A Multivariate Threshold GARCH Model with Time-varying Correlations

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Abstract

In this article, a Multivariate Threshold Generalized Autoregressive Conditional Heteroscedasticity model with time-varying correlation (VC-MTGARCH) is proposed. The model extends the idea of Engle (2002) and Tse & Tsui (2002) in a threshold framework. This model retains the interpretation of the univariate threshold GARCH model and allows for dynamic conditional correlations. Extension of Bollerslev, Engle and Wooldridge (1988) in a threshold framework is also proposed as a by-product. Techniques of model identification, estimation and model checking are developed. Some simulation results are reported on the finite sample distribution of the maximum likelihood estimate of the VC-MTGARCH model. Real examples demonstrate the asymmetric behaviour of the mean and the variance in financial time series and that the VC-MTGARCH model can capture these phenomena.

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1 Introduction

During the last two decades, the modelling of conditional volatility in finance has been widely discussed in the literature. As a model for financial data with a changing conditional variance, Engle (1982) first proposed the autoregressive conditional heteroscedasticity (ARCH) model. Bollerslev (1986) extended this into a generalized ARCH (GARCH) model. Engle & González-Rivera (1991) further extended the GARCH model to a semiparametric GARCH model which does not assume a parametric form of the noise distribution. A tremendous literature now exists for the GARCH model, for instance Li, Ling & McAleer (2002).

Incidentally, there have been growing interests in the nonlinear time series, for instance, the self-exciting threshold autoregressive (SETAR) model of Tong (1978, 1980, 1983) and Tong & Lim (1980). Various tests for nonlinearity have since been developed. Keenan (1985) constructed a test for linearity which is an analogue of Tukey's one degree of freedom for nonadditivity test. Petruccelli (1986) proposed a portmanteau test for self-exciting threshold autoregressive nonlinearity model. Moreover, Tsay (1989) proposed an efficient procedure for testing threshold nonlinearity and successfully illustrated its use via the analysis of high-frequency financial data. During the time, many researchers have also extended the ARCH model to a nonlinear ARCH model, for example Li & Lam (1995). Li & Li (1996) extended the threshold ARCH model to a double-threshold ARCH model, which can handle the situation where both the conditional mean and the conditional variance specifications are piecewise linear given previous information. Brooks (2001) further extended the double-threshold ARCH model to a double-threshold

GARCH model.

After the development in univariate ARCH model, the study of multivariate ARCH models becomes the next important issue. Bollerslev, Engle and Wooldridge (1988) suggested a basic structure for a multivariate GARCH (MGARCH) model. Engle & Kroner (1995) proposed a BEKK model which is a class of MGARCH model. Numerous applications of the multivariate GARCH models have been applied to financial data. For instance, Bollerslev (1990) studied the time-varying variance structure of the exchange rate in the European Monetary System. Kroner & Claessens (1991) applied the models to evaluate the optimal debt portfolio in multiple currencies. Thereafter, Tsay (1998) proposed a procedure for testing multivariate threshold nonlinearity models and successfully illustrated its use via the analysis of monthly U.S. interest rates and two daily river flow series of Iceland. In order to satisfy the necessary conditions presented by Engle, Granger and Kraft (1984) for the conditional-variance matrix of an estimated MGARCH model to be positive definite, Bollerslev (1990) suggested a parsimonious constant-correlation MGARCH model. The necessary conditions for positive definiteness can be easily imposed during the optimization of the log-likelihood function. Engle & Susmel (1993) investigate some international stock markets that have similar time-varying volatility. The recent work of Tse & Tsui (2002), Engle (2002) and that of Pelletier (2003) describe a parsimonious MGARCH model that allows a time-varying correlation instead of a constant-correlation formulation for the conditional variance equation. It is found that the time-varying model could provide interesting and more realistic empirical results.

In this paper, a multivariate threshold GARCH (MTGARCH) model with time-varying correlation (VC-MTGARCH) is proposed. The proposed model is an extension of the threshold approach for nonlinearity to the time-varying correlation model of Tse & Tsui (2002). In

Section 2, the construction of a time-varying correlation MTGARCH model is discussed. A nonlinearity test for model building is presented in section 3. Model identification and estimation procedures of the proposed model are given in Section 4 and Section 5. Here, model identification includes estimating the AR orders, GARCH orders, delay parameter and threshold parameter. Simulation results are provided in Section 6. In Section 7, some empirical examples of the proposed model using some real data sets are presented. These are the exchange rate data and national stock market price data considered in Tse & Tsui (2002). Finally some concluding remarks are given in the last section.

2 A time-varying correlation MTGARCH Model

In this section, time-varying correlation Multivariate Threshold GARCH models are presented. Consider an n-dimensional multivariate time series $Z_t = (Z_{1t}, \ldots, Z_{nt})$, where $t = 1, \ldots, T$. The conditional variance matrix of Z_t follows a time-varying structure,

$$Var(Z_t|F_{t-1}) = H_t,$$

where F_{t-1} is the information set $\{Z_{t-1}, \ldots, Z_1\}$ at time $t-1$. Rewrite $H_t = H_t^{\frac{1}{2}} H_t^{\frac{1}{2}}$, where $H_t^{\frac{1}{2}}$ is the symmetric square-root matrix based on the spectral decomposition. Let $e_t = H_t^{\frac{1}{2}} \epsilon_t$, where $\epsilon_t \sim N(0, I)$. Here, ϵ_t=$(\epsilon_{1t}, \ldots, \epsilon_{nt})'$ is assumed to be independently distributed and $e_t = (e_{1t}, \ldots, e_{nt})'$ is conditionally normally distributed with mean zero and variance-covariance matrix H_t. Here, v' denotes the transpose of v.

In this article, a time-varying correlations MGARCH model with threshold structure (VC-MTGARCH) is mainly discussed. Pelletier (2003) introduced a regime switching model of constant correlations within each regime. In this work, an extension of the VC-MGARCH model of Tse & Tsui (2002) using the threshold approach is discussed. This model will have an appealing property of dynamic correlations within a regime. In particular, the time varying conditional variance matrix H_t is defined as follows:

$$H_t = D_t \Gamma_t D_t.$$

Denote the variance elements of H_t by σ_{it}^2, for $i = 1, \ldots, n$, and the covariance elements by σ_{ijt}^2, where $1 \leq i < j \leq n$. Define D_t as a $n \times n$ diagonal matrix where the ith diagonal element is σ_{it}. Then, Γ_t is the correlation matrix of Z_t. Let $l_0 < l_1 < \ldots < l_{s-1} < l_s$ be a partition of the real line, where $l_0 = -\infty$ and $l_s = \infty$. Let d be the delay parameter and r_{t-d} be a real-valued threshold variable. The j-th regime of a VC-MTGARCH$(p_1, \ldots, p_s; P_1, \ldots, P_s; Q_1, \ldots, Q_s; s)$ model is given by

$$Z_{i,t} = \Phi_{i,0}^{(j)} + \sum_{k=1}^{p_j} \Phi_{i,k}^{(j)} Z_{i,t-k} + e_{i,t}, \qquad l_{j-1} < r_{t-d} \leq l_j, \tag{1}$$

with

$$\sigma_{it}^2 = c_i^{(j)} + \sum_{k=1}^{P_j} \alpha_{i,k}^{(j)} \sigma_{i,t-k}^2 + \sum_{k=1}^{Q_j} \beta_{i,k}^{(j)} e_{i,t-k}^2, \quad j = 1, \ldots, s, \tag{2}$$

where $c^{(j)}$, $\alpha_{i,k}^{(j)}$ and $\beta_{i,k}^{(j)}$ are non-negative and subject to

$$\sum_{k=1}^{P_j} \alpha_{i,k}^{(j)} + \sum_{k=1}^{Q_j} \beta_{i,k}^{(j)} < 1.$$

The corresponding time-varying conditional correlation matrix Γ_t in the j-th regime follows

$$\Gamma_t = (1 - \theta_1^{(j)} - \theta_2^{(j)})\Gamma + \theta_1^{(j)} \Gamma_{t-1} + \theta_2^{(j)} \Psi_{t-1}, \tag{3}$$

where $\Gamma = \{\rho_{ij}\}$ is a time-invariant $n \times n$ positive definite parameter matrix with unit diagonal elements and Ψ_{t-1} is a $n \times n$ matrix whose elements are functions of the lagged standardized residuals $\hat{u}_{i,t} = \dfrac{e_{i,t}}{\sigma_{i,t}}$. The parameters $\theta_1^{(j)}$ and $\theta_2^{(j)}$ are non-negative subject to $\theta_1^{(j)} + \theta_2^{(j)} \leq 1$. Denote $\Psi_t = \{\Psi_{ij,t}\}$. In Tse & Tsui (2002), the matrix Ψ_{t-1} follows

$$\Psi_{ij,t-1} = \frac{\sum_{h=1}^{M} \hat{u}_{i,t-h}\hat{u}_{j,t-h}}{\sqrt{(\sum_{h=1}^{M} \hat{u}_{i,t-h}^2)(\sum_{h=1}^{M} \hat{u}_{j,t-h}^2)}}, \tag{4}$$

for $M \geq n$.

Tse & Tsui (2002) showed that $M \geq n$ is a necessary condition for Ψ_{t-1} to be positive definite. Thus, Γ_t would also be a positive definite correlation matrix with unit diagonal elements. As a result, H_t is a positive-definite matrix and hence, $H_t^{\frac{1}{2}}$ is also a positive definite matrix.

A threshold structure multivariate GARCH model with time-varying covariance (VCOV-MTGARCH) is also proposed as a by-product. This can be seen as a simplified extension of the MGARCH model with time-varying covariance of Bollerslev, Engle and Wooldridge (1988). Let $l_0 < l_1 < \ldots < l_{s-1} < l_s$ be a partition of the real line, where $l_0 = -\infty$ and $l_s = \infty$. Let d be the delay parameter and r_{t-d} be a real-valued threshold variable. Under the same assumption of e_t and ϵ_t as above, the j-th regime of a VCOV-MTGARCH$(p_1, \ldots, p_s; P_1, \ldots, P_s; Q_1, \ldots, Q_s; s)$is given by

$$Z_{i,t} = \Phi_{i,0}^{(j)} + \sum_{k=1}^{p_j} \Phi_{i,k}^{(j)} Z_{i,t-k} + e_{i,t}, \qquad l_{j-1} < r_{t-d} \leq l_j, \tag{5}$$

with

$$H_t = C^{(j)} + \sum_{k=1}^{P_j} A_k^{(j)} H_{t-k} A_k^{(j)} + \sum_{k=1}^{Q_j} B_k^{(j)} [e_{t-k}e'_{t-k}] B_k^{(j)}, \quad j = 1, \ldots, s,$$

where $A_k^{(j)}$, $B_k^{(j)}$ and $C^{(j)}$ are diagonal matrices with non-negative entries. This model is a simplified threshold BEKK model. The positive definiteness of the matrix H_t will be guaranteed under some useful restrictions derived from the BEKK representation, introduced by Engle and Kroner (1995). According to the discussion of Engle and Mezrich (1996), this model can be estimated subject to the variance targeting constraint by which the long run variance covariance matrix is the sample covariance matrix.

For simplicity of notation, the VC-MTGARCH($p_1, \ldots, p_s; P_1, \ldots, P_s; Q_1, \ldots, Q_s; s$) will be rewritten as VC-MTGARCH($p; P; Q; s$) if $p_j = p$, $P_j = P$ and $Q_j = Q$ for any $j = 1, \ldots, s$; a similar approach will also be applied on the VCOV-MTGARCH model. As in Tse & Tsui (2002), the number of parameters is parsimonious and also the conditional correlations are not restricted to be constants.

The above models have s regimes and are piecewise linear in the threshold space r_{t-d}. The times series will be nonlinear in time when s is greater than 1. The threshold variable r_{t-d} is assumed to be known, however the delay parameter d, the number of regimes s, and the threshold values l_j are unknown.

The VC-MTGARCH model extends both Tong's (1990) threshold model and Tse & Tsui's (2002) time varying multivariate generalized autoregressive conditional heteroscedasticity model, VC-MGARCH($p; P; Q$), in a natural way. It is shown in Tong & Lim (1980) that the threshold model can capture various nonlinear phenomena.

3 A Nonlinearity test

A nonlinearity test for multivariate GARCH time series models is proposed. The proposed test follows the idea of Tsay (1998). For ease of exposition, the threshold structure of equations (1) and (2) are assumed to be the same (i.e. the same threshold variable r_{t-d} is employed for equations (1) and (2)). The null hypothesis $H_0 : Z_t$ is linear versus the alternative hypothesis $H_1 : Z_t$ follows a multivariate threshold GARCH model, (i.e. $H_0 : s = 1$ versus $H_1 : s > 1$).

Suppose observations $\{Z_t\}$ are given, where $t = 1, \ldots, T$. Setting the model in a regression framework,

$$Z_t' = Y_t'\mathbf{\Phi} + e_t', \qquad t = \tau + 1, \ldots, T, \tag{6}$$

where $\tau = \max(p, d)$, $Y_t = (1, Z_{t-1}', \ldots, Z_{t-p}')'$ is a $(pn+1)$-dimensional regressor and $\mathbf{\Phi}$ denotes a parameter matrix. Under the null hypothesis of linearity of the conditional mean, there is only one mean model for Z_t and the least squares estimates of (6) are consistent and unbiased. However, the least squares estimates are asymptotically biased under the alternative hypothesis. Equation (6) therefore provides information about the threshold structure if the ordering of the setup is suitably rearranged. For the arranged autoregression, the observations are grouped such that all of the data in a group is assumed to follow the same linear AR model. Define S to be the set of values taken by the threshold variable r_{t-d}, i.e. $S = \{r_{\tau+1-d}, \ldots, r_{T-d}\}$. Denote $r_{(i)}$ be the ith smallest element of S, and $\mu(i)$ be the corresponding time index of $r_{(i)}$. The arranged autoregression based on the increasing order r_{t-d} is

$$Z_{\mu(i)+d}' = Y_{\mu(i)+d}'\mathbf{\Phi} + e_{\mu(i)+d}', \qquad i = 1, \ldots, T - \tau. \tag{7}$$

Let $\hat{\mathbf{\Phi}}_k$ be the least squares estimate of $\mathbf{\Phi}$ of equation (7) with $i = 1, \ldots, k$. Let

$$\hat{e}_{\mu(k+1)+d} = Z_{\mu(k+1)+d} - \hat{\mathbf{\Phi}}_k' Y_{\mu(k+1)+d} \tag{8}$$

and

$$\hat{\xi}_{j,\mu(k+1)+d} = \frac{\hat{e}_{j,\mu(k+1)+d}}{\sqrt{\hat{\sigma}_j^2 + Y_{\mu(k+1)+d}' U_{j,k} Y_{\mu(k+1)+d}}}, \tag{9}$$

be respectively the predictive residual and the standardized predictive residual of regression (7), where

$$\hat{\sigma}_j^2 = \sum_{i=1}^{k} \frac{\hat{e}_{j,\mu(i)+d}^2}{k - np - 1}$$

is the residual mean squared error of the jth element of Z_t and

$$U_{j,k} = (\sum_{i=1}^{k} Y_{\mu(i)+d} Y'_{\mu(i)+d})^{-1} (\sum_{i=1}^{k} \hat{e}^2_{j,\mu(i)+d} Y_{\mu(i)+d} Y'_{\mu(i)+d}) (\sum_{i=1}^{k} Y_{\mu(i)+d} Y'_{\mu(i)+d})^{-1}.$$

Consider the regression

$$\hat{\xi}'_{\mu(i)+d} = Y'_{\mu(i)+d} \mathbf{\Psi} + \eta'_{\mu(i)+d}, \qquad i = s_0 + 1, \ldots, T - \tau, \tag{10}$$

where s_0 denotes the starting point of the recursive least squares estimation. Tsay (1998) stated that the value of $s_0 \approx 3\sqrt{T}$ for the unit-root series and $s_0 \approx 5\sqrt{T}$ for the stationary case, where T is the sample size. The procedure is then to test the hypothesis $H_0 : \mathbf{\Psi} = 0$ versus the alternative $H_a : \mathbf{\Psi} \neq 0$ in regression (10). We consider as in Tsay (1998) the test statistic

$$R(d) = (T - \tau - s_0 - (np + 1)) \times (\ln(det A_0) - \ln(det A_1)), \tag{11}$$

where d, the delay parameter, indicates that the test depends on the threshold variable r_{t-d},

$$A_0 = \frac{1}{T - \tau - s_0} \sum_{l=s_0+1}^{T-\tau} \hat{\xi}_{\mu(l)+d} \hat{\xi}'_{\mu(l)+d}$$

and

$$A_1 = \frac{1}{T - \tau - s_0} \sum_{l=s_0+1}^{T-\tau} \hat{\eta}_{\mu(l)+d} \hat{\eta}'_{\mu(l)+d},$$

and $\hat{\eta}_t$ is the least squares residual of regression (10). Based on the theorem of Tsay (1998, Thm. 2) and the theorem of Lai & Wei (1982, Thm. 1), it can be shown that $R(d)$ defined in (11) is asymptotically a chi-squared random variable with $n(np+1)$ degrees of freedom under the null hypothesis.

Remark. In general, threshold structure might not be found in the mean equation, the method of Li & Li (1996) would be applied. The squared residuals from the best-fitting AR model would be adopted in identifying the threshold structure of the conditional variance equation.

4 Model Identification

The next tasks to be carried out are model identification and parameter estimation. Model identification will be illustrated in this section and parameter estimation will be given in the next section. For a simple linear AR model, model identification can be easily handled by examining the process of autocorrelation function (ACF) and partial autocorrelation function (PACF). However, when identifying a VC-MTGARCH model, it will not be the case as autocorrelations are uninformative about asymmetric in the model. Arranged autoregression are used as in Tsay (1989) for identifying the threshold model. In the previous section, procedures for testing the presence of threshold nonlinearity are given. Tsay (1989) pointed out that scatterplot of various statistics versus the specified threshold variable could provide useful information in locating the thresholds. The statistics used should demonstrate the special features of the threshold model. A detail discussion of the scatterplot would be given in the next section. The AR orders of each regime can be identified by using the Akaike's information criterion (AIC).

Consider an AR-GARCH(p; P, Q) process, for simplicity, assuming the GARCH order P and Q are the same. An AR-GARCH process is a process Z_t given by

$$Z_{i,t} = \Phi_{i,0} + \sum_{k=1}^{p} \Phi_{i,k} Z_{i,t-k} + e_{i,t},$$

with conditional variance given by

$$\sigma_{i,t}^2 = c_i + \sum_{k=1}^{P} \alpha_{i,k} \sigma_{i,t-k}^2 + \sum_{k=1}^{Q} \beta_{i,k} e_{i,t-k}^2,$$

Let $v_{i,t} = e_{i,t}^2 - \sigma_{i,t}^2$. Then we have

$$e_{i,t}^2 = c_i + \sum_{k=1}^{Max(P,Q)} (\alpha_{i,k} + \beta_{i,k}) e_{i,t-k}^2 + v_{i,t} - \sum_{k=1}^{P} \beta_k v_{i,t-k}. \tag{12}$$

Gouriéroux (1997, p.37) shows that $E(v_{i,t} - \sum_{k=1}^{P} \beta_k v_{i,t-k} | F_{t-1}) = 0$. Therefore, the MGARCH model can be rewritten in an ARMA representation. This is useful in identifying the GARCH orders P and Q.

The model identification procedure mainly involves two parts:

- Applying Tsay's procedure to identify the delay parameters, the threshold parameters and the AR orders in the conditional mean.
- Identify the GARCH orders of the conditional variances.

The overall procedure is as follows.

1. Select the AR order p, the GARCH order P, Q.
2. Fit arranged autoregressions for a given p and each possible delays d, and perform the threshold nonlinearity test. When nonlinearity is detected, choose the delay parameter d which maximizes the test statistics.
3. For given p and d, locate the value of the threshold parameter by using Tsay' arranged autoregression based on the scatterplots of the elements of Φ versus the threshold variable.
4. If the threshold structure is identified, calculate the residuals $\hat{e}_t$ of the threshold AR model and identify the GARCH order using $\hat{e}_t \hat{e}_t'$ and equation (12). Then fit the entire VC-MTGARCH model.
5. Use an information criterion like AIC or Bayesian information criterion (BIC) to refine the AR orders, the GARCH orders, the delay and threshold parameters by repeating steps (1) - (4), if necessary.

5 Estimation and Model checking

The specification of the threshold variable is a major issue in modelling threshold model, as it plays a key role in the nonlinear structure of the model. In this section, the conditional least squares estimation of the threshold model in (1) and (2) is considered. Assuming the order p of the mean equation is known, Tsay (1998) indicates that the nonlinearity test will have good power when the delay d is correctly specified. The delay parameter is estimated by the value $\hat{d}$ that provides the most significant result of $R(d)$ of (11) in the testing for threshold nonlinearity.

After obtaining the delay parameter, estimating the threshold values will be the next important issue. For ease of presentation, and without loss of generality, the case of $s = 2$ is considered below. Then model (1) and (2), the VC-MTGARCH model becomes,

$$Z_{i,t} = \begin{cases} \Phi_{i,0}^{(1)} + \sum_{k=1}^{p_1} \Phi_{i,k}^{(1)} Z_{i,t-k} + e_{i,t}, & r_{t-d} \leq l, \\ \Phi_{i,0}^{(2)} + \sum_{k=1}^{p_2} \Phi_{i,k}^{(2)} Z_{i,t-k} + e_{i,t}, & r_{t-d} > l, \end{cases} \tag{13}$$

with

$$\sigma_{i,t}^2 = \begin{cases} c_i^{(1)} + \sum_{k=1}^{P_1} \alpha_k^{(1)} \sigma_{i,t-k}^2 + \sum_{k=1}^{Q_1} \beta_k^{(1)} e_{i,t-k}^2, \\ c_i^{(2)} + \sum_{k=1}^{P_2} \alpha_k^{(2)} \sigma_{i,t-k}^2 + \sum_{k=1}^{Q_2} \beta_k^{(2)} e_{i,t-k}^2. \end{cases} \tag{14}$$

and

$$\Gamma_t = \begin{cases} (1 - \theta_1^{(1)} - \theta_2^{(1)})\Gamma^{(1)} + \theta_1^{(1)} \Gamma_{t-1} + \theta_2^{(1)} \Psi_{t-1}, \\ (1 - \theta_1^{(2)} - \theta_2^{(2)})\Gamma^{(2)} + \theta_1^{(2)} \Gamma_{t-1} + \theta_2^{(2)} \Psi_{t-1}. \end{cases} \tag{15}$$

Chan (1993) has shown the strong consistency of the estimator of a threshold model. In particular, the threshold value is super-consistent in the sense that, $\hat{l} = l + O_p(1/N)$. We now propose a method for estimating the threshold values. For simplicity, the same threshold structure of the mean and conditional variance equations are considered. When there is a threshold structure in the mean equation, it is noted in Section 3 that the least squares estimates of the mean equation are biased. Then we can take this into account by the arranged autoregression method.

The next step is to locate the threshold value l, so that we can divide observations into regimes. To do this, the true value of l satisfies $r_{\mu(s)} \leq l < r_{\mu(s+1)}$. In general, there are several methods to estimate the threshold values. Tsay (1989) considered the empirical percentiles as candidates for the threshold values. In this articles, all the values in the set S are considered to be candidates for the threshold values. We also adopt the approach of Tsay (1989) by considering scatterplots of various statistics versus the specified threshold variable as method for locating the threshold. In the arranged autoregression framework, the threshold model consists of various model changing at each candidate threshold value. For simplicity, the AR order of the mean equation is usually taken to be of low order, say $p = 1$. Therefore, the values of the lag-1 AR coefficients are biased at the threshold values. A scatterplot of the lag-1 AR coefficients versus the threshold variable thus may reveal the locations of the threshold values.

At each candidate threshold value, the lag-1 AR coefficients in the first and second regime, $\Phi_1^{(1)}$ and $\Phi_1^{(2)}$ can be calculated respectively. However, the lag-1 AR coefficients have n different values in each regime. In order to obtain a relevant scatterplot, we therefore have to consider a real value deterministic function which can differentiate between $\Phi_1^{(1)}$ and $\Phi_1^{(2)}$. Here, the deterministic function will be defined as the mean of all the entries of the lag-1 AR coefficients.

A scatterplot can then be obtained by plotting the values of the suggested deterministic function against the values of the threshold variable. Following Tsay's (1989) approach, the threshold can be obtained.

After locating the threshold value, the series in each regime of model (13) becomes linear. Moreover, the threshold structure also applies to (14). The remaining task is to estimate the parameters in (14). Assuming normality, $e_t|F_{t-1} \sim N(0, H_t)$, and the conditional loglikelihood at time t, L_t is given by

$$\begin{aligned} L_t &= -\frac{1}{2}(n \log 2\pi + \log |H_t| + e_t' H_t^{-1} e_t) \\ &= -\frac{1}{2}(n \log 2\pi + \log |\Gamma_t| + \log \sigma_{i,t}^2 + e_t' D_t^{-1} \Gamma_t^{-1} D_t^{-1} e_t) \end{aligned}$$

and thus the loglikelihood function, $L = \sum_{t=1}^{T} L_t$ can be obtained.

Remark. Asymptotic normality of the estimated parameter $\hat{\theta}$ can be established as in Chan (1993). In actual estimation, numerical derivative will be applied in the conjugate gradient method instead. It is because the size of the data series usually are large enough so that the estimated results using numerical derivative are still appropriate. In addition, the number of operations in estimation required for the numerical derivative is much less than that for the theoretical derivative. However, the derivatives of L will be useful if the sample size is small because more information about the gradient is often required for speedy convergence.

In checking the adequacy of the ARMA models with homogeneous conditional covariance over time, residual autocorrelations has been widely applied. Li (1992) proposed the asymptotic distribution of residual autocorrelations of a general threshold nonlinear time series model. Li & Mak (1994) provided the asymptotic distribution of squared residual autocorrelations of a general conditional heteroscedastic nonlinear time series model. Tse (2002) proposed an asymptotic distribution of his residual-based diagnostics for conditional heteroscedasticity models. However, the asymptotic covariances of the standardized residual autocorrelations and the squared residual autocorrelations are all very complicated. In order to simplify the complexity, Ling & Li (1997) proposed and derived the asymptotic distribution of the lag j sum of squared residual autocorrelations $\hat{R}_j$ of the model with $j = 1, \ldots, M$. Here, the lag l sum of squared residual autocorrelations of i-th regime, $\hat{R}_j^{(i)}$ is defined as

$$\hat{R}_j^{(i)} = \frac{\sum_{k=l+1}^{n_i} (\hat{e}'_{\mu_i(k)} \hat{H}^{-1}_{\mu_i(k)} \hat{e}_{\mu_i(k)} - \tilde{e})(\hat{e}'_{\mu_i(k-l)} \hat{H}^{-1}_{\mu_i(k-l)} \hat{e}_{\mu_i(k-l)} - \tilde{e})}{\sum_{k=1}^{n_i} (\hat{e}'_{\mu_i(k)} \hat{H}^{-1}_{\mu_i(k)} \hat{e}_{\mu_i(k)} - \tilde{e})^2}$$

with

$$\tilde{e} = \frac{1}{n_i} \sum_{k=1}^{n_i} \hat{e}'_{\mu_i(k)} \hat{H}^{-1}_{\mu_i(k)} \hat{e}_{\mu_i(k)}$$

where n_i is the number of observations in i-th regime and $\mu_i(k)$ denotes the time index of the kth smallest threshold variable in the ith regime. Intuitively, $\hat{R}_l^{(i)}$ is the lag l sum of squared residual autocorrelations within ith regime. The quantity $n_i \left[\hat{R}_l^{(i)}\right]^2$ is assumed to be asymptotically a chi-square random variable with one degree of freedom which is the correct asymptotic distribution when $\hat{e}_t$ are replaced by their population counter-part.

6 Simulations

Simulated realization of the VC-MTGARCH(1;1;1;2) model are used to investigate the finite sample performance of the identification and estimation procedure in this section. In the simulation, 100 independent replications with sample sizes $1,000$ and $2,000$ are generated. The initial value for every parameter is set to be zero. For simplicity, the threshold structure of the mean equation and the conditional equation are the same. The threshold variable, r_{t-d} is considered to be the first entry of the series with delay parameter equals to one. Also, the threshold value is set equal to zero. In table 1, the parameters in the simulation model are shown.

As stated in Section 5, a real value deterministic function should be defined for differentiating between $\Phi_1^{(1)}$ and $\Phi_1^{(2)}$. In the estimation process, the deterministic function is the mean of the elements of Φ_1. The average estimated threshold values are 0.1407 and 0.1365 of the sample sizes $1,000$ and $2,000$ respectively. The results are close to the original value. The average estimated results of the simulated models are summarized in the tables below. Values inside parenthesis are the standard errors of the estimated results.

From Tables 2 and 3, the estimates are in general fairly close to the true value. The proportion of rejections based on the upper fifth percentile of the corresponding asymptotic χ^2 distribution is summarized in Table 4. The simulation was performed assuming that d and the threshold value are known. The overall empirical size seems acceptable. It is observed that the estimates are closer to the true value while the standard errors becomes smaller as the sample size is larger. The result agrees with Chan (1993)'s strong consistency result on the estimators. It can be seen that the estimates have very small biased and standard errors. Note that the threshold values are well-estimated which is consistent with Chan's univariate result that these estimates are super-consistent with a rate of n^{-1}.

7 Empirical Results

Empirical examples of the time-varying correlation multivariate threshold GARCH model are presented for two interesting series considered by Tse & Tsui (2002). These two sets of data are transformed to first order differences of log value in percentage. The first data set consists

Table 1: *Parameters of the simulated model*

regime	Variable	Φ_0	Φ_1	C	α	β	θ_1	θ_2	ρ
First	1	0.1	0.7	0.1	0.7	0.1	0.4	0.2	0.5
	2	0.1	0.4	0.1	0.7	0.1			
Second	1	0.1	0.4	0.1	0.5	0.1	0.3	0.2	0.4
	2	0.1	0.2	0.1	0.5	0.1			

Table 2: *Estimation result from 100 simulated series with series length 1000*

regime	Variable	Φ_0	Φ_1	C	α	β	θ_1	θ_2	ρ
First	1	0.0905 (0.0320)	0.6804 (0.0766)	0.1012 (0.0036)	0.6809 (0.0035)	0.0866 (0.0120)	0.3708 (0.1059)	0.2024 (0.0090)	0.5085 (0.0131)
	2	0.0970 (0.0227)	0.3926 (0.0656)	0.1006 (0.0022)	0.6808 (0.0962)	0.0860 (0.0123)			
Second	1	0.0960 (0.0193)	0.4042 (0.0511)	0.1132 (0.0098)	0.4990 (0.0318)	0.0927 (0.0173)	0.3130 (0.0504)	0.2168 (0.0291)	0.4258 (0.0136)
	2	0.0988 (0.0136)	0.1990 (0.0396)	0.1139 (0.0090)	0.4908 (0.0230)	0.0918 (0.0161)			
	Threshold	0.1407 (0.0347)							

Table 3: *Estimation result from 100 simulated series with series length 2000*

regime	Variable	Φ_0	Φ_1	C	α	β	θ_1	θ_2	ρ
First	1	0.1006 (0.0209)	0.6975 (0.0531)	0.1004 (0.0012)	0.6805 (0.0021)	0.0845 (0.0091)	0.3869 (0.0968)	0.2003 (0.0022)	0.5029 (0.0293)
	2	0.1023 (0.0169)	0.3846 (0.0468)	0.1003 (0.0013)	0.6805 (0.0024)	0.0849 (0.0102)			
Second	1	0.1004 (0.0147)	0.3961 (0.0338)	0.1154 (0.0074)	0.4883 (0.0213)	0.0909 (0.0151)	0.3108 (0.0099)	0.2042 (0.0105)	0.4183 (0.0036)
	2	0.1002 (0.0109)	0.1960 (0.0278)	0.1146 (0.0063)	0.4881 (0.0211)	0.0883 (0.0137)			
	Threshold	0.1365 (0.0303)							

Table 4: *Empirical Size of the diagnostic statistics with 100 replications*

Series Length = 1000

		Lag	1	2	3	4	5	6
Regime	First		0.06	0.07	0.04	0.03	0.05	0.02
	Second		0.04	0.05	0.07	0.06	0.05	0.04

Series Length = 2000

		Lag	1	2	3	4	5	6
Regime	First		0.06	0.06	0.04	0.03	0.03	0.02
	Second		0.03	0.04	0.03	0.06	0.05	0.04

of the stock market indices of the Hong Kong and the Singapore markets, the Hang Seng Index (HSI) and the Straits Time Index (SES) for Hong Kong and Singapore respectively. These series represent 1,942 daily (closing) prices for each series from January 1990 through March 1998. The second data set consists of two exchange rate (versus U.S. dollar) series, namely the Deutsche Mark and the Japanese Yen. There are 2,131 daily observations covering the period from January 1990 through June 1998. Tse & Tsui (2002) suggested a parsimonious AR order of the conditional mean equation to fit these two data sets. According to the model checking discussed in Section 5, the lag-l sum of squared (standardized) residual autocorrelations are also given below assuming $M = 6$ for each of the considered data set.

7.1 Hang Seng Index and Straits Time Index

The dramatic rise in the HSI over the latter 1990s puzzled many portfolio managers. Tse & Tsui (2002) pointed out that the national stock markets in Hong Kong experienced different phases of bulls and bears over the 1990s. It is also found that the HSI has always been more volatile than the SES but that this gap widens at the end of this sample. Tse & Tsui (2002) showed that significant serial autocorrelations are present.

Here, the VC-MTGARCH(1;1;1;2) model is used to fit this data set. Let $Z_{1,t}$ represents the HSI and $Z_{2,t}$ represents the SES. Graphically, it suggests that the stock market indices in Hong Kong have a larger volatility than Singapore. During the observed period, it is known that the HSI has a giant rise before 1997, while there is a huge drop in 1998. It is believed that there may be a structural change in the economy and hence a threshold model would be relevant. Let the lagged values ($d = 1$) of the HSI, $Z_{1,t-1}$ be the threshold variable. The nonlinearity test statistic, $R(d) = 80.13$. The asymptotic distribution of the statistics is χ^2 with 6 df. Therefore the test strongly suggests threshold nonlinearity. Choosing the mean of all the entries of lag-1 AR estimates as the deterministic function. A scatterplot of the deterministic function against the threshold variable is given in Figure 1. Using the proposed method suggested in Section 5, the threshold value is estimated at 0.3423. It is found that there are 1164 observations belonging to the first regime. A threshold model can be obtained and the estimated parameters are given in the Table 5. It is shown in the second regime that the estimated parameters in the mean equation are nearly all positive and the elements of the GARCH parameter in the conditional variance equation are larger than those in the first regime. It is due to the rapid growth of the economy in Hong Kong and Singapore. In the first regime, the estimated conditional mean parameters in the mean equation of HSI are negative. It is suggested that there is frequent reversals of HSI over the years. This is consistent with Li & Lam (1995). The first regime represents economic slowdown or recession, especially the period experiencing the Asian financial crisis in 1997 and the giant shock of the stock market in 1998. In the second regime, which represents economic expansion period. Owing to the rapid growth of the Hong Kong economy in 1994 to 1997, the HSI increased substantially with a high volatility. It is found that the second regime in the threshold model captures this phenomenon. However, the volatility of the first regime is still larger than that of the second regime. It is believed that the HSI and SES experienced a higher volatility in the economic slowdown or recession period, and should be related to the Asian financial crisis in 1997. From table 6, squared residual autocorrelations at lags 1 and 4 are slightly significant under the reference χ_1^2 distribution. However, this seems acceptable when compared with the highly significant lag 2 squared residual autocorrelation of the VC-MGARCH (1;1;1) model in table 8. The corresponding chi-square statistic has a value of about 39.6. The LLF of the non-threshold model is also somewhat smaller than the total

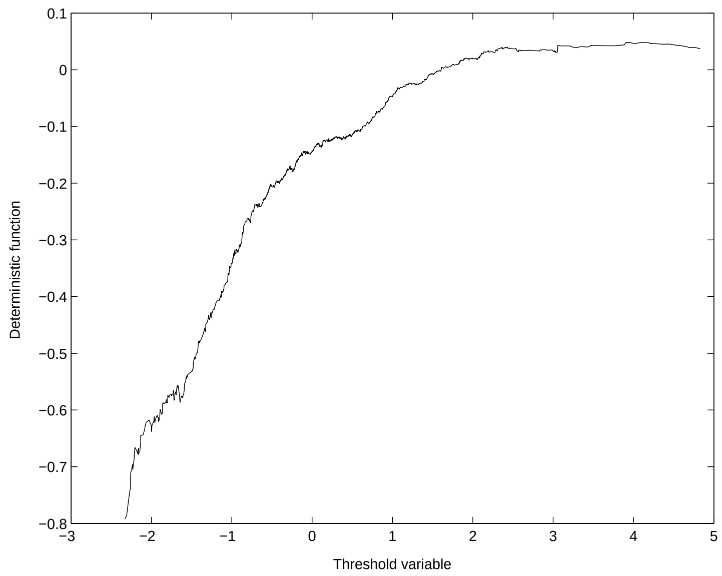

Figure 1: *Threshold value plot of National stock market data.*

sum of the LLF's of the threshold model. In figure 2, the time-varying correlation pattern is shown. It is observed that HSI and SES are highly correlated after 1994. It suggests that the economic relationship between Hong Kong and Singapore after 1994 is closely linked. Table 7 summarizes the estimation result of the VC-MGARCH(1;1;1) model for the stock return data set. It seems that the VC-MTGARCH model captures better the movement of these two time series.

7.2 Japanese Yen and Deutsche Mark

The second empirical example is the exchange rate data of the Japanese Yen and the Deutsche Mark with respect to the U.S. Dollar. As there is a huge economic recession in Japan during the observed period, it is believed that changes in the relationship between the economic variables and the exchange rates may follow a threshold model. The VC-MTGARCH(1;1;1;2) model seems a natural choice to be considered for this data set. Let $Z_{1,t}$ be the Japanese Yen and $Z_{2,t}$

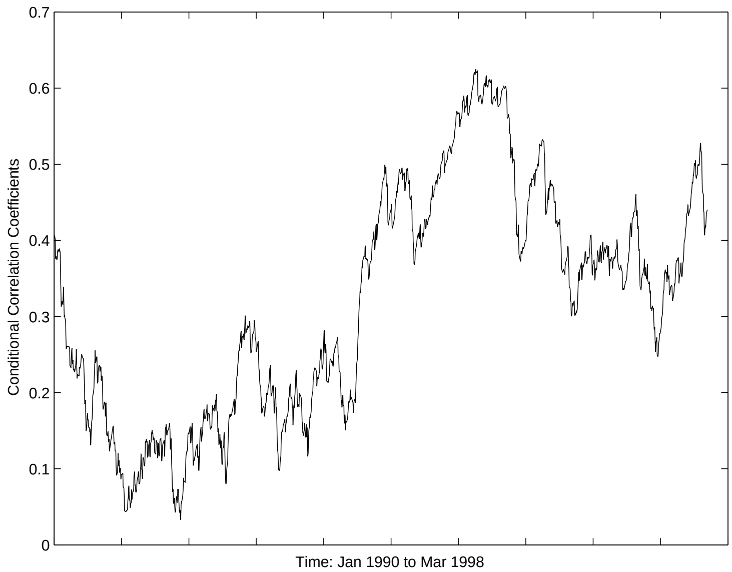

Figure 2: *Conditional Correlation Coefficients of (H,S), VC-MTGARCH*

Table 5: *National stock index data, Hang Send Index vs SES Index (VC-MTGARCH(1;1;1;2))*

regime	Variable	Φ_0	Φ_1	C	α	β	θ_1	θ_2	ρ
1	H	−0.1643 (0.0634)	−0.1200 (0.0409)	0.1827 (0.0529)	0.8048 (0.0328)	0.1808 (0.0185)	0.9944 (0.0073)	0.0026 (0.0040)	0.6918 (0.1428)
	S	−0.0442 (0.0326)	0.1750 (0.0301)	0.0868 (0.0251)	0.6726 (0.0410)	0.2570 (0.0244)			
2	H	0.3181 (0.0820)	−0.0368 (0.0427)	0.1738 (0.0614)	0.8443 (0.0359)	0.0212 (0.0095)	0.9802 (0.0072)	0.0188 (0.0043)	0.6018 (0.0511)
	S	0.0605 (0.0376)	0.1799 (0.0353)	0.1189 (0.0263)	0.7727 (0.0379)	0.0850 (0.0170)			

Table 6: *The squared standardized residual autocorrelation and LLF of the National stock index data (VC-MTGARCH(1;1;1;2))*

regime	$\left[\hat{R}_1^{(i)}\right]^2$	$\left[\hat{R}_2^{(i)}\right]^2$	$\left[\hat{R}_3^{(i)}\right]^2$	$\left[\hat{R}_4^{(i)}\right]^2$	$\left[\hat{R}_5^{(i)}\right]^2$	$\left[\hat{R}_6^{(i)}\right]^2$	LLF
1	4.2855×10^{-3}	5.0103×10^{-4}	1.6301×10^{-4}	2.5835×10^{-4}	4.9904×10^{-3}	1.7854×10^{-3}	-4.7872×10^{3}
2	1.4197×10^{-4}	9.7563×10^{-5}	3.2380×10^{-4}	3.4338×10^{-4}	6.5263×10^{-6}	8.9405×10^{-6}	-3.3026×10^{3}

Table 7: *National Stock index data, Hang Seng Index vs SES Index (VC-MGARCH(1;1;1))*

Variable	Φ_0	Φ_1	C	α	β	θ_1	θ_2	ρ
H	0.0709 (0.0389)	0.0122 (0.0227)	0.1793 (0.0175)	0.8029 (0.0145)	0.1241 (0.0111)	0.9547 (0.0091)	0.0314 (0.0062)	0.4260 (0.0183)
S	-0.0008 (0.0240)	0.1876 (0.0223)	0.1029 (0.0090)	0.6985 (0.0175)	0.2061 (0.0176)			

Table 8: *The squared standardized residual autocorrelation and LLF of National stock index data (VC-MGARCH(1;1;1))*

$\hat{R}_1^2$	$\hat{R}_2^2$	$\hat{R}_3^2$	$\hat{R}_4^2$	$\hat{R}_5^2$	$\hat{R}_6^2$	LLF
7.8970×10^{-4}	2.0415×10^{-2}	5.4042×10^{-7}	9.6642×10^{-6}	8.6647×10^{-5}	3.1976×10^{-4}	-8.1959×10^{3}

be the Deutsche Mark. From the graph in Tse & Tsui (2002), it is found that the Deutsche Mark has a smaller variation than the Japanese Yen. Choosing lagged value ($d = 1$) of the Japanese Yen itself, $Z_{1,t-1}$ as the threshold variable. The nonlinearity test statistic, $R(d) = 80.26$. The asymptotic distribution of the statistics is χ^2 with 6 df. Therefore, as expected, the test strongly suggests threshold nonlinearity. Using the mean of the estimated lag-1 AR parameter as the deterministic function, a scatterplot of the deterministic function against the threshold variable is given in Figure 3. Using the proposed method suggested in Section 5, the threshold value is estimated to be 0. There are 1037 observations belonging to the first regime. A threshold model is identified and the estimated results are given in Table 9. It can be observed that the exchange rate data set has a double threshold structure. A VC-MGARCH model, without the threshold phenomenon, for the exchange rate data set is shown in Table 11.

It is found that the estimated non-threshold model is similar to the first regime model as shown above. Starting from 1993, the Japanese economy has been experiencing a great

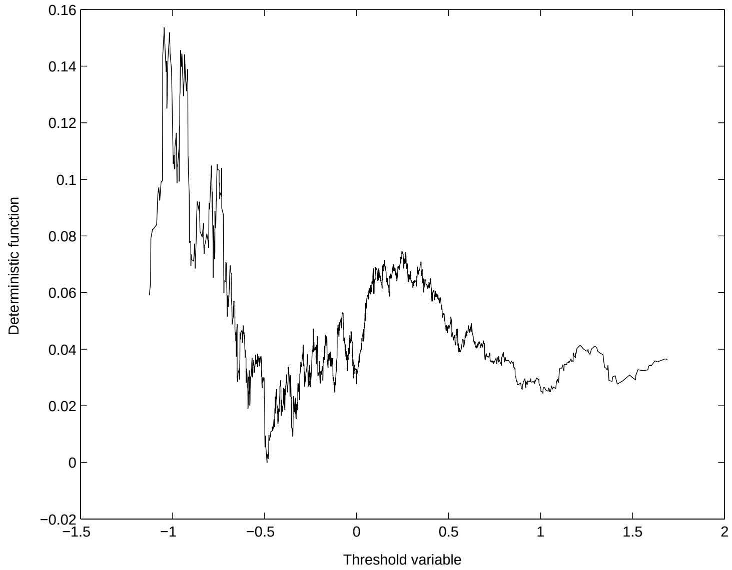

Figure 3: *Threshold value plot of Forex market data.*

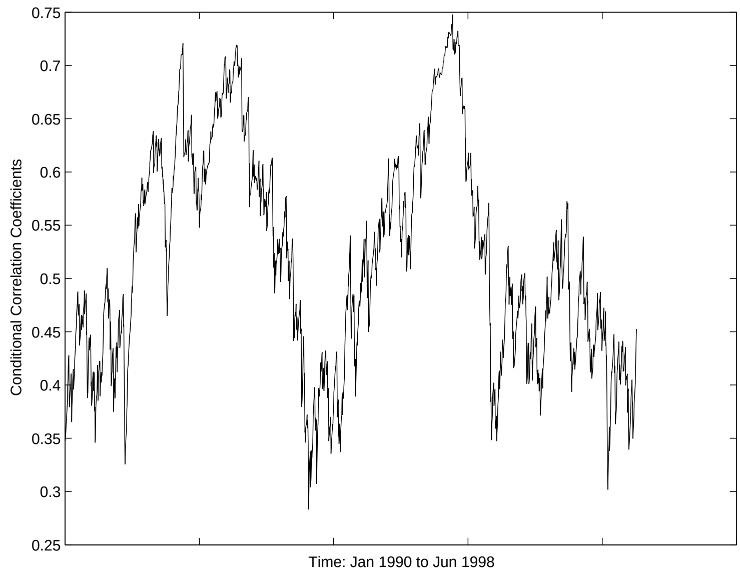

Figure 4: *Conditional Correlation Coefficients of (D,J), VC-MTGARCH*

Table 9: *Forex market data, Japanese Yen vs Deutsche Mark (VC-MTGARCH(1;1;1;2))*

regime	Variable	Φ_0	Φ_1	C	α	β	θ_1	θ_2	ρ
1	J	-0.0170 (0.0308)	0.0323 (0.0431)	0.0217 (0.0173)	0.9251 (0.0491)	0.0369 (0.0080)	0.9680 (0.0106)	0.0258 (0.0073)	0.4607 (0.0253)
	D	0.0054 (0.0232)	0.0489 (0.0341)	0.0079 (0.0337)	0.9601 (0.1069)	0.0389 (0.0109)			
2	J	0.0236 (0.0299)	0.0084 (0.0170)	0.0010 (0.0238)	0.9335 (0.0637)	0.0598 (0.0107)	0.9866 (0.0220)	0.0010 (0.0197)	0.7211 (0.1368)
	D	0.0067 (0.0227)	0.0170 (0.0338)	0.0073 (0.0191)	0.9032 (0.0386)	0.0664 (0.0351)			

Table 10: *The squared standardized residual autocorrelation and LLF of Forex market data (VC-MTGARCH(1;1;1;2))*

regime	$\left[\hat{R}_1^{(i)}\right]^2$	$\left[\hat{R}_2^{(i)}\right]^2$	$\left[\hat{R}_3^{(i)}\right]^2$	$\left[\hat{R}_4^{(i)}\right]^2$	$\left[\hat{R}_5^{(i)}\right]^2$	$\left[\hat{R}_6^{(i)}\right]^2$	LLF
1	2.3667×10^{-3}	2.2564×10^{-3}	3.0883×10^{-6}	3.0802×10^{-5}	7.0402×10^{-5}	2.0952×10^{-3}	-1.9545×10^{3}
2	2.1167×10^{-4}	2.2666×10^{-3}	6.8026×10^{-5}	4.7915×10^{-3}	4.6243×10^{-4}	1.3124×10^{-3}	-1.7083×10^{3}

Table 11: *Forex market data, Japanese Yen vs Deutsche Mark (VC-MGARCH(1;1;1))*

Variable	Φ_0	Φ_1	C	α	β	θ_1	θ_2	ρ
J	-0.0020 (0.0146)	0.0431 (0.0217)	0.0151 (0.0027)	0.9122 (0.0107)	0.0557 (0.0069)	0.9666 (0.0191)	0.0148 (0.0036)	0.5750 (0.0191)
D	0.0021 (0.0146)	0.0343 (0.0217)	0.0181 (0.0028)	0.8834 (0.0110)	0.0765 (0.0073)			

Table 12: *The squared standardized residual autocorrelation and LLF of Forex market data (VC-MGARCH(1;1;1))*

$\hat{R}_1^2$	$\hat{R}_2^2$	$\hat{R}_3^2$	$\hat{R}_4^2$	$\hat{R}_5^2$	$\hat{R}_6^2$	LLF
9.4099×10^{-4}	6.9398×10^{-4}	1.1401×10^{-5}	2.8431×10^{-4}	1.8954×10^{-6}	4.6954×10^{-5}	-3.7391×10^{3}

Table 13: *Forex market data, Japanese Yen vs Deutsche Mark (VCOV-MTGARCH(1;1;1;2))*

regime	Variable	Φ_0	Φ_1	C	A	B
1	J	−0.0170 (0.0308)	0.0323 (0.0431)	0.2564 (0.0781)	0.5419 (0.1741)	0.4481 (0.3859)
	D	0.0054 (0.0232)	0.0489 (0.0341)	0.2200 (0.1041)	0.6041 (0.2160)	0.3859 (0.0416)
2	J	0.0236 (0.0299)	0.0084 (0.0170)	0.1792 (0.1424)	0.5877 (0.2892)	0.4023 (0.0613)
	D	0.0067 (0.0227)	0.0170 (0.0338)	0.2008 (0.0351)	0.4820 (0.0838)	0.5080 (0.0348)

recession. The Japanese Yen had a huge drop during the observed period. During this period the volatility of the exchange rate market is high. The first regime in the threshold model captures this phenomenon. It is observed that the parameters in the mean equation of the second regime are all positive. The second regime represents a moderate or stable economy. The period before the Japanese economic recession and also the period for some rebound of the Japanese Yen in 1995 and the end of 1997 to 1998 are represented in the second regime. From Table 9, it is observed that the mean parameters are not significant in both regimes but the rho between the Japanese Yen and the Deutsche Mark of the two regimes are quite different. It is found that all the parameters in the second regime is positive. It is due to the rising of the Japanese Yen. In Tables 10 and 12, it is shown that the total sum of the LLFs in the threshold model is greater than the LLF in the non-threshold model. In figure 4, it can be seen that the conditional correlations of the VC-MTGARCH model have a sharper pattern than that of the VC-MGARCH model of Tse and Tsui (2002). Note that all the squared residual ACFs of both the threshold model and non-threshold model are not significant under the reference χ_1^2 distribution. It is believed that the threshold model better represents the data.

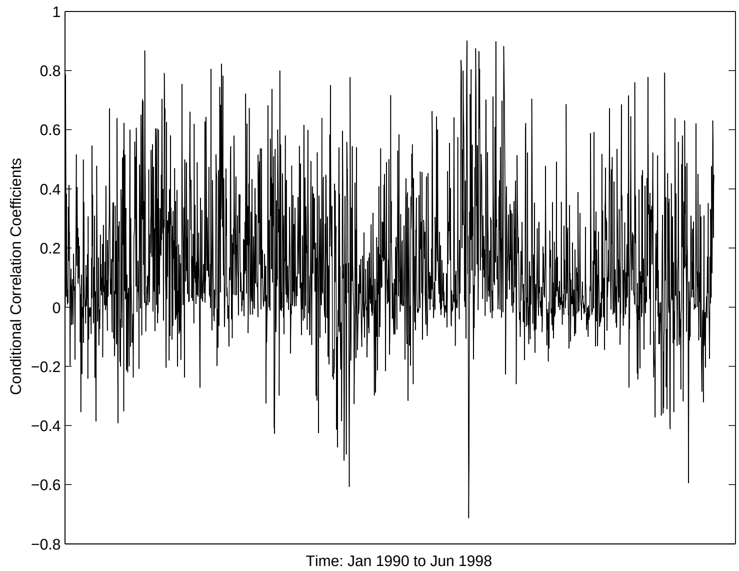

Figure 5: *Conditional Correlation Coefficients of (D,J), VCOV-MTGARCH*

Table 14: *The squared standardized residual autocorrelation and LLF of Forex market data (VCOV-MTGARCH(1;1;1;2))*

regime	$\left[\hat{R}_1^{(i)}\right]^2$	$\left[\hat{R}_2^{(i)}\right]^2$	$\left[\hat{R}_3^{(i)}\right]^2$	$\left[\hat{R}_4^{(i)}\right]^2$	$\left[\hat{R}_5^{(i)}\right]^2$	$\left[\hat{R}_6^{(i)}\right]^2$	LLF
1	2.3706×10^{-3}	1.0787×10^{-3}	2.0389×10^{-3}	1.6325×10^{-3}	4.0664×10^{-4}	3.1034×10^{-4}	-2.3890×10^{3}
2	9.6025×10^{-4}	1.1517×10^{-6}	1.4708×10^{-3}	3.9979×10^{-4}	1.4009×10^{-4}	1.8405×10^{-4}	-1.9986×10^{3}

Table 15: *Forex market data, Japanese Yen vs Deutsche Mark (VCOV-MGARCH(1;1;1))*

Variable	Φ_0	Φ_1	C	A	B
J	-0.0020 (0.0146)	0.0431 (0.0217)	0.2186 (0.0278)	0.5671 (0.0651)	0.4429 (0.0242)
D	0.0021 (0.0146)	0.0343 (0.0217)	0.2145 (0.0432)	0.5364 (0.0876)	0.4536 (0.0244)

Table 16: *The squared standardized residual autocorrelation and LLF of Forex market data (VCOV-MGARCH(1;1;1))*

$\hat{R}_1^2$	$\hat{R}_2^2$	$\hat{R}_3^2$	$\hat{R}_4^2$	$\hat{R}_5^2$	$\hat{R}_6^2$	LLF
1.0733×10^{-3}	1.1100×10^{-3}	2.3475×10^{-3}	6.1053×10^{-4}	1.4375×10^{-3}	6.6775×10^{-5}	-4.4124×10^{3}

As an illustration of the VCOV-MTGARCH model, the VCOV-MTGARCH(1;1;1;2) model is used to fit this data set. Similar to the approach of the VC-MTGARCH(1;1;1;2) model, let $Z_{1,t}$ be the Japanese Yen and $Z_{2,t}$ be the Deutsche Mark. Estimation result is given in Table 13. It is found that the threshold structure of the conditional variance equation in the VCOV-MTGARCH model is significant. However, the loglikelihood is smaller than that obtained in the VC-MTGARCH model. A non-threshold structure VCOV-MGARCH model is also shown in Table 15. It is found that the total sum of the LLFs of the VCOV-MTGARCH model is greater than the LLF of the VCOV-MGARCH model in table 14 and 16. However, the total sum of the LLFs of the VC-MTGARCH model is still greater than that of the VCOV-MTGARCH model. Besides, in figures 4 and 5, it can be seen that the correlation pattern of the VCOV-MTGARCH model is not as clear as that of the VC-MTGARCH model. As a result, the VC-MTGARCH model seems to be a better model in representing the data set.

8 Conclusion

The model structure of the VC-MTGARCH is an extension and a synthesis of the work of Tong, Tsay, Tse & Tsui (2002) and Engle (2002). The conditional variance matrix is easily guaranteed to be positive definite and the conditional correlations are allowed to be non-constants. The number of parameters of the model is also parsimonious. A modelling methodology is proposed for the VC-MTGARCH model. Extensions of Tsay's identification procedures are made to identify the AR orders, GARCH orders, delay parameters and threshold parameters. Some simulation results are presented. As a by-product of the VC-MTGARCH model, the multivariate threshold GARCH model with time-varying covariance (VCOV-MTGARCH) is also defined. For empirical applications, the VC-MTGARCH and VCOV-MTGARCH models are applied to the data sets in Tse & Tsui (2002). The obtained VC-MTGARCH and VCOV-MTGARCH model seem to capture well the threshold structure in the series. However, the correlation pattern of the VC-MTGARCH model is more clear than that obtained by the VCOV-MTGARCH model. Moreover, the loglikelihood of the VC-MTGARCH model is greater than that of the VCOV-MTGARCH model. This suggests that the proposed VC-MTGARCH model should be a potentially useful tool in modelling financial time series.

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