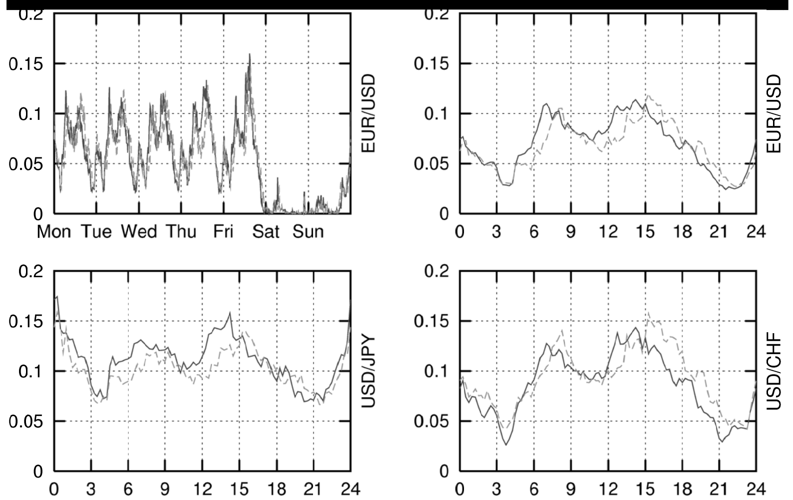
HOURLY RETURNS



INTRADAY AND INTRAWEEK VOLATILITY PATTERNS

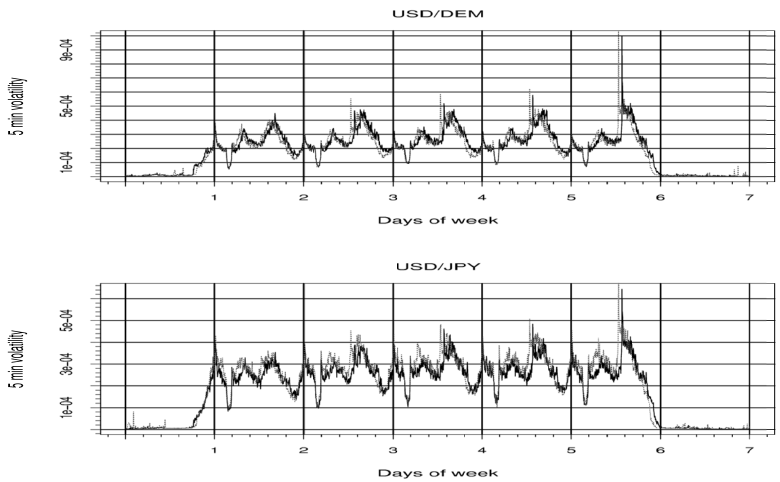
- There are many different patterns are observed.
- Extreme cases:
  - OTC markets as the FX spot market
  - Localized, exchange-traded markets as equity indices.
- The patterns may change when the institutional setting is changing (**example**: change of the opening hours of a market).

SEASONAL VOLATILITY PATTERN FOR FX SPOT RATES

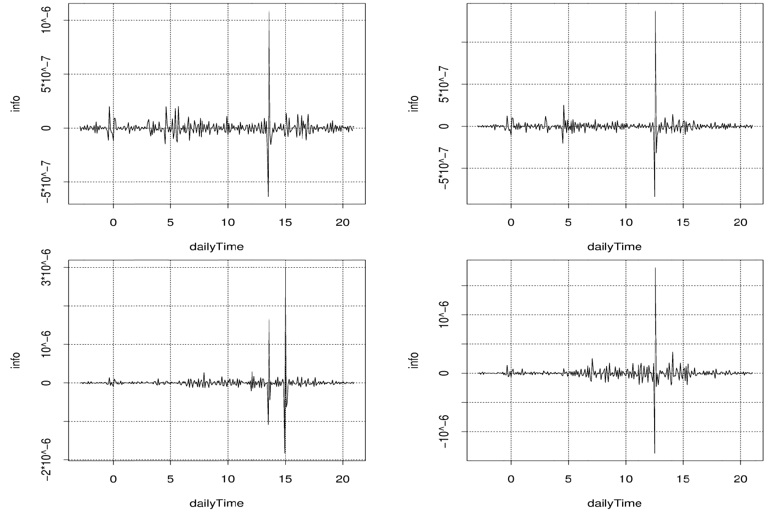


Full, red line: summer time. Dashed, green line: winter time.

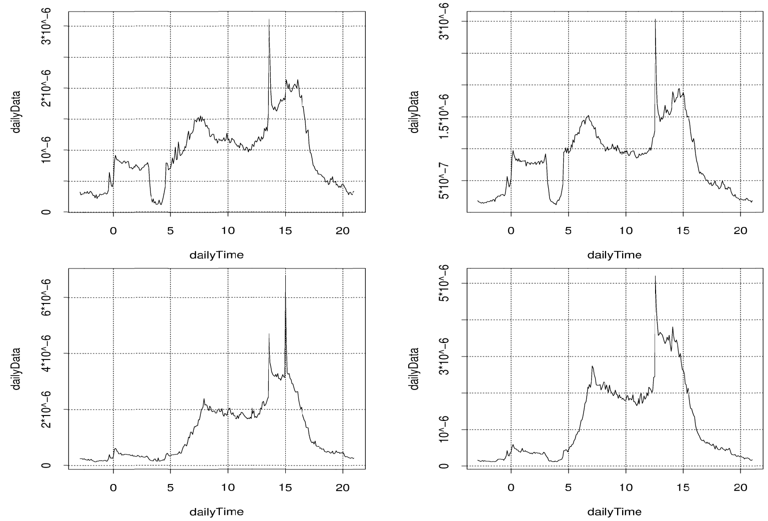
THE WEEKLY VOLATILITY PATTERN



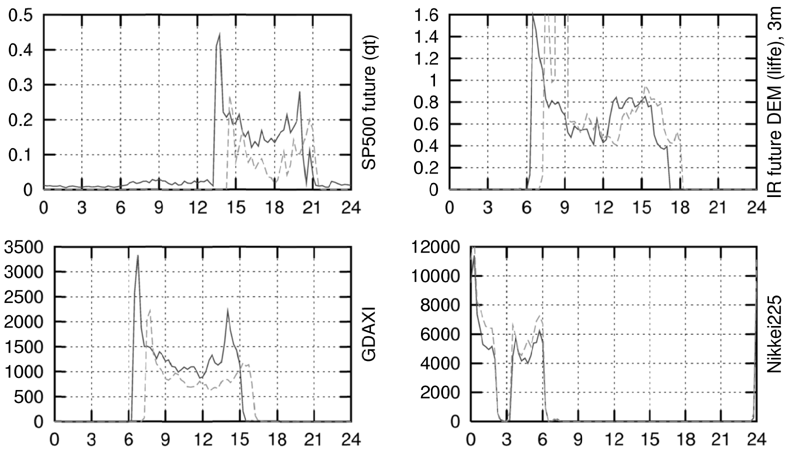
INFORMATION EFFECTS FOR FX SPOT RATES II



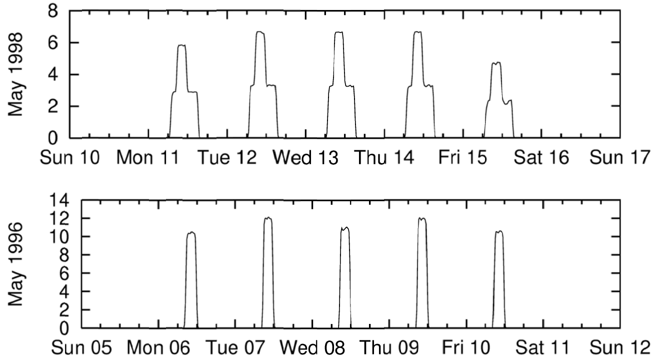
SEASONAL VOLATILITY PATTERN FOR FX SPOT RATES II



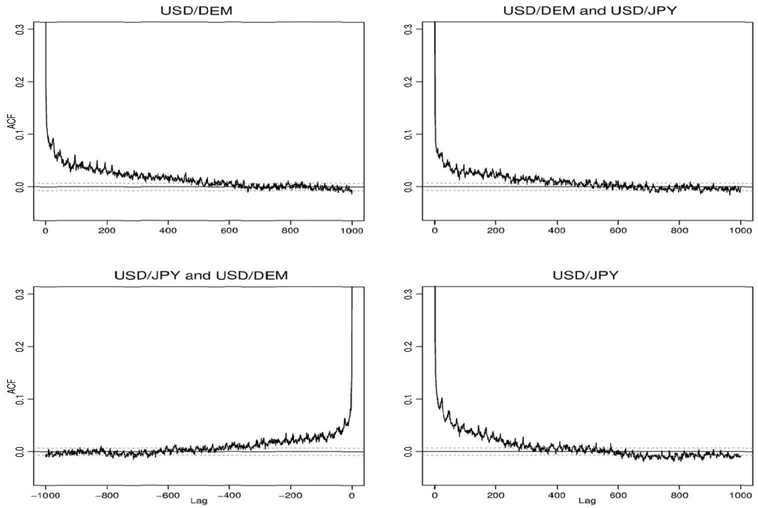
SEASONAL VOLATILITY PATTERN FOR EXCHANGE-TRADED INSTRUMENTS



CHANGE OF DAILY ACTIVITY PERIOD FOR DAX



ACF OF DESEASONALISED HOURLY ABS. RETURNS



LONG RANGE VOLATILITY AUTOCORRELATIONS

- After deseasonalization:
  Autocorrelation function of absolute returns displays long-time dependence.

S.F.: SCALING LAWS

- Empirical scaling behavior of first moments of returns
- An application of scaling behavior analysis
- Empirical scaling behavior of arbitrary moments
- “True” vs. apparent scaling

Anomalous scaling behavior can be explained by multi-fractal properties.

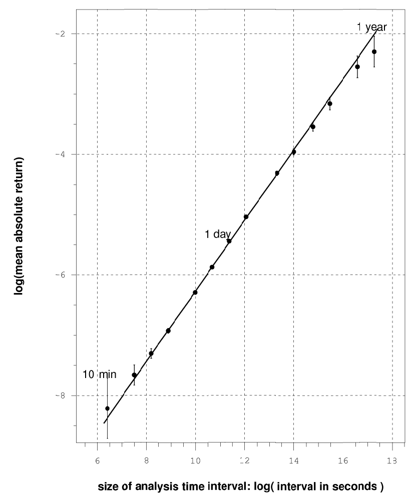
SCALING LAW

- Dependence of mean absolute return on the size of the time interval on which the return is observed.
- Empirically, we find a power law for mean absolute returns:

  $$\overline{|r[\Delta t]|} = C \cdot (\Delta t)^D$$

  Δt is the interval size and D the **drift exponent** (a constant).
- The drift exponent is around $D = 0.58$ for freely floating FX rates, with a confidence of typically ± 0.01.
  A Gaussian i. i. d. random walk would have $D = 0.5$.

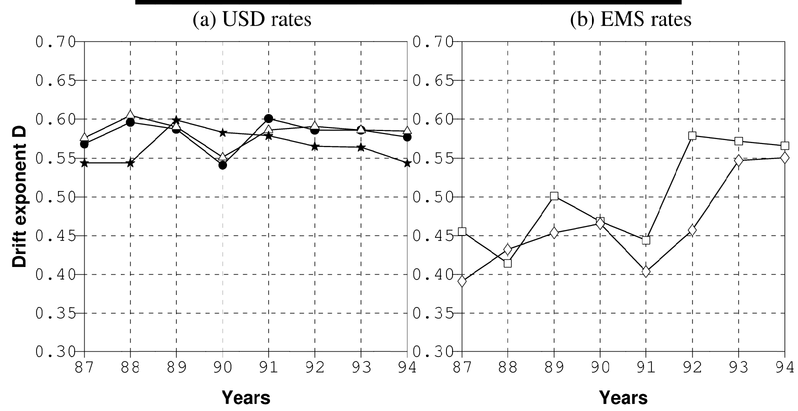
SCALING LAW GRAPH



Mean absolute change of the logarithmic middle price as a function of the time interval (in seconds). for USD/DEM spot rates (Period: Feb 1986 to Sep 1993, ca. 10 million quotes).
Vertical bars indicate the total error (stochastic and bid-ask uncertainty).

Scaling law:
$\langle r[\Delta T]\rangle = \left(\frac{\Delta T}{11230 \text{ days})}\right)^D$
with drift exponent
$D = 0.59$.

DRIFT EXPONENT VERSUS TIME



Yearly estimation of the drift exponent of the scaling law for USD rates (a) on the left (DEM (circle), FRF (triangle), JPY (star)) and (b) EMS rates against the DEM on the right (ITL (box) and FRF (diamond)).
The drift exponents reflects the **institutional framework** of FX rates.

GENERALIZED SCALING BEHAVIOR

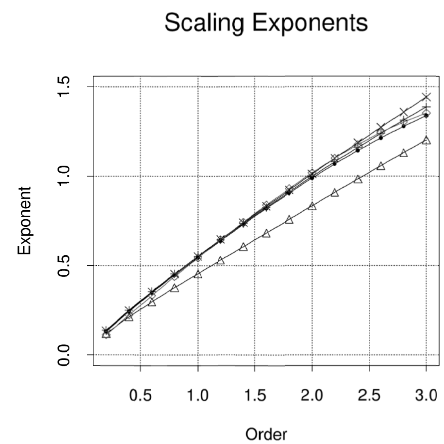
- Dependence of the moments of absolute returns on the size of the time interval on which the return is observed.
- Empirically, we find a power law for mean absolute returns:

  $$\overline{|r[\Delta t]|^n} = C \cdot (\Delta t)^{\xi_n}$$

  Δt is the interval size and ξ_n the **scaling exponent**.
- The scaling exponent for absolute returns is around $\xi_1 = 0.58$ for freely floating FX rates
- The scaling exponent for squared returns is around 1.

Scaling behavior of the absolute moments of returns for USD/DEM spot rates. (Feb 1986 to Sep 1998, more than 10 million quotes).

SCALING EXPONENTS VS. ORDER



CURRENCIES: (): USD/DEM; (+): USD/CHF; (x): GBP/USD; (triangle): DEM/FRF; (diamond, red): SCM model.

SF: INFORMATION FLOW FROM LONG TO SHORT HORIZONS

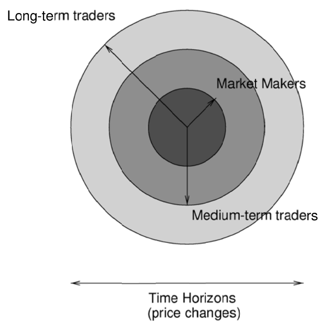
- Heterogeneous objects.
- Not only the market markers but many participants.
- Different people with different interests, different geographical locations, and different time horizons.
- Time scale (frequency of interventions) is the aspect for which the heterogeneity becomes the most obvious.
- Information flow from long to short time scales.

CATEGORIES OF MARKET PARTICIPANT

- Intraday dealer
- Fund manager
- International Companies
- Central banks

They influence the market on different time scales.

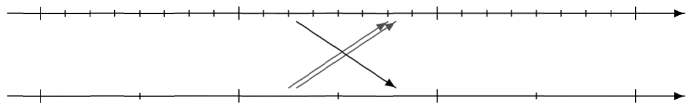
A NON-TRADITIONAL VIEW OF A FINANCIAL MARKET



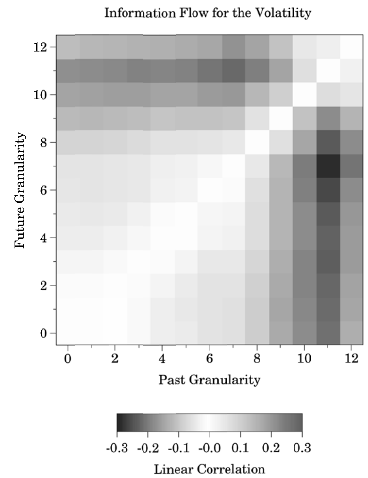
News hit the market at the center. If they are important the price changes will hit different market components and feedback effects will happen in the center. Price changes and time horizons are related through the scaling law $|\Delta x| \propto \Delta t^D$.

AN ASSYMMETRIC INFORMATION FLOW

Coarse volatility predicts fine volatility better than the other way around:



INFORMATION FLOW ASYMMETRY, THE HARCH EFFECT



$$\boxed{I(n, n') = \mathrm{Corr}(\sigma[T, \frac{T}{2^n}](t), \sigma[T, \frac{T}{2^{n'}}](t+T)) - \mathrm{Corr}(\sigma[T, \frac{T}{2^n}](t+T), \sigma[T, \frac{T}{2^{n'}}](t))}$$