# The Value of Liquidity and Option Timing from a Simple Game

*Niall Whelan, ScotiaBank, Toronto and*
*Ranjan Bhaduri, AlphaMetrix Alternative Investment Advisors, Chicago*

*Niall Whelan is the director of analytics and structured transactions at ScotiaBank in Toronto. Ranjan Bhaduri is the managing director and the head of research at AlphaMetrix Alternative Investment Advisors in Chicago.*

## Introduction

In a previous paper, Bhaduri and Whelan (2008), we presented a simple model of hedge fund liquidity. That paper explored the fact that not being able to pull one's money out of an investment instantaneously at a fair price can have a powerful impact on the portfolio. Real world examples include the 1993 Metallgesellschaft debacle (Smithson, 1998) and the recent difficulties experienced by the Bank of Montreal (Perkins and Stewart, 2007), as they attempt to exit from some thinly traded OTC energy derivatives. As we pointed out in our earlier paper, liquidity is a growing issue in the hedge funds arena; increased regulatory pressure has led various hedge funds to extend their lock-up period to avoid more scrutiny. Locking up investments represents a loss of liquidity to the investors and, despite its growing importance, very little quantitative work has been done to understand it in the context of hedge funds. In this paper, we explore the impact of liquidity in a simple model.

To address this question in a tractable, heuristic setting we created a model which is intended as an analogy to the liquidity question. It is not difficult to understand that the value of liquidity can also be thought of as an option value, since liquidity gives the holder of a position the right, but not the obligation, to act, which is the classic feature of an option. Whether one chooses to think in terms of liquidity value or option value is largely dictated by context. At the end of this paper, we make the connection between the two interpretations explicit in the context of our specific model.

To recall our model, we consider a hat with $b$ black balls worth -\$1 each and $w$ white balls worth \$1 each. On each turn, the player chooses whether to draw a ball from the hat and gains \$1 if a white ball is drawn, and loses \$1 if a black ball is drawn. The selection is done without replacement of balls and the player can elect to stop playing whenever they like. We would like to determine the expected payoff of this game. The role of liquidity is that the player can choose to exit as they please. We showed in Bhaduri and Whelan (2008) that the value of this game exceeds the intrinsic value of the hat itself, which is $w - b$. Subsequent to that paper, we learned of the seminal work of Shepp (Shepp, 1969) who analysed the same game and attained the same expected payoff. (Interestingly, the motivation related to accounting for biases in ESP experiments arising from preferential stopping.) The work we present here extends from that by considering higher moments and the full distribution of values, as well as relating it to topics in finance.

In this paper, we focus on analyses of the payoff and risk-return profiles. We work almost exclusively in the asymptotic limit of large hat size. This is for the three reasons: i) that limit is tractable; ii) it is most meaningful since it relates to long time horizons; and iii) it is indicative of what happens even for small hats, as we showed in Bhaduri and Whelan (2008).

First, we determine the expected value of the game and then introduce the concept of a 'strategy' of play. We then determine the standard deviation of the returns from playing, which permits an exploration of the Sharpe ratio. We then generalise the analysis by deriving the entire distribution of returns by way of the characteristic function. We find highly nonnormal value distributions, which argues against use of the Sharpe Ratio. We explore use of the Ω function (Keating and Shadwick, 2002) applied to this problem. We conclude the paper with a brief reinterpetation of the problem as an asset option. This then singles out one of the strategies as corresponding to the risk neutral value.

## The game

We start by presenting some numerical results. Each cell in Table 1 represents the expected value of employing an optimal strategy to the corresponding hat. Here, optimal strategy is defined as playing if the expected value is greater than zero, and stopping if the expected value is zero. The table indicates how to effectively play the game and can obviously be extended to larger hats.

Consider the hat with six black balls and four white balls, highlighted in Table 1. Intuitively, one might think that it is not worth playing since there are more black balls than white balls. However, we observe the counterintuitive result that the expected value is positive (equal to 1/15). Thus, it makes sense

to play even with a significant preponderance of black balls. The reason for the positive value is that the right to stop playing at any time overcomes the negative imbalance of black balls to white balls. The player's advantage of getting to choose to stop picking balls at any time is clearly large. We identify this to be a liquidity/option effect.

The best way to calculate the expected value of a given hat is by iteration, as we argued in Bhaduri and Whelan (2008). Owing to the fact that the probabilities of getting white or black balls is directly related to their numbers, the recursion relation is

$$E(b,w) = \max\left[\frac{w}{b+w}(E(b,w-1)+1) + \frac{b}{w+b}(E(b-1,w)-1), 0\right]. \quad (1)$$

The expression above, together with the boundary conditions that $E(0,w) = w$ and $E(b,0) = 0$, completely specifies the problem. Starting with the boundary cases, we can solve recursively for arbitrary values of $w$ and $b$. These are the results presented in Table 1.

## Asymptotic analysis of the value

It is natural to inquire what happens as the number of balls gets larger. For this purpose, it is useful to reparameterise as $N = b+w$ and $r = b/N$. In particular, $N \to \infty$ will play the role of an asymptotic parameter as we hold $r$ fixed. When expressed in the new variables, we call the expected value

**Table 1: The expected results of playing the game for a variety of hats.**

| w/b | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1.00 | 0.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 2.00 | 1.33 | 0.67 | 0.20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 3.00 | 2.25 | 1.50 | 0.85 | 0.34 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 4.00 | 3.20 | 2.40 | 1.66 | 1.00 | 0.44 | 0.07 | 0 | 0 | 0 | 0 |
| 5 | 5.00 | 4.17 | 3.33 | 2.54 | 1.79 | 1.12 | 0.55 | 0.15 | 0 | 0 | 0 |
| 6 | 6.00 | 5.14 | 4.29 | 3.45 | 2.66 | 1.91 | 1.23 | 0.66 | 0.23 | 0 | 0 |
| 7 | 7.00 | 6.13 | 5.25 | 4.39 | 3.56 | 2.76 | 2.01 | 1.33 | 0.75 | 0.30 | 0 |
| 8 | 8.00 | 7.11 | 6.22 | 5.35 | 4.49 | 3.66 | 2.86 | 2.11 | 1.43 | 0.84 | 0.36 |
| 9 | 9.00 | 8.10 | 7.20 | 6.31 | 5.43 | 4.56 | 3.75 | 2.95 | 2.21 | 1.52 | 0.92 |
| 10 | 10.00 | 9.09 | 8.18 | 7.28 | 6.39 | 5.52 | 4.66 | 3.83 | 3.04 | 2.30 | 1.61 |

$E_N(r)$. We can think of $E_N(r)$ as a function of $r$, parameterised by $N$. The recursion relation in the new representation is

$$(2) \quad E_N(r) = \max\left[(1-r)\left(E_{N-1}(r_1)+1\right)+r\left(E_{N-1}(r_2)-1\right),0\right],$$

where $r_1 = b/(b+w-1)$ and $r_2 = (b-1)=(b+w-1)$. The leading order expression (in $N$) are $r_1 \approx r+r/N$ and $r_2 \approx r-(1-r)/N$.

Now we expand the first argument of the max function. We account for the small differences in $N$ and $r$ by way of Taylor series expansions. To be consistent in our powers of $N$, we need to expand to first order in the derivatives with respect to $N$ and to second order in the derivatives with respect to $r$. The consistency of this will be demonstrated below.

$$(3) \quad \text{argument} \approx E_N(r) + (1-2r) + \left((1-r)(r_1-r)+r(r_2-r)\right)\frac{\partial E}{\partial r} + \frac{1}{2}\left((1-r)(r_1-r)^2+r(r_2-r)^2\right)\frac{\partial^2 E}{\partial r^2} - \frac{\partial E}{\partial N}$$

The term involving the first derivative with respect to $r$ vanishes since the two terms inside the bracket sum to zero. The second derivative does not vanish, however, so the above expression simplifies to

$$(4) \quad \text{argument} \approx E_N(r) + (1-2r) - \frac{\partial E}{\partial N} + \frac{1}{2}\frac{r(1-r)}{N^2}\frac{\partial^2 E}{\partial r^2}.$$

Henceforth, we dispense with the approximation symbol, the large $N$ approximation being implicit. Intuitively, if $r$ is close to ½, the process of drawing balls means that the cumulative winnings is similar to a Brownian process. We expect that in $O(N)$ draws, the winnings will diffuse by an amount which scales as $O(\sqrt{N})$. Equivalently, after $O(N)$ draws, $r$ will diffuse by an amount of order $O(1=\sqrt{N})$. We are, therefore, motivated to apply the ansatz:

$$(5) \quad E_N(r) = \sqrt{N}f\left(\sqrt{N}(r-1/2)\right)$$

where $f(t)$ is a function to be determined. The consistency of this ansatz will be shown in the subsequent analysis. If we apply that expression to the relation above, we find that the derivate with respect to $N$ gives two terms, while the second derivative with respect to $r$ gives one term:

$$(6) \quad \frac{\partial E}{\partial N} = \frac{f}{2\sqrt{N}} + \frac{r-1/2}{2}f'$$
$$\frac{\partial^2 E}{\partial r^2} = N^{3/2}f''$$

where primes denote derivatives.

We also define $t = \sqrt{N}(r-1/2)$ so that

$$(7) \quad r(1-r) = \frac{1}{4} - \frac{t^2}{N} \rightarrow \frac{1}{4}$$
$$1-2r = -\frac{2t}{\sqrt{N}}.$$

Putting this together, we have

$$(8) \quad \text{argument} = E_N(r) + \frac{1}{\sqrt{N}}\left(-2t - \frac{f}{2} - \frac{tf'}{2} + \frac{f''}{8}\right).$$

If we now assume that $E_N(r) > 0$ or equivalently that $f > 0$, this is the term that dominates the max function. The $E_N(r)$ cancels from each side of the recursion relation. What remains has a common power of $N$, which justifies the use of our ansatz and the consistency of Taylor expanding to two terms in $r$ and only one term in $N$. Had we included more terms of either, or expanded $r_1$ and $r_2$ to higher order, we would have generated correction terms of higher order in $N$.

The expression then simplifies to

$$(9) \quad f'' - 4tf' - 4f = 16t.$$

To be precise, (9) is only valid when $f>0$; otherwise, $f$ is identically zero. The equation above has a particular solution, which is $f_p(t) = -2t$, corresponding to $E = N(1-2r) = w-b$. This is, of course, just the value of the game when there is no choice to exit. To this solution we can add any solution to the homogeneous differential equation

$$(10) \quad G'' - 4tG' - 4G = 0.$$

The equation (10) is the Hermite differential equation (Abramowitz and Stegun,1965) with index -1. The first solution has various equivalent expressions

$$g(t) = 4e^{2t^2}\int_{-\infty}^{t}\mathrm{d}s\,e^{-2s^2}$$
$$(11) \quad = \sqrt{2\pi}e^{2t^2}\left(\mathrm{erf}(\sqrt{2}t)+1\right)$$
$$= \sqrt{8\pi}e^{2t^2}\phi(2t),$$

where $\phi$ is the standard normal cdf function. The prefactor of 4 is for normalisation purposes. This particular combination is sometimes referred to as the Mill's ratio. The second independent solution of (10) is simply $\tilde{g}(t) = g(-t)$. For $t$ large and negative, $g(t)$ decreases like a power law, while for $t$ large and positive, it increases exponentially. The converse is true for $\tilde{g}(t)$. In particular, $\lim_{t\to\infty}g(t) \approx 1/|t|-1/4|t|^3$ and $\lim_{t\to\infty}g(t) \approx \sqrt{8\pi}e^{2t^2}$.

As yet, we have no boundary conditions. For $t$ large and negative, the function $f$ should approach the function $f_p$. By the previous discussion, this means that we can have no amount of the second solution $\tilde{g}(t)$. The second boundary condition is that for the value of $t = t_c$ such that $f(t_c) = 0$ we should also have $f'(t_c) = 0$, as we show in the next section (where we also argue that the strategy we have outlined is optimal). The value for $t > t_c$ is then identically zero and our global solution is $C^1$.

Our solution to (9) (globally extended as per the clarification immediately after (9)) is

(12)
$$f(t) = -2t + Ag(t) \text{ for } t < t_c$$
$$= 0 \qquad\qquad \text{for } t \geq t_c$$

where $A$ is a constant to be determined. (While we have explicitly put in the condition of being zero for $t \geq t_c$, we will forego doing so henceforth for $f(t)$ and other quantities depending on $t$.) The final step is to vary the coefficient $A$, so that the second boundary condition is satisfied. We find that the solution is $A \approx 0.14653$ for which $t_c \approx 0.42$. In conclusion, our solution is

(13)
$$f(t) = -2t + \frac{2t_c}{g(t_c)}g(t).$$

The second term has the interpretation of the additional value associated with the option to stop playing at a time of the player's choosing. We can also express this in terms of how many black balls there can be for a given number of white balls such that the value is positive. It is not difficult to show that this is $b \leq w + 2\sqrt{2t_c}\sqrt{w} = w + 1.189\sqrt{w}$.

In Figure 1, we show the value for a hat with 100 balls obtained from exact implementation of the recursion relation as well as via the large $N$ approximation derived above. Clearly, the approximation has converged very well to the exact result. For reference, we also show the result from playing only as long as $w > b$, which returns $\max(w-b,0)$. That can be thought of as a 'risk-free' approach, since the outcome is certain. It can also be understood as the pay-off of an option to take possession of the entire hat.

There are two noteworthy features about our solution. The first is that the game has nonzero value even when there are more black balls than white balls. However, the domain over which the value is nonzero is limited. The other feature is that when there are more white balls than black balls, the game has more value than we might have otherwise thought.

The amount by which the value exceeds the risk-free value of $w-b$ is simply $A/(1/2-r)$, as long as $r$ is not too close to either 1 or 1/2. We get this by considering the ansatz (5), and the fact that $g \to 1/|t|$ as $t \to -\infty$. In particular, this deviation is universal and independent of $N$ (although there are higher order corrections which depend on $N$).

## The second boundary condition and specifying the strategy

Let us consider what happens to the value function when the value of $r$ (equivalently $t$) is right at the value for which $f = 0$. We seek to demonstrate

**Figure 1: The value as a function of $b$ for an $N = 100$ hat**
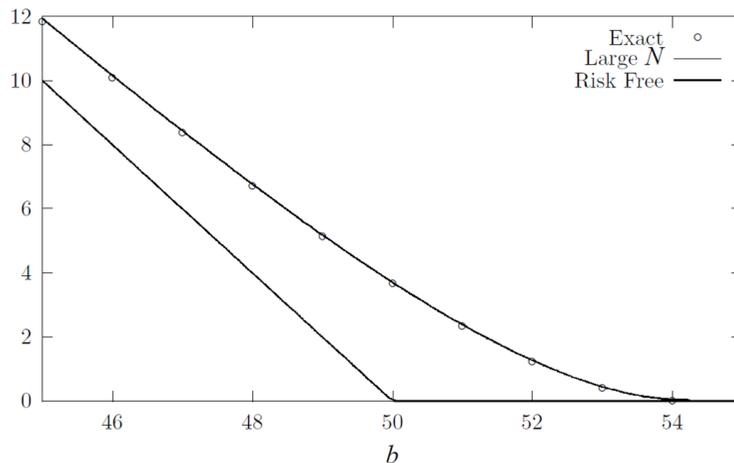
that the derivative is also zero at that point (the same conclusion is reached in a different manner in Shepp, 1969).

For a given (large) value of $N$, we evaluate Eq. (2). We imagine that the left hand side is evaluated exactly at the smallest value of $r$ for which $E_N(r) = 0$. Then it is clearly true that taking a white ball away will mean that $E_{N-1}(r_1)$ must also be zero, since it can be no more valuable than $E_N(r)$. However, $E_{N-1}(r_2) > 0$, since we have removed a black ball and the value before was just critical so, for this term, it must be slightly larger than zero.

Therefore Eq. (2) becomes

$$(14) \qquad 0 = \max[1 - 2r + rE_{N-1}(r_2), 0].$$

For this to be exactly critical requires the two arguments of the max function be equal, so that

$$(15) \qquad E_{N-1}(r_2) = -\frac{1-2r}{r}$$

To leading order in $N$, $1-2r = -2t/\sqrt{N}$, $1/r = 2$ and $r_2 = r - (1-r)/N$. We also make use of the ansatz (5), so that this equation can be expressed as

$$(16) \qquad \begin{aligned} \frac{2t}{\sqrt{N}} &= \sqrt{N}f\left(\sqrt{N}\left(r - \frac{1-r}{N} - \frac{1}{2}\right)\right) \\ &= \sqrt{N}f\left(t - \frac{1-r}{\sqrt{N}}\right) \end{aligned}$$

We then proceed by Taylor expanding the right hand side. We note that by assumption $t$ is such that $f(t) = 0$ and to leading order in $N$, we can assume $r = 1/2$. We then have

$$(17) \qquad \frac{2t}{\sqrt{N}} \approx -\frac{1}{2}f'(t)$$

so that $f'(t) \to 0$ in the limit of large $N$. Therefore, we have demonstrated that when $t$ is such that $f(t) = 0$, we also require $f'(t) = 0$ in order for the recursion relation to be satisfied at the boundary.

While the demonstration above is complete, it is not particularly enlightening. We now provide a more heuristic interpretation of that boundary condition and, in the course of doing so, provide an argument that the strategy we have devised for playing this game is optimal. To proceed, consider risk-free strategy which is simply that we keep playing as long as $r < 1/2$. As we have demonstrated, it is possible to play with positive value even when $r > 1/2$ so that we are in a suboptimal strategy. Nevertheless, there is a value associated with it.

In fact, all of the analysis holds in terms of the differential equations. The only difference is in the boundary conditions. For the risk-free strategy, the value would necessarily go to zero for $r = 1/2$ or equivalently for $t = 0$. In that case. the solution of the ODE is simply the pure particular solution with no component of $g$ or, equivalently, $A = 0$. In other words, the value is $w-b$ for $w > b$ and zero otherwise.

That example suggests a powerful paradigm. Associated with each strategy there is a value function which is determined by imposing a specific boundary condition. The optimal strategy is the one with the largest value for all $r$ or, equivalently, the one with the largest value of $A$.

Suppose we then devise a third strategy where we keep playing as long as $t$ is less than some specified value, 0.2 for the sake of argument. In that case, we would again have the same differential equation but with the boundary condition that $f(0.2)=0$ which would entail some value of $A$. However, this would be less than the $A$ associated with the solution described in the main text. As a result, the payoff would be smaller than that in the text for all values of $r$.

Therefore, we imagine a family of strategies, parameterised by the critical value of $t$ where we are willing to keep playing (this is presented graphically in Figure 2 on the following page). However, if we get greedy and try for too large a value of $t$ (0.5 for the sake of argument), we will find that the solution will be such that $f$ will be negative for some values of $t$ and will be less than the optimal strategy for all values of $t$. Clearly, the optimal strategy is the one for which the critical value of $t$ is as large as possible, while still keeping $f$ nonnegative. This largest value occurs when the function is just tangent to the $t$ axis, which is the boundary condition we have specified. In the last section of this paper, we will demonstrate how this strategy can also be understood as a risk-neutral strategy in the sense of derivative valuations.

We can define a strategy function $s(b,w)$ which equals unity if based on the value of $b$ and $w$, the player continues. Equivalently, we can use the alternate variables and define the function to be $s_N(r)$. Then the recursion relation (2) can be generalised as

$$E_N(r) = \begin{cases} (1-r)(E_{N-1}(r_1)+1) + r(E_{N-1}(r_2)-1) & s_N(r) = 1 \\ 0 & s_N(r) = 0. \end{cases} \quad (18)$$

The strategy we have focused on, which we call the optimal strategy, has $s = 1$, if the first argument of (1) is larger than the second argument, and $s = 0$ otherwise. There are other strategies one can envisage:

i) the player chooses never to play; $s(b,w) = 0$;
ii) the player chooses always to play; $s(b,w) = 1$; and
iii) the player chooses to play if $w > b$; $s(b,w) = 1$ for $w > b$ and zero otherwise.

The third of these is the risk-free strategy and is clearly preferable to the other two, since it is never less than either of them in all situations. In the language of game theory, the third strategy (weakly) dominates the other two (Thomas, 1986). The second strategy has the interpretation of being the intrinsic value of the hat. The other interesting aspect of these three strategies is that there is nothing random about the outcome. Under these strategies, the player will finish with 0, $w{-}b$ and $\max(w{-}b,0)$ respectively. Also, in general, we see that there is nothing preventing the value of the game from being negative, for example, with the second strategy when $w < b$. The condition of having a positive value is a property of the strategy and not of the game itself. Of course, any reasonable strategy will have this property. We also point out that the second and third strategies exactly correspond to the particular solution $f_p(t)$; the only difference being that, for the third strategy, we impose the additional condition that $f \equiv 0$ for $t > 0$ so that the solution has a kink at $t = 0$. The first strategy of course has a trivial solution of $f(t) \equiv 0$ for all values of $t$.

Altogether, then, we have explicitly explored four strategies: the 'optimal' one, as well as the three enumerated in the previous paragraph. The optimal strategy is constructed to maximise the expected payoff of the game. However, it can be risky. For example, with six black balls and four white balls, there is a small positive expected payoff but one is assured of getting this only on average. In any given play the player may get more or less. A risk-averse player may not feel that the small expected payoff is worth the large downside risk of losing a substantial sum. Therefore, there may be risk-averse strategies intermediate between risk-free one and the optimal one. We explore this question in the following sections.

In Figure 2, we show the function $f(t)$ for a variety of choices of $t_c$. Clearly, the optimal strategy exceeds all others for all $t$, while the risk-free strategy is less than all others for all $t$. We show two intermediate values of $t_c$. From these examples, it is simple to understand the general behaviour.

## The second moment

As we discuss above, a risk-averse player may be willing to forego some expected payoff in order to lessen the downside risk of his play. The classical way to quantify this is through the Sharpe Ratio (Sharpe, 1966 and later clarified in Sharpe, 1994), ie, the ratio of expected earnings (relative to some 'risk-free' benchmark) to the standard deviation of earnings. This requires that we determine the second moment, which is the focus of this section. A more refined approach is to consider the $\Omega$

**Figure 2: $f(t)$ for a variety of strategies**
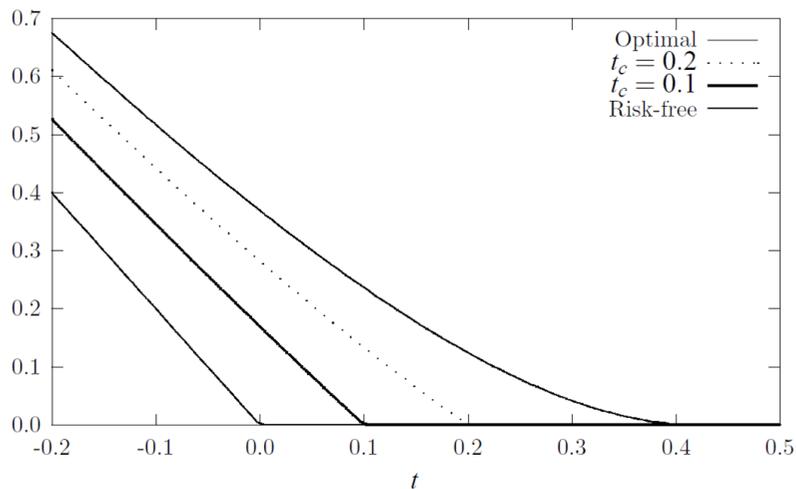
function (Keating and Shadwick, 2002) invented by mathematicians Keating and Shadwick in 2002, which is sensitive to the entire distribution and which we explore in a subsequent section.

To pursue the calculation, we extend the notation slightly. We introduce the variable $U_N(r)$ which is the actual realised value of playing the game. It is a random variable which has $E_N(r)$ as its expectation. We also introduce a Boolean random number $x$, which equals one if the next draw is black and equals zero if the next draw is white. The $U_N(r)$ satisfy the recursion relation

$$(19) \quad U_N(r) = (1-x)(U_{N-1}(r_1)+1) + x(U_{N-1}(r_2)-1)$$

as long as the strategy function says to proceed, otherwise $U_N(r) = 0$.

We can then take the expectation of both sides of the relation above. Since subsequent draws are independent, $x$ is independent of both $U_{N-1}(r_1)$ and $U_{N-1}(r_2)$, so that the expectation of the products is the product of the expectations. Using the fact that the probability that $x = 1$ is $r$, we recover Eq.(2).

This slightly more general foundation allows us to extend the calculation to other statistical measures than the expectation. A natural question is to determine the second moment of the value. We start by considering the square of the Eq.(19). By the nature of Boolean random variables (specifically, $x(1-x) \equiv 0$), we do not need to consider the cross term so that

$$(20) \quad \begin{aligned} U_N^2(r) &= (1-x)\left(U_{N-1}^2(r_1)+2U_{N-1}(r_1)+1\right) \\ &\quad + x\left(U_{N-1}^2(r_2)-2U_{N-1}(r_2)+1\right). \end{aligned}$$

The fact that $x^2 = x$ is also useful. We then take the expectation of both sides, defining $W_N(r) \equiv E\left[U_N^2(r)\right]$ so that

$$(21) \quad \begin{aligned} W_N(r) &= (1-r)\left(W_{N-1}(r_1)+2E_{N-1}(r_1)+1\right) \\ &\quad + r\left(W_{N-1}(r_2)-2E_{N-1}(r_2)+1\right). \end{aligned}$$

We now invoke the same large $N$ expansion as before. Also, we use a similar ansatz as before: $W_N(r) = Nh(\sqrt{N}(r-1/2))$ (the rationale being that the scale of $r$ should as before, but the magnitude is now of order $O(N)$ since $W_N$ is related to the square of the $E_N$, which is $O(\sqrt{N})$). We arrive at the equation for $h$ which is

$$(22) \quad h'' - 4th' - 8h = 32tf - 8f' - 8,$$

where $f$ is already determined from the first moment. As before, we can think in terms of a particular plus homogeneous solution. The left hand side is, in fact, the Hermite differential equation with index $-2$.

We first focus on the particular solution, which requires some analysis of the right hand side. We start by focusing on the first two terms of the right hand side of (22). Note that

$$\frac{d}{dt}\left(4tf - f'\right) = -\left(f'' - 4tf' - 4f\right)$$
$$(23)$$
$$= -16t$$

where, in the second line, we have Eq.(9). We integrate both sides of the equation and pick up an integration constant which must equal $-f'(0)$, as determined by considering $t = 0$. Consideration of Eq.(12) and Eq.(11) differentiated at zero argument shows that $f'(0) = -2+4A$. We conclude that

$$h'' - 4th' - 8h = -64t^2 - 32A + 8. \quad (24)$$

This has a particular solution of $h_p(t) = 4t^2+4A$.

Now, of course, to $h_p$ we can add any amount of a homogeneous solution to the differential equation for $h$. As we mentioned, that equation is the Hermite equation of index $-2$. We seek a similar boundary condition as for $f$, that the homogeneous solution vanish as $t \to -\infty$. It is not difficult to see that the function $g'(t)$ is a solution to the homogeneous equation where $g(t)$ is the solution for the first moment discussed above.

Therefore, in total, we have a solution of

$$h(t) = 4t^2 + 4A + 4Bg'(t). \quad (25)$$

We fix $B$ such that $h(t_c) = 0$ where $t_c$ is the boundary condition which defines where we stop playing since, if we do not play, all moments are trivially zero. $B$ is defined with a factor of 4 for later convenience (whereas for $f(t)$, we recall that $h(t) \equiv 0$ for $t > t_c$).

We can also express the variance (divided by $N$) as $\sigma^2 = (h-f^2)$. An explicit expression is

$$\sigma^2 = 4A + 4Bg' + 4Atg - A^2g^2. \quad (26)$$

If we want to follow strategy 3, which is risk-free, then we need to set $t_c = 0$ which, in turn, requires both $A$ and $B$ to be zero which, in turn, implies the variance is zero. This is consistent since, with that strategy, we will play until the number of each

colour is the same, at which point we will have realised exactly *w−b*, with no uncertainty. It is gratifying that this has dropped out of the analysis in such a natural manner.

It is fair to say that the 'risk-free' return is that from playing strategy 3, which is simply max(−2*t*,0). Then, using that as the point of reference, we can express the Sharpe ratio as

$$\text{(27)} \qquad \begin{aligned} \text{Sharpe} &= \frac{Ag}{\sigma} & \text{for } t < 0 \\ &= \frac{Ag - 2t}{\sigma} & \text{for } 0 < t < t_c \end{aligned}$$

In Figure 3, we show the Sharpe Ratio as a function of $t$ for a variety of choices of $t_c$. We observe a kink at $t = 0$, due to the abrupt change in the definition of the risk-free return. We also note that the curve for the 'risk-free' Sharpe is defined as a limit as $t_c \to 0$. In practise, this is not a useful investment, since both the reward and the risk go to zero. The others all show a more meaningful behaviour. In particular, at $t = 0$, we observe that the Sharpe Ratio decreases with $t_c$. (We have also confirmed agreement with direct application of the recursion relation (21) as in Figure 1 but we forego presenting results.)

In all cases, the Sharpe Ratio goes to zero at $t = t_c$, typically, as a square root in $t_c − t$. This is because the numerator has a linear behaviour, but the denominator has a square root behaviour (since $s(t)$ is defined as the square root of a linear function). For $t_c = 0.42$, the numerator has a quadratic

behaviour and the approach to zero is as a 3/2 power. The fact that the Sharpe Ratio approaches zero means that it is impossible to define a strategy that mandates playing, as long as the Sharpe Ratio exceeds some critical value. The reason is that just prior to quitting, the Sharpe Ratio will always be arbitrarily small, whatever the strategy. It is then impossible to find a self-consistent strategy other than the risk-free strategy or the optimal strategy (the latter being realised if the critical Sharpe Ratio is zero.) As we shall see, the Sharpe Ratio is actually not a particularly good metric for performance anyway, so this behaviour is not a serious shortcoming.

We can also consider the limit of the Sharpe Ratio in the limit of $t$ large and negative. In the limit, we recall that $g \approx -1/t + 1/4t^3$ so

$$\text{Sharpe} \to \frac{A}{\sqrt{4B + A(1 - A)}}, \qquad \text{(28)}$$

which is nonzero corresponding to a possibly good investment. This limiting value decreases as a function of $t_c$ from $\sqrt{\pi}/(\pi - 2) \approx 1.66$ for $t_c = 0$ to about 0.82 for $t_c = 0.42$. However, in this limit, both reward and risk go to zero so this is not particularly meaningful.

## The characteristic function

If we are interested in the entire distribution, as opposed to the first couple of moments, one efficient way to proceed is to determine the characteristic

**Figure 3: The Sharpe Ratio as a function of $t$ for a variety of strategies**
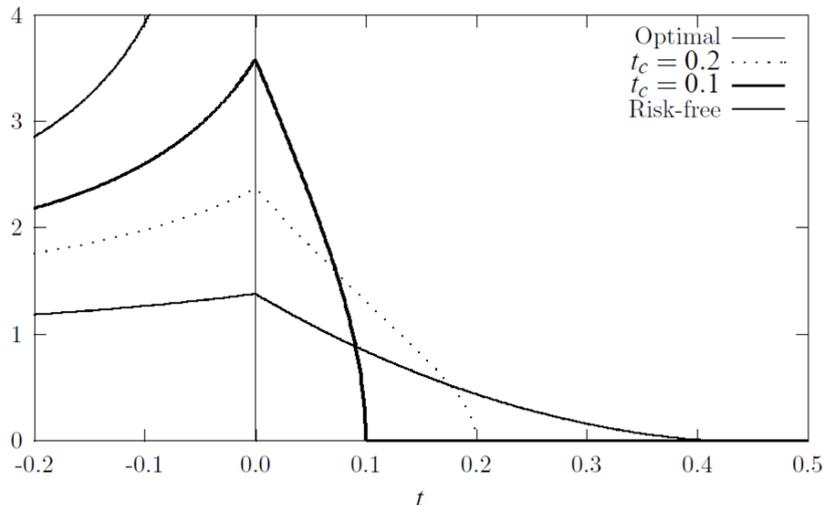
function. It has the double interpretation of being the Fourier transform of the probability distribution function (pdf) and of being the expectation of $\exp(iz U_N(r))$. Eq. (19) implies that

$$(29) \quad e^{iz U_N(r)} = e^{iz(1-2x)} e^{iz(1-x) U_{N-1}(r_1)} e^{izx U_{N-1}(r_2)}.$$

Noting that there are only two possible values for $x$ and further defining $F_N(z,r) \equiv E[\exp(iz U_N(r))]$, we conclude

$$(30) \quad F_N(z,r) = (1-r)e^{iz} F_{N-1}(z,r_1) + re^{-iz} F_{N-1}(z,r_2).$$

We know that the risk-free strategy would have a characteristic function of $\exp(izN(1-2r))$ so we will define

$$(31) \qquad F_N(z,r) = e^{izN(1-2r)} G_N(z,r)$$

and seek a solution for $G_N$.

Substitution of the previous equation into the earlier recursion relation gives the expression

$$(32) \quad G_N(z,r) = (1-r)G_{N-1}(z,r_1) + rG_{N-1}(z,r_2).$$

Clearly, $G$ is of order unity (and must be identically unity for the risk-free strategy). We have already established that a natural scale for $r$ is in terms of $\sqrt{N}$ times the deviation from $1/2$. Also, the scale of $U$ is $\sqrt{N}$, so we expect $z$ to have a corresponding scale. This motivates an ansatz of

$$(33) \qquad G_N(z,r) = H\left(\sqrt{N}z, \sqrt{N}(r-1/2)\right),$$

where $H$ is a universal function. We again define $t = \sqrt{N}(r-1/2)$, as well as $\zeta = \sqrt{N}z$, and invoke a similar Taylor expansion as for the first and second moments to arrive at the partial differential equation in the limit of large $N$

$$(34) \qquad \frac{\partial^2 H}{\partial t^2} - 4t\frac{\partial H}{\partial t} - 4\zeta\frac{\partial H}{\partial \zeta} = 0.$$

Now, we need to define the boundary conditions. As $r \to 0$ (equivalently $t \to -\infty$), $F_N$ is just $\exp(izN)$. Together, this implies $H(\zeta,-\infty) = 1$. The other limit is for $t = t_c$ for which $F \to 1$, meaning that $H(\zeta,t_c) = \exp(2i\zeta t_c)$. An interesting result is that from the boundary conditions alone and the nature of (34), we can infer the entire behaviour for $\zeta = 0$. In that limit, the third term of (34) drops out and the boundary conditions imply $H(0,-\infty) = H(0,t_c) = 1$. The only function consistent with

the reduced differential equation, and the two boundary conditions just mentioned, is unity so we can immediately state $H(0,t) \equiv 1$. Of course, this could also be derived directly from the fact that the distribution is normalised, but it is reassuring that it also falls out of the analysis.

To solve (34) we make a separability ansatz that $H(\zeta,t) = Z(\zeta)T(t)$. Dividing through by $H$, we conclude

$$(35) \qquad \zeta\frac{Z'}{Z} = \frac{T''}{4T} - t\frac{T'}{T} = k,$$

where $k$ is a separation constant. Clearly the solution to the first equation is $Z(\zeta) = \zeta^k$. The second equation can be recast as

$$(36) \qquad T'' - 4tT' - 4kT = 0.$$

This is the Hermite differential equation with index $-k$. This only has solutions consistent with $t \to -\infty$ for $k \in \mathbb{N}$. We will label the solution corresponding to $k$ as $T_k$. We note that $T_0 = 1$ and, from our analysis of the first two moments, we see that $T_1(t) = g(t)$, and $T_2(t) = g'(t) = 4tT_1 + 4$. In general, we find that

$$(37) \qquad T_k(t) = \frac{d^{k-1}}{dt^{k-1}}g(t),$$

which can easily be seen by recursion. Another useful recursion relation comes from repeated differentiation of $T_2(t)$:

$$(38) \quad T_k(t) = 4tT_{k-1}(t) + 4(k-2)T_{k-2} \quad \text{for} \quad k > 2.$$

In general, any linear combination of solutions (corresponding to different values of $k$) is possible, so we write

$$(39) \qquad H(\zeta,t) = \sum_{k=0}^{\infty} a_k \zeta^k T_k(t).$$

By virtue of the solutions $T_k$ which we selected, we have already imposed the $t \to -\infty$ boundary condition, so we only need to consider the one at $t = t_c$. We recall that $H(\zeta,t) = \exp(2it_c\zeta)$ and, by matching powers of $z$, we conclude

$$(40) \qquad a_k = \frac{(2it_c)^k}{k!T_k(t_c)}$$

leading to the final compact expression

$$(41) \qquad H(\zeta,t) = \sum_{k=0}^{\infty} \frac{(2it_c\zeta)^k}{k!}\frac{T_k(t)}{T_k(t_c)}$$

There are a number of interesting things to be drawn out of this solution. The first is that for the

risk-free strategy of $t_c = 0$, only the $k = 0$ term is nonzero and we conclude $H = 1$, as we expect. For the same reason, we find that $H(0,t) = 1$, which we earlier argued must be the case.

Among other features, the characteristic function can be thought of as a moment generating function. Due to the rescalings we have applied, $H(\zeta,t)$ is the characteristic function corresponding to the random variable $V = U/\sqrt{N} + 2t$. Therefore, we identify $E[V^k] = (2t_c)^k T_k(t)/T_k(t_c)$. This is consistent with our derivations of the first and second moment, but is more general since it includes all moments.

## Numerically determined distributions

The solution of the distribution of values discussed in the previous section, while gratifying, is not easy to implement numerically. The reason being that the characteristic function is oscillatory in $\zeta$, while our representation of it is in terms of a power series in $\zeta$. For large values of $\zeta$, this requires precise cancellation of alternating terms which are very large in absolute value. Such an exercise is notoriously difficult and, in practice, we lose all numerical precision for $|\zeta|$ larger than about 40.

Our difficulty is related to the famous 'moment problem' (Shohat and Tamarkin, 1943) of determining a distribution from its moments (since knowledge of the powers series in $\zeta$ is one and the same as knowledge of all moments of the distribution). This is a notoriously difficult problem. For example, we could represent the distribution as a sum over finely spaced delta functions whose coefficients will be fixed by specifying the moments. Such a process leads to needing to invert a Vandermonde matrix, which is intrinsically ill-conditioned. The problem is not specific to the delta function representation. Any representation in terms of a sum over representative basis functions will lead to inversion of a near-singular matrix.

What is required is to derive a complementary representation of the characteristic function valid for $|\zeta| \gg 1$. While we have not succeeded in doing that completely, we have determined the general functional form, and by 'glueing' it onto the small $|\zeta|$ representation, we have an approximate representation valid for all $\zeta$. The details are presented in the appendix and theoretical (as well as simulated) distributions are presented in Figure 4 for $t = 0$ and two values of $t_c$.

Leaving aside the numerical considerations, the distributions of values are clearly very interesting. They have the following features:

i) they have finite support, specifically $0 < V < 2t_c$;
ii) they have peaks at the two edges of the domain of support; and
iii) they are strongly bimodal.

**Figure 4: The probability distribution of the scaled value $V$ for $t = 0$ and two choices of strategy, the optimal strategy and the $t_c = 0.20$ strategy**
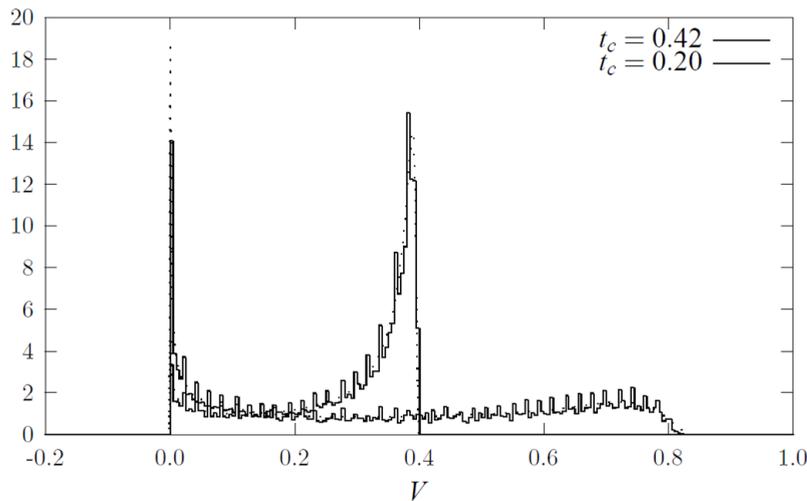


*Please note that for each, we show the result of a simulation as well as the theoretically derived distributions.*

The first point is understood by recalling that players always have the right to continue playing. Playing to exhaustion of the hat will lead to $V = 0$ and the player can never do worse than that. Similarly, the largest possible outcome is to draw a string of white balls at inception until the threshold is hit. It is not difficult to show that this will lead to $V = 2t_c$. The second and third points are, of course, related, and show that the expected value is actually not a typical value. Instead, we will typically have either substantially more or less than the expected value in any given simulation.

To explore the second two features a bit more closely, in Figure 5, we show three scenarios for an $N = 100$ hat. These are representative of finishing with large value, near-average value and small value (the final scaled values were 0.8, 0.4 and 0, respectively). Of course, just seeing these three representative trajectories on their own do not immediately explain the relative frequencies of the different results. For that, it is useful to consider the limiting cases of $t \to \infty$ and $t \to t_c$. In the first case, the trajectories start well off to the right and it is exceedingly improbable for any one of them to encounter the threshold. The bulk of them will finish at the origin with a game value equal to the initial value of $w-b$. In the other limit, the problem looks locally like a Brownian motion with a stopping time given by hitting a threshold. It is well known that, with probability unity, a given path will hit a given threshold and, in our case, this will result in a scaled value very close to $2t_c$. For general values of $t$ (such as $t = 0$), the paths can be crudely classified into one of the two types corresponding to the limiting behaviours and the resulting distribution is bimodal.

In Figure 4, we show the distributions for two choices of $t_c$. We also show the results of direct simulation. For these, we ran with very large hats consisting of 100,000 balls of each colour. For each case, we ran 100,000 simulations and binned the results. We make a few comments about the distributions. Firstly, both distributions are bounded between $V = 0$ and $V = 2t_c$, for reasons discussed above. We also note that both distributions are bimodal, but that $t_c = 0.42$ is more symmetric whereas $t_c = 0.20$ is more weighted towards the upper end. This is due to the greater relative ease of hitting the threshold as discussed in the context of Figure 5, specifically, the $t \to t_c$ limit. For both cases, there is a finite probability of finishing with zero value, which is reflected in a large weight in the first bin at $V = 0$ (representing 4.3% and 0.7% of the simulations for $t_c = 0.42$ and $t_c = 0.20$, respectively). These are from simulations which never hit the threshold. We observe this fraction to decrease with $N$ and we expect it to approach zero as $N \to \infty$, but extremely slowly since it is related to the probability distribution of a Brownian stopping time, which is extremely fat-tailed. The final observation is that there is a low amplitude, high-frequency oscillation in the distribution. This we also believe to be a finite $N$ effect and choose not to explore in this article.

**Figure 5: Three different realisations of the game with the optimal strategy**
(the smooth curve is the threshold boundary where play is stopped)
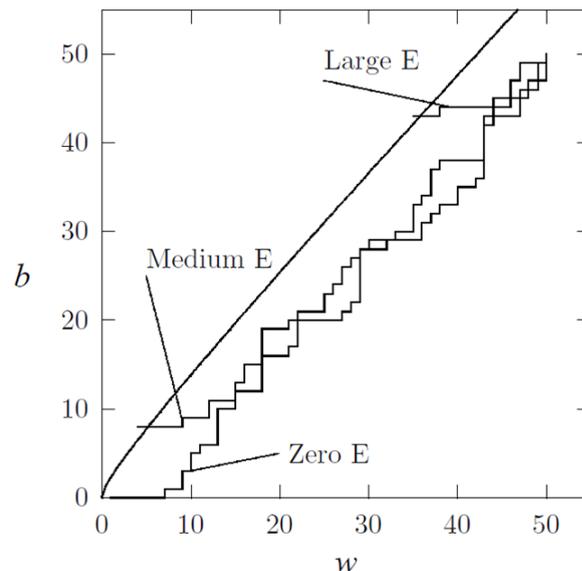
From a risk-return perspective, these results are quite interesting. While a straight analysis of the Sharpe Ratio would indicate that $t_c = 0.20$ is the superior strategy, as seen in Figure 3, this clearly misses the fact that the optimal strategy has much more upside potential, while no worse downside potential. This is actually a tailor-made application for the more sophisticated $\Omega$ analysis and we explore that in the following section.

## $\Omega$ analysis

As discussed above, the Sharpe Ratio is a popular performance measure for risk-adjusted return in the finance community. It is a rankings function to compare investment strategies. Increasingly, the investment community is recognising that it has serious shortcoming. By only making use of the first two moments of the distribution, it is not sensitive to other important statistical features such as skew, fat-tails or if the distribution is bimodal or has only finite support, as in our problem. Therefore, the Sharpe Ratio can be misleading if the underlying investment's return distribution is non-normal, and it is well known that alternative investments demonstrate non-normal distributions. For example, employing the Sharpe Ratio for a distribution that is bimodal can clearly be misleading. Consequently, the use of the Sharpe Ratio in this arena is not ideal. Moreover, the Sharpe Ratio does not distinguish between upside and downside volatility and, thus, penalises for upside volatility.

The Omega function does not suffer from any of the problems of the Sharpe Ratio. That is, the Omega function is a rankings function, which naturally incorporates all features of the distribution, and it does not penalise for upside volatility. It is defined as

$$(42) \qquad \Omega(r) = \frac{\int_r^\infty (1 - F(x))\, dx}{\int_r^\infty F(x) dx}$$

$F(x)$ refers to the cumulative distribution function of the investment and $r$ is the selected threshold. In addition to its other properties, by being a function of the threshold $r$, it is 'tunable' in a way that the Sharpe Ratio is not. In particular, the relative rankings of different strategies is a function of the target threshold $r$ ($\Omega$ has itself been generalised to a still more general statistic called Kappa (Kaplan and Knowles, 2004), but we defer from an analysis of it).

The Omega function captures upside versus downside in a mathematically precise way that has finance intuition. Kazemi, Schneeweis, and Gupta (Kazemi, et al, 2003) showed that the mathematical definition of the Omega value is equivalent to the value of a call over the value of a put. More precisely,

$$\Omega(r) = \frac{C(r)}{P(r)} \qquad (43)$$

where $r$ is the threshold value, $C(r)$ and $P(r)$ are prices of European call and put prices with strike $r$. Normally, variable $r$ refers to the return of the strategy. We work, instead, in terms of the value of the strategy. The return is defined as the value divided by the initial value. However, that is not a well-defined notion for our game, so we work exclusively in terms of value. One nice aspect of that is that the interpretation in terms of the ratio of option values is even more immediate than usual. As it turns out, working in terms of instrument values and not returns does not affect the rankings of distinct strategies. $\Omega(r)$ can be thought of as the quality of a bet on a return above a given level; 'quality' is upside versus downside. No special assumptions, such as those needed for the Black-Scholes-Merton option-pricing model, are needed.

This elegant result of math-finance duality demonstrates that the Omega function has strong mathematical foundations and a solid finance meaning. The use of the Omega function is not limited as a performance measure. By plotting Omega as a function of threshold, one can gain insights about the interplay between risk and return. We will see some simple examples of this soon.

Recall that the Sharpe ranking for $t_c = 0.2$ was higher than the Sharpe ranking for $t_c = 0.42$. One should expect the Omega and Sharpe rankings to be different in cases of high non-normality or upside volatility, though of course the choice of the threshold is an important factor as well. The results are shown in Figure 6, on the following page, and indicate that for a very risk-averse investor, $t_c = 0.2$ is more suitable, but for most risk appetites, $t_c = 0.42$ is superior.

We have also plotted the Omega graphs for the corresponding normal distributions with the same means and standard deviations. Note that, by definition, $\Omega = 1$ when evaluated at the mean of the distribution (a result which can be understood as a manifestation of put-call parity), so the

**Figure 6: Ω(V) for two choices of strategy**
(for reference we show corresponding Ω curves for normal value distributions with the
same means and standard deviations; the means are indicated by vertical bars)
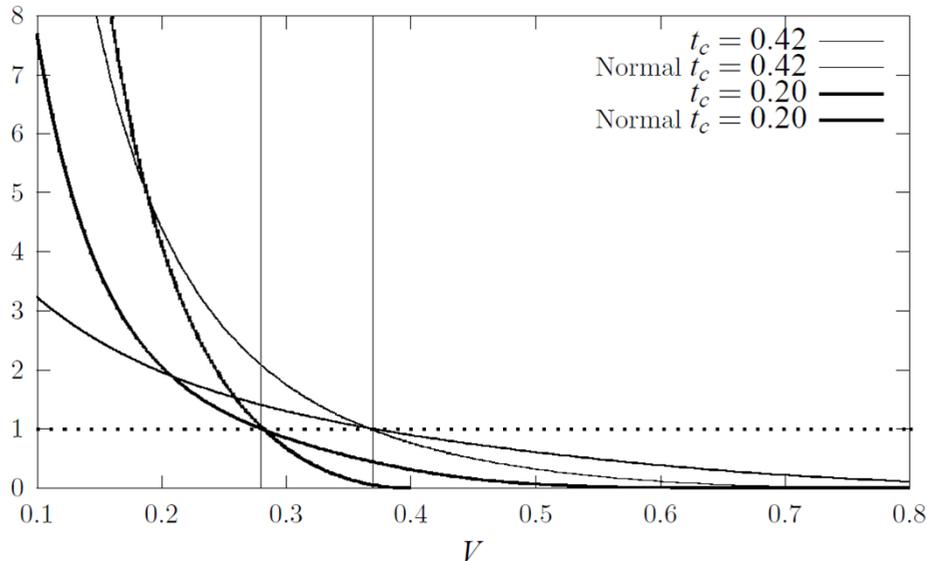


normal Omega graphs cross the actual ones at the respective means (indicated by vertical bars in the graph.) Interestingly, the cross-over value of $V$ is virtually the same ($V \approx 0.2$) for the actual and normal distributions. Also, the actual Omega graphs are much larger than the normal ones for small argument (a consequence of the fact that they diverge for $V = 0$) and contrariwise for large argument. It is interesting to note that for the actual distributions, the values for small argument are not very different, in contrast to the large differences between the corresponding normal values. This means that the Omega analysis suggests that generally the $t_c = 0.42$ strategy is never very much worse than the $t_c = 0.2$ strategy, but where it is better, it is very much better. This is a consequence of the highly non-normal value distribution.

The Omega graph of $t_c = 0.42$ has a smaller slope than that of $t_c = 0.2$. Consequently, the upside potential for $t_c = 0.42$ is larger. We define the robustness at a given threshold as the exponential of the derivative of the Omega function, so defined so that it is between zero and unity and the higher the value the more robust the result (ie, the smaller the slope.)

Use of the robustness coefficient could serve to deter hedge fund managers from smoothing results.

Smoothing is a problem that can be associated with hedge funds that trade in illiquid instruments (such as distressed securities or OTC derivatives). Hedge funds that trade in highly liquid instruments do not have the scope to smooth their results. Smoothing refers to adjusting the return series to decrease the perceived volatility. For example, in a positive month, the hedge fund might report a slightly lower return than it actually achieved, while in a negative month, it might report a slightly better return. By decreasing the volatility of the reported return series, this in turn increases the Sharpe Ratio. Smoothing is not ethical and the majority of hedge funds do not smooth. However, there still exists the problem that some do — and it is not always easy to detect.

While the effect of smoothing is to spuriously increase the Sharpe Ratio, the effect on the Omega function is mixed. (Here we envisage that the distribution from which Omega is determined is obtained by consideration of the history of returns.) Smoothing increases the Omega value for lower thresholds, but decreases the Omega value for higher thresholds. The Omega graph then has a steeper slope than it otherwise would and, thus, a lower robustness. Consequently, if analysis is using Omega, then a hedge fund manager has less incentive to smooth their results.

## Interpretation as an asset option

We now briefly outline how this game can be thought of as an option on an asset. We envisage an asset whose value is $A_n(\omega)$ where $n$ indicates the number of timesteps (where there is one draw per timestep) and $\omega \in \Omega$ is a label indicating the particular path in our probability space. In particular, $A_n$ equals the number of white balls which have been drawn up to time $n$.

We then further imagine that there is an option on the asset whose payoff function is

$$(44) \qquad V_n = \max(2A_n - n, 0).$$

In particular, the strike is time dependent. To complete the model, we need to specify the dynamics of $A_n$. In any given timestep, it either increases in value by one dollar or stays the same with probabilities $p_n(\omega)$ and $1-p_n(\omega)$, respectively, where

$$(45) \qquad p_n(\omega) = \frac{w - A_n(\omega)}{N - n},$$

and $w$ and $N$ represent the initial attributes of the corresponding hat. Of course, this problem can be thought of without reference to any hat, in which case, $w$ and $N$ are simply values parameterising the initial moneyness and term. In particular, the dynamics is only defined as long as $n < N$, after which everything is frozen. Henceforth, we assume the option maturity is also $N$.

If the option is European in style, then the option value is the same as the value of the hat game following the risk-free strategy discussed above. If the option is American in style, meaning we can stop playing whenever we like, then the option value is the same as the value of the hat game following the optimal strategy. This follows from the fact that to evaluate we can use standard arbitrage-free hedging arguments, where we use the asset to hedge the option. The smooth boundary condition at $t_c \approx 0.42$ is a standard result of risk-neutral pricing (Dixit, 1993). Therefore, in this interpretation, where we can trade the asset as well as the derivative, we find that the only strategies of interest are the risk-free and the risk neutral (optimal) strategies.

This suggests an extension of the problem as well, where the option maturity is at some time $M < N$. In that case, even the European option will have a nontrivial value. This generalisation will break the scaling of Eq. (5) and we will need an additional parameter to represent remaining term in the expression for the value. Assuming this can be done, we then have an interesting limit, since for $1 \ll M \ll N$, we recover a standard binary tree model for which the relative value of an American put option is a non-trivial problem.

## Conclusion

We have presented results for an interesting single player game with analogies to finance. In particular, while the rules of the game are simple to state, the results are extremely nontrivial and display a rich set of behaviours. In particular, we have derived the first and second moments as well as the characteristic function. Doing so required derivation of a differential equation. Assigning boundary conditions to that equation led to a rich question about the strategies which a player would invoke which in turn depends on his risk-return profile.

We have demonstrated that this is a good laboratory for exposition of the $\Omega$ function as a performance measure as opposed to the Sharpe Ratio. This is a highly nonnormal context where downside risk is inherently limited but upside risk can be substantial. The Sharpe Ratio fails to properly account for these features.

We have also mapped the problem onto one of a standard option valuation. The usefulness of this is firstly to further motivate the optimal strategy as deserving of special attention since it results in the arbitrage-free risk neutral option price. This mapping also provides an interpretation of the game's value in terms of the difference between American-style and European-style option values.

This problem displays a surprising array of rich behaviours. Possibilities for further study include exploring finite $N$ effects, including understanding the apparent delta function peak in the distributions at $V = 0$ and the oscillations observed in the numerically simulated distributions. Better numerical determination of the probability densities is also an interesting problem as is extending the problem to allow for the term of the game to be smaller than the number of balls in the hat, as mentioned in the final section.

## Appendix A:
## Numerical determination of the characteristic function

As we discussed, numerical determination of the characteristic function is not trivial. Our solution (41) is a Taylor expansion in $\zeta$, which is well adapted for small to intermediate values of $\zeta$. While it is formally convergent for all values of $\zeta$, that actually does not help much for numerically determining $F(\zeta,t)$ for large values of $\zeta$. What is required is a representation adapted for large values of $\zeta$. For a general discussion of these sort of issues see Press *et al*, 1992.

We appeal back to the differential equation (34) and express $H$ with the ansatz $H = \exp(2it_c\zeta+a(t-t_c)\sqrt{|\zeta|})$ where $a$ is a constant to be determined. Then to leading order in $\zeta$, the second derivative with respect to $t$ and the first derivative with respect to $\zeta$ both generate terms of order $\zeta$. These are the leading order terms, but there are corrections of higher order in $1/\zeta$. Further, imposing the condition that the solution should vanish for $t$ large and negative gives $a = \pm \sqrt{8it_c}$ where the sign of the root is the same as the sign of $\zeta$. As mentioned, this is not a complete solution, but is only the leading order solution. We empirically observe a second component which behaves as $b(t)/\sqrt{|\zeta|}$ where $b$ is a solution of the Hermite equation of order $1/2$. Unfortunately, there is no way to make this solution consistent with the boundary condition at $t \to -\infty$ so this is, at best, a leading order term in an asymptotic expansion of a solution which does satisfy that boundary condition. Therefore, we can say that approximately

$$H(\zeta,t) = \exp\left(2it_c\zeta + a(t-t_c)\sqrt{|\zeta|}\right) + \frac{b(t)}{\sqrt{|\zeta|}}. \tag{A.1}$$

In practise, we use the power series expansion (41) out to some large value of $|\zeta|$ and then fit the value of $b(t)$ to the above expression which we then use for larger values of $|\zeta|$. This provides us with an expression for the characteristic function which is more universally valid. This is not perfect, since it is based only on a leading order asymptotic expansion in $\zeta$ where ideally we would have it to arbitrary order. Also, we have no self-contained manner of determining the coefficient $b(t)$. The ultimate problem is that the implicit condition that $|\zeta| \gg |t|$ is inconsistent with a boundary condition imposed at $t \to -\infty$.

## References

M. Abramowitz and I. Stegun, editors. *Handbook of Mathematical Functions*. Dover Publications, New York, 1965.

R. Bhaduri and N. D. Whelan. 'The value of liquidity'. *Wilmott Magazine*, January, 2008.

A. Dixit. *Art of Smooth Pasting, Fundamentals of Pure and Applied Economics 55*. Routledge, May 1993. ISBN 3718653842.

P. D. Kaplan and J. A. Knowles. 'Kappa: A generalized downside risk-adjusted performance measure'. *Journal of Performance Measurement*, 8, 2004. URL: www.edge-fund.com/KaKn04.pdf.

H. Kazemi, T. Schneeweis, and R. Gupta. 'Omega as a performance measure'. EHDEC working paper, 2003. URL: http://cisdm.som.umass.edu/research/pdffiles/omega.pdf.

C. Keating and W. Shadwick. 'Omega: A universal performance measure'. *Journal of Performance Measurement*, Spring, 2002.

T. Perkins and S. Stewart. 'BMO staggers under $680-million loss'. *Globe and Mail*, 19 May 2007.

W. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes in C*. Cambridge University Press, Cambridge, 2nd edition, 1992.

W. F. Sharpe. 'Mutual fund performance'. *Journal of Business*, 39:119–138, 1966.

W. F. Sharpe. 'The Sharpe Ratio'. *Journal of Portfolio Management*, 20, Fall: 49–58, 1994.

L. A. Shepp. 'Explicit solutions to some problems of optimal stopping'. *The Annals of Mathematical Statistics*, 1969.

J. A. Shohat and J. D. Tamarkin. The problem of moments. *American Mathematical Society Mathematical Surveys*, II, 1943.

C. Smithson. *Managing Financial Risk*, 3rd Edition. McGraw Hill, 1998.

L. C. Thomas. Games, *Theory and Applications*. John Wiley & Sons, 1986.

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**Niall Whelan**
Director, Analytics & Structured Transations
ScotiaBank
niall_whelan@scotiacapital.com

**Ranjan Bhaduri**
Managing Director, Head of Research
AlphaMetrix Alternative Investment Advisors
rbhaduri@alphametrix.com