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Socially efficient discounting under ambiguity aversion

Workshop on the pricing and hedging of environmental and energy-related financial derivatives

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Ellsberg game: Ambiguity aversion

- Consider two urns.
  - The unambiguous urn has 50 back balls and 50 white balls.
  - The ambiguous urn has 100 balls, some are black and the other are white.
- Take an urn; select a colour; take a ball at random; if its colour is the colour on which you bet, you get 10 000 $.
- On which colour do you want to bet?
- What is your willingness to pay to play this game?

Introduction

- Which discount rate should be used for the distant future?
- Applications: Nuclear wastes, global warming, pension systems, public debt,…
- “There must be something wrong with discounting”: 1,000,000 € in 200 years discounted at 5% is valued 58 € today. At 1.4%, it goes up to 62 000 €.
- Two problems:
  - the level of the discount rate;
  - its constancy with respect to time horizon.
- A standard consumption-based model of the yield curve to determine its level and its shape, adding smooth ambiguity aversion into the picture.

Discounted marginal damage of the tCO2

	Discount rate	Social value of CO2
Nordhaus	5%	8 $/tC02
Stern/Hope	1.4%	85 $/tC02

The three determinants of the discount rate

- Ethical dimension: intergenerational Pareto weights in the SWF: $\underline{\delta}$

- Preference for consumption smoothing over time + positive growth rate ($\underline{\mu}$) of GDP per capita (+): the marginal utility of 1 unit of consumption next period is smaller than the marginal utility of 1 unit of consumption now.

- Prudence + uncertain growth rate ($\underline{\sigma}$) (-)

(shadow) price
=
MRS

→ Ramsey rule:

$$r_t = \delta + \gamma\mu - 0.5\gamma^2\sigma^2$$

Ambiguous growth and ambiguity aversion

- Two new ingredients:
  - Ambiguity on the μ and σ^2 for the next 200 years.
  - People are ambiguity-averse. The following two situations are *not* equivalent: The GDP in 10 years will be $(50, p; 150, 1\text{-}p)$ with
    - p= ½;
    - p random with mean ½.
- This paper: Role of ambiguity and ambiguity aversion on
  - The term structure of equilibrium interest rates;
  - The term structure of the socially efficient discount rates.
- Conjecture: Ambiguity aversion should reduce the discount/interest rate.

Bibliography

- On the socially efficient discount rate:
  - Weitzman (1999, 2007, 2009)
  - Gollier (2002, 2007, 2008)
- On ambiguity aversion (AA):
  - Gilboa and Schmeidler (1987)
  - Klibanoff, Marinacci and Mukerji (2005, 2008), hereafter KMM
- On the effect of AA on asset prices:
  - Gollier (2009)
  - Ju and Miao (2008), Ju, Chen and Miao (2009)
  - Collard, Mukerji, Sheppard and Tallon (2009)

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The basic model

The model

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- Lucas tree economy: Each agent is endowed with a tree which produces c_t fruits at date t, $t=0, 1,\ldots$.

- The growth of trees is governed by an unknown parameter θ which can take value $1,2,\ldots,n$ with probability $q_1,q_2,\ldots,q_n$.

- There is a credit market at date 0, with risk-free zero-coupon bonds for the different maturities $t=1, 2, \ldots$. Let r_t denote the interest rate associated to maturity t.

- First Theorem of Welfare Economics applies here: The equilibrium interest rates are socially efficient.

The decision problem

$$\alpha^* \in \arg\max_{\alpha} \; u(c_0 - \alpha) + e^{-\delta t} V_t(\alpha)$$

- Classical EU model:

$$V_t(\alpha) = \sum_{\theta=1}^{n} q_\theta E u(\tilde{c}_{t\theta} + \alpha e^{r_t t})$$

- Following KMM, we assume that agents are averse to mean-preserving spreads in the space of probability distributions.

$$\phi(V_t(\alpha)) = \sum_{\theta=1}^{n} q_\theta \phi(E u(\tilde{c}_{t\theta} + \alpha e^{r_t t}))$$

Concavity of V ?

$$V_t(\alpha) = \phi^{-1}\left(\sum_{\theta=1}^{n} q_\theta \phi\left(Eu(\tilde{c}_{t\theta} + \alpha e^{r_t t})\right)\right)$$

- Proposition: *Suppose that ϕ has a concave absolute ambiguity tolerance, i.e., -ϕ'(U)/ϕ "(U) is concave in U. This implies that V_t is concave in α.*

- Proof: Theorem 106 in Hardy, Littlewood and Polya (1934).

Asset pricing formula

- First-order condition: $u'(c_0 - \alpha^*) = e^{-\delta t} V_t'(\alpha^*)$

$$V_t'(\alpha) = e^{r_t t} \frac{\sum_{\theta=1}^{n} q_\theta \phi'(Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t})) Eu'(\widetilde{c}_{t\theta} + \alpha e^{r_t t})}{\phi'(V_t(\alpha))}$$

- Equilibrium condition: $\alpha = 0$.

- Asset pricing formula:

$$\boxed{r_t = \delta - \frac{1}{t} \ln\left[\sum_{\theta=1}^{n} q_\theta \frac{\phi'(Eu(\widetilde{c}_{t\theta}))}{\phi'(V_t)} \frac{Eu'(\widetilde{c}_{t\theta})}{u'(c_0)} \right]}$$

The effect of ambiguity aversion

AA= More patience + More pessimism

$$r_t = \delta - \frac{1}{t}\ln\left[\sum_{\theta=1}^{n} q_\theta \frac{\phi'(Eu(\tilde{c}_{t\theta}))}{\phi'(V_t)} \frac{Eu'(\tilde{c}_{t\theta})}{u'(c_0)}\right]$$

$$a = \frac{\sum_{\theta=1}^{n} q_\theta \phi'(Eu(\tilde{c}_{t\theta}))}{\phi'(V_t(0))}$$

$$r_t = (\delta - \frac{1}{t}\ln a) - \frac{1}{t}\ln\left[\sum_{\theta=1}^{n} q_\theta \frac{\phi'(Eu(\tilde{c}_{t\theta}))}{\sum_{\tau=1}^{n} q_\tau \phi'(Eu(\tilde{c}_{t\tau}))} \frac{Eu'(\tilde{c}_{t\theta})}{u'(c_0)}\right]$$

$$\hat{q}_\theta = q_\theta \frac{\phi'(Eu(\tilde{c}_{t\theta}))}{\sum_{\tau=1}^{n} q_\tau \phi'(Eu(\tilde{c}_{t\tau}))}$$

$$r_t = (\delta - \tfrac{1}{t}\ln a) - \tfrac{1}{t}\ln\left[\sum_{\theta=1}^{n} \hat{q}_\theta \frac{Eu'(\tilde{c}_{t\theta})}{u'(c_0)}\right]$$

$$r_t^{AN} = \delta - \tfrac{1}{t}\ln\left[\sum_{\theta=1}^{n} q_\theta \frac{Eu'(\tilde{c}_{t\theta})}{u'(c_0)}\right]$$

The ambiguity prudence effect

$$a = \frac{\sum_{\theta=1}^{n} q_\theta \phi'(Eu(\tilde{c}_{t\theta}))}{\phi'(V_t(0))} \qquad r_t = (\delta - \frac{1}{t}\ln a) - \frac{1}{t}\ln\left[\sum_{\theta=1}^{n} \hat{q}_\theta \frac{Eu'(\tilde{c}_{t\theta})}{u'(c_0)}\right]$$

- Condition $a>1$ is necessary to guarantee that the discount rate is reduced by AA.
- Under risk neutrality, it is necessary and sufficient.
- Risk neutrality switches off the wealth effect and the standard precautionary effect.
- Condition $a>1$ tells us that the uncertainty about the mean growth rate raises the willingness to save of the ambiguity-averse consumer.
- By analogy to EUT, we coin the term “ambiguity prudence effect”.

The ambiguity prudence effect

$$a = \frac{\sum_{\theta=1}^{n} q_\theta \phi'(Eu(\tilde{c}_{t\theta}))}{\phi'(V_t(0))}$$

- $a>1$ requires that

$$\sum_{\theta=1}^{n} q_\theta \phi'(u_\theta) \geq \phi'(V) \text{ whenever } \sum_{\theta=1}^{n} q_\theta \phi(u_\theta) = \phi(V)$$

- What is the condition on ϕ such that any expected-ϕ-preserving risk raises the willingness to save?

- Analogy with DARA: any expected-utility-preserving risk raises the willingness to save.

The case of risk neutrality

- Proposition:
  - Decreasing ambiguity aversion => $a>1$;
  - Constant ambiguity aversion => $a=1$;
  - Increasing ambiguity aversion => $a<1$.
- Proposition: Suppose that the representative agent is risk-neutral. Then,
  - Decreasing ambiguity aversion => discount rate reduced;
  - Constant ambiguity aversion => discount rate unchanged;
  - Increasing ambiguity aversion => discount rate increased.

Twisted beliefs

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$$\widehat{q}_\theta = q_\theta \frac{\phi'(Eu(\widetilde{c}_{t\theta}))}{\sum_{\tau=1}^{n} q_\tau \phi'(Eu(\widetilde{c}_{t\tau}))}$$

- The representative agent uses twisted beliefs to estimate the future expected marginal utility of wealth.
- She puts more weight on the scenarios yielding a smaller conditional expected utility. Extreme case is maxmin.
- Technically, it means that
- Lemma: The twisted beliefs $\hat{q}$ are MLR-dominated by the original beliefs q.

Effect of twisted beliefs on r

- The pessimism effect reduces the discount rate if

$$\sum_{\theta=1}^{n} q_\theta \frac{\phi'(Eu(\tilde{c}_{t\theta}))}{\sum_{\tau=1}^{n} q_\tau \phi'(Eu(\tilde{c}_{t\tau}))} \frac{Eu'(\tilde{c}_{t\theta})}{u'(c_0)} \geq \left[\sum_{\theta=1}^{n} q_\theta \frac{\phi'(Eu(\tilde{c}_{t\theta}))}{\sum_{\tau=1}^{n} q_\tau \phi'(Eu(\tilde{c}_{t\tau}))}\right]\left[\sum_{\theta=1}^{n} q_\theta \frac{Eu'(\tilde{c}_{t\theta})}{u'(c_0)}\right]$$

- By the covariance rule, this inequality holds if the distortion weights and *Eu'* are comonotone, i.e., if

$Eu(\tilde{c}_{t\theta})$ and $Eu'(\tilde{c}_{t\theta})$ are anti-comonotone.

Result with FSD

- Suppose that $c_{t1} \prec_{FSD} c_{t2} \prec_{FSD} \dots \prec_{FSD} c_{tn}$.
- By risk aversion, Eu and Eu' are anti-comonotone.
- This is a case where the pessimism effect always reduces the discount rate.
- In fact, the shift in distribution of c_t is FSD-deteriorating.
- Proposition: *Suppose that priors can be ranked according to the FSD order. Under DAAA, ambiguity aversion reduces the discount rate.*

Result with SSD

- Suppose that $c_{t1} \prec_{SSD} c_{t2} \prec_{SSD} \dots \prec_{SSD} c_{tn}.$
- Under risk prudence ($u''' > 0$), Eu and Eu' are anti-comonotone.
- This is another case where the pessimism effect always reduces the discount rate.
- In fact, the shift in distribution of c_t is SSD-deteriorating.
- Proposition: *Suppose that priors can be ranked according to the SSD order. Under DAAA and risk prudence, ambiguity aversion reduces the discount rate.*

Jewitt order

- X is riskier than Y in the sense of the Jewitt order if, for all increasing and concave u,
  - if agent u prefers (the less risky) Y to X,
  - then all agents more risk-averse than u also prefer Y to X.
- This is weaker than SSD.
- Remember that DARA means that $-u'$ is more concave than u.
- Proposition: *The pessimism effect reduces the socially efficient discount rate if the set of posteriors* $(c_{t1},\ldots,c_{tn})$ *can be ranked according to Jewitt order and* u *exhibits decreasing absolute risk aversion.*

Numerical illustrations

An analytical solution:
Power –power normal-normal case

- Specification:

$$\ln c_t \,|\, \theta \sim N(\ln c_0 + \theta t, \sigma^2 t) \qquad (d \ln c_t = \theta dt + \sigma dz)$$

$$\theta \sim N(\mu, \sigma_0^2)$$

$$u(c) = c^{1-\gamma} / (1-\gamma)$$

$$\phi(V) = V^{1-\eta} / (1-\eta) \qquad \text{(when } \gamma \text{ is smaller than unity)}$$

- Solution:

$$r_t = \delta + \gamma\mu - 0.5\gamma^2(\sigma^2 + \sigma_0^2 t) - 0.5\eta\left|1-\gamma^2\right|\sigma_0^2 t$$

Numerical illustration

$$\ln c_t \sim N(\ln c_0 + \theta t, \sigma^2 t)$$
$$\theta \sim N(\mu, \sigma_0^2)$$
$$u(c) = c^{1-\gamma}/(1-\gamma)$$
$$\phi(V) = V^{1-\eta}/(1-\eta)$$

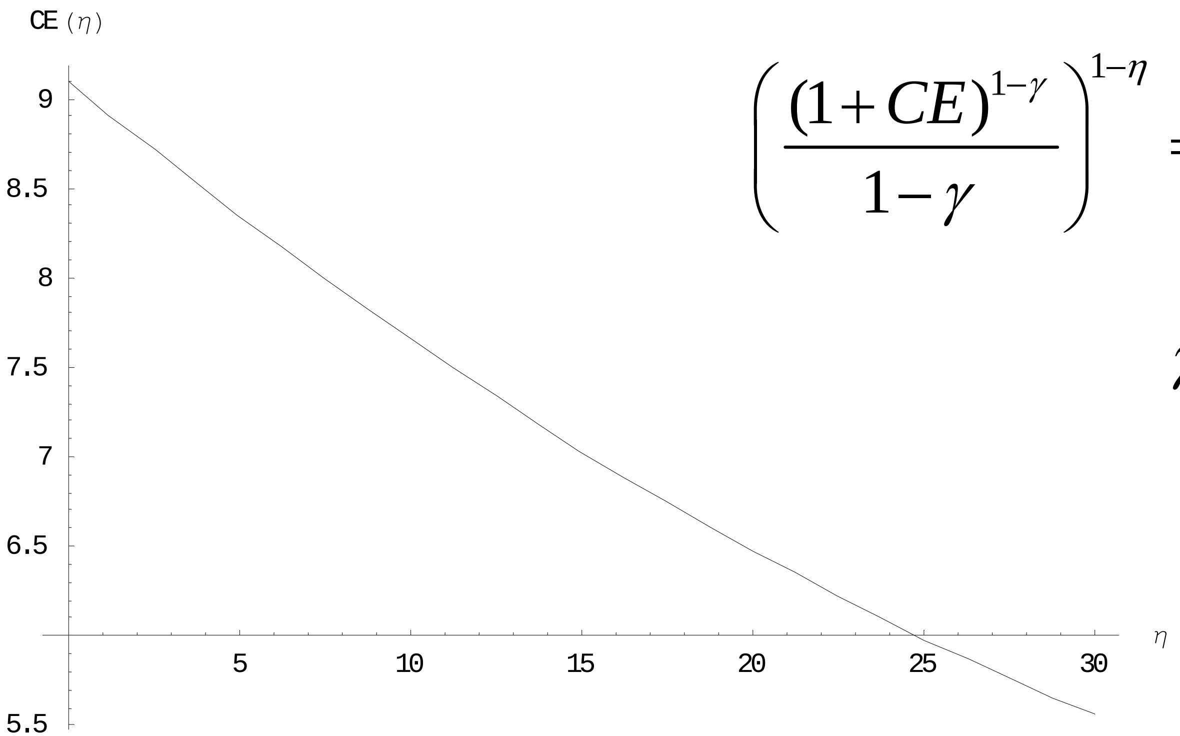
- Power-power, normal-normal.
  - δ=2%; γ=2, μ=2%, σ=2% implies $r_t = 5.88\% - 3\sigma_0^2 t(1+\eta/2)$
  - σ_0=1%.

t	$\eta = 0$	$\eta = 5$	$\eta = 10$
10	5.58%	4.83%	4.08%
30	4.98%	2.73%	0.48%

Evaluate your own CRAA

- *Suppose that the growth rate in the next 20 years is either 20% with prob θ, or 0% with prob $1-\theta$. Suppose that θ is uniformly distributed on [0,1].*

- *What is the certainty equivalent (CE) growth rate?*



$$\left(\frac{(1+CE)^{1-\gamma}}{1-\gamma}\right)^{1-\eta} = \int_0^1 \left(\theta \frac{1.2^{1-\gamma}}{1-\gamma} + (1-\theta)\right)^{1-\eta} d\theta$$

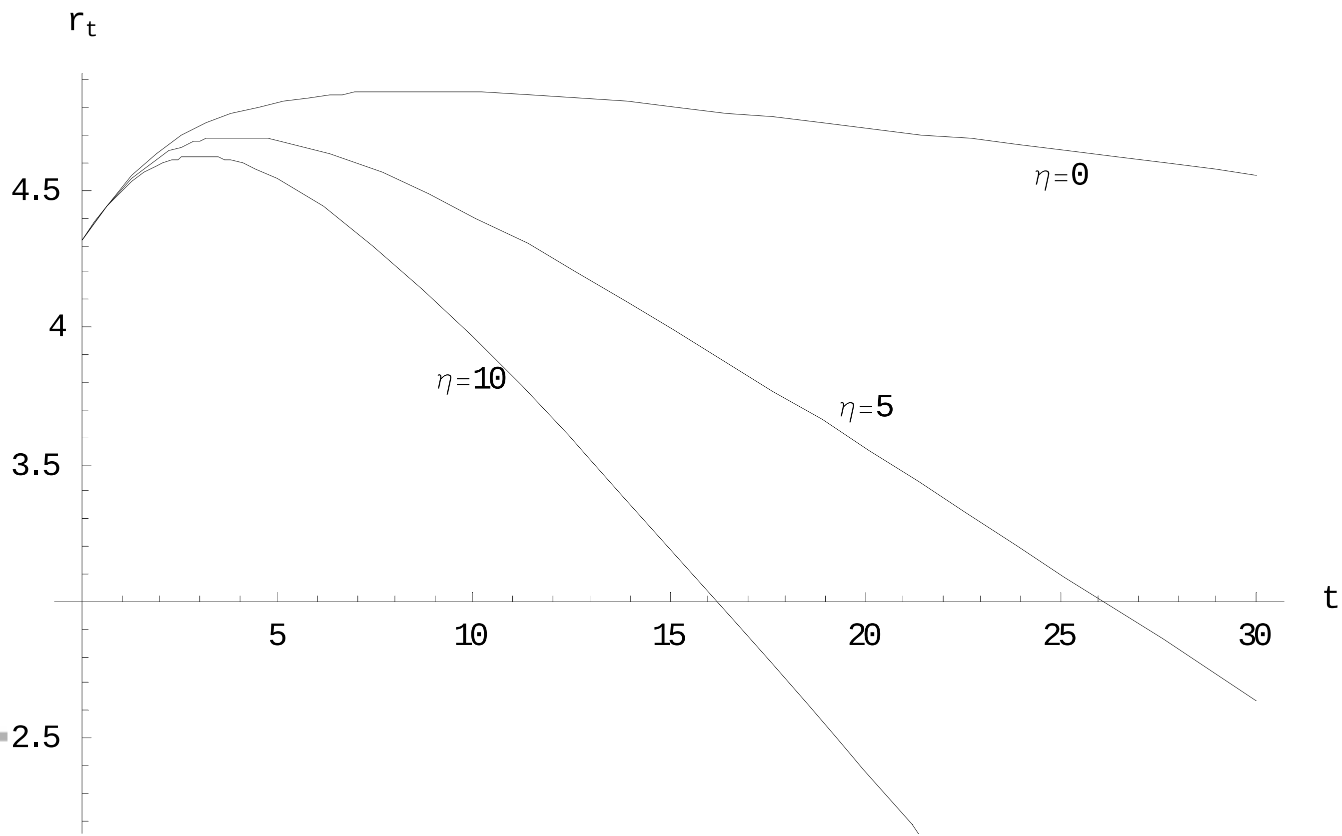
$\gamma=2$

An AR(1) process for log consumption with an ambiguous long-term trend

$$\ln c_{t+1} = \ln c_t + x_t$$
$$x_t = \xi x_{t-1} + (1-\xi)\mu + \varepsilon_t$$
$$\varepsilon_t \sim N(0,\sigma^2),\ \varepsilon_t \perp \varepsilon_{t'}$$
$$\mu \sim N(\mu_0, \sigma_0^2),$$

$\delta = 2\%$, $\gamma = 2$, $\mu_0 = 2\%$, $\sigma = 2\%$, $\sigma_0 = 1\%$, and $x_{-1} = 1\%$

$\xi = 0.7$ year^{-1}



More ambiguity aversion

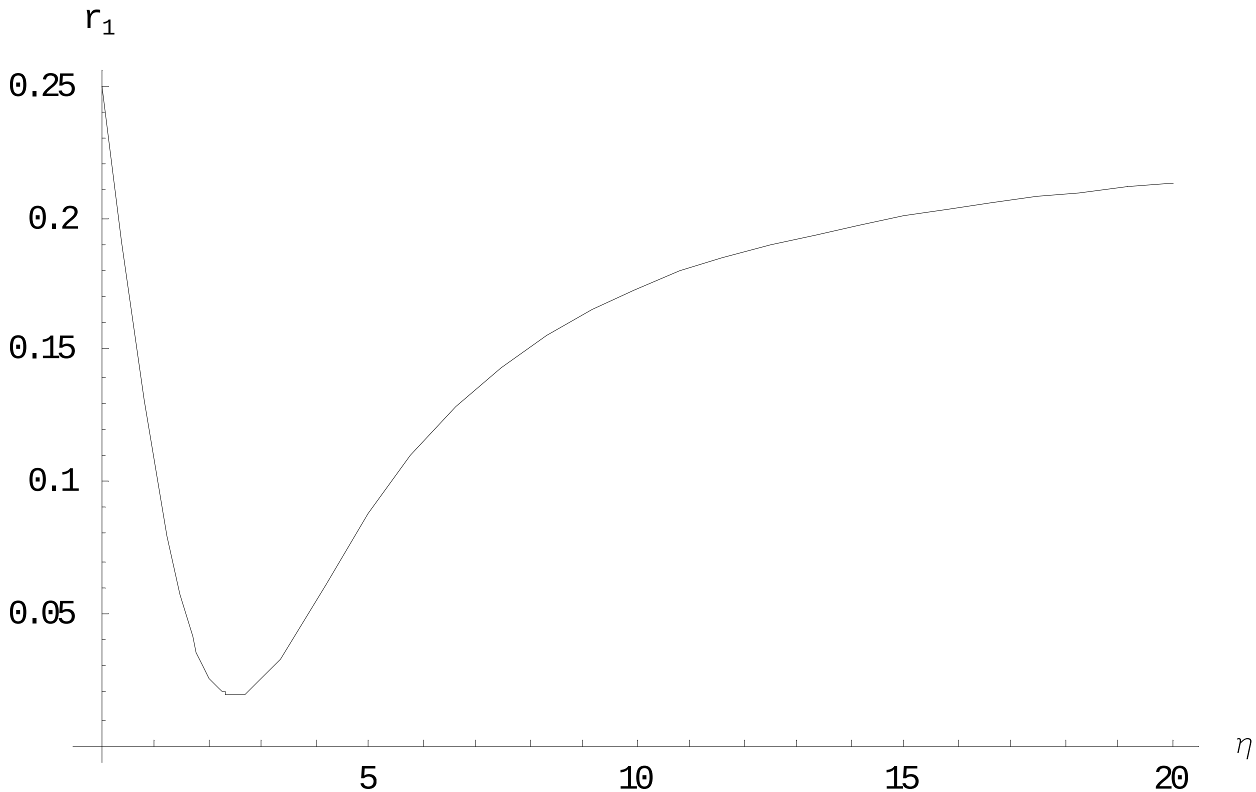
- In economy 2, the representative agent is more AA than in economy 1.

- *Pessimism effect*: This implies that the twisted q in economy 2 is MLR-dominated by the twisted q in economy 1. If the conditionals can be ranked according to FSD, then the pessimism effect reduces the discount rate.

- *Ambiguity prudence effect*: a_2 is larger than a_1 if the degree of ambiguity is small and if $\frac{\partial}{\partial V}\left(-\frac{\phi_2''(V)}{\phi_2'(V)}\right) \geq \frac{\partial}{\partial V}\left(-\frac{\phi_1''(V)}{\phi_1'(V)}\right)$.
- Not true when the degree of ambiguity is not small.

Counterexample

- $$u(c) = c$$
  $$\phi(U) = U^{1-\eta} / (1-\eta)$$
  $$E\left[\tilde{c}_1 \middle| \theta = 1\right] = 0.5c_0; \quad E\left[\tilde{c}_1 \middle| \theta = 2\right] = 1.5c_0$$
  $$\delta = 0.25$$



Conclusion

- The growth process is ambiguous.
- Human beings are ambiguity-averse.
- These two ingredients raises the willingness to save, and reduces interest rates.
- Many projects in the agenda of research:
  - Recursive approach;
  - Dynamic portfolio choices;
  - Conditions for decreasing risk/uncertainty aversion;
  - Aggregation of preferences and beliefs;
  - …