# Representation of the penalty function

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Notation (one period model)

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Only bounded random variables $L^\infty$

$u : L^\infty \to \mathbb{R}$ satisfies

1. $u(0) = 0$, $\xi \geq 0$ implies $u(\xi) \geq 0$

2. $u$ is concave

3. $u(\xi + a) = u(\xi) + a$ (monetary)

4. The set $\mathcal{A} = \{\xi \mid u(\xi) \geq 0\}$ is weak* closed: if $\xi_n \to \xi$ in probability and $\sup_n \|\xi_n\| < \infty$, then $u(\xi) \geq \limsup u(\xi_n)$.

Coherent means $u(\lambda\xi) = \lambda u(\xi)$ for $\lambda \geq 0$

The structure is well known (Föllmer-Schied)

$\mathbf{P}$ represents the set of all probability measures $\mathbb{Q} \ll \mathbb{P}$

**Theorem 1** *There is a convex, lsc function* $c : \mathbf{P} \to [0, +\infty]$ *such that*

1. $\inf_{\mathbb{Q}\in\mathbf{P}} c(\mathbb{Q}) = 0$

2. $u(\xi) = \inf_{\mathbb{Q}\in\mathbf{P}} \left(\mathbb{E}_{\mathbb{Q}}[\xi] + c(\mathbb{Q})\right)$

If $u$ is coherent $c$ is the "indicator" of a convex closed set $\mathcal{S} \subset L^1$.

It is convenient to add the hypothesis: there is $\mathbb{Q} \sim \mathbb{P}$ with $c(\mathbb{Q}) = 0$.

What about multiperiod functions?

We need a filtration $(\mathcal{F}_t)_t$ and stopping times

$u_\sigma(\xi) \in L^\infty(\Omega, \mathcal{F}_\sigma, \mathbb{P})$

$u_\sigma(0) = 0$ and $u_\sigma(\xi) \geq 0$ for $\xi \geq 0$

$0 \leq \lambda \leq 1, \lambda \in L^\infty(\mathcal{F}_\sigma)$, $\xi_1, \xi_2$ then

$u_\sigma(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \lambda u_\sigma(\xi_1) + (1-\lambda)u_\sigma(\xi_2)$

$\eta \in L^\infty(\mathcal{F}_\sigma)$ then $u_\sigma(\xi + \eta) = u_\sigma(\xi) + \eta$

$\xi_n \downarrow \xi$ then $u_\sigma(\xi_n) \downarrow u_\sigma(\xi)$

This gives a family of penalty functions

Detlefsen-Scandolo, Frittelli and Rosazza-Gianin

$c_\sigma(\mathbb{Q}) \in L^0(\mathcal{F}_\sigma; \overline{\mathbb{R}_+})$.

This time we need to incorporate the time interval in the definition of $c$.

So we talk about

$$c_{[\sigma,\tau]}(\mathbb{Q}) = \text{ess.sup}\left\{ \mathbb{E}_{\mathbb{Q}}[-\xi \mid \mathcal{F}_\sigma] \,\middle|\, \begin{array}{l} \xi \in L^\infty(\mathcal{F}_\tau) \\ u_\sigma(\xi) \geq 0 \end{array} \right\}$$

and for $\xi \in L^\infty(\mathcal{F}_\tau)$ we have

$$u_\sigma(\xi) = \text{ess.inf}_{\mathbb{Q}\sim\mathbb{P}} \left( \mathbb{E}_{\mathbb{Q}}[\xi \mid \mathcal{F}_\sigma] + c_{[\sigma,\tau]}(\mathbb{Q}) \right)$$

$$\mathcal{A}_{\sigma,\tau} = \{\xi \in L^\infty \mathcal{F}_\tau \mid u_\sigma(\xi) \geq 0\}$$

Time consistency

Koopmans (1960 — 1961)

For $\xi, \eta \in L^\infty(\mathcal{F}_T)$, two stopping times $\sigma \le \tau$, then

$u_\tau(\xi) \ge u_\tau(\eta)$ implies $u_\sigma(\xi) \ge u_\sigma(\eta)$

This implies conditions on the family $c_\sigma$. For $\sigma \le \tau$ we need

$$\mathcal{A}_{\sigma,T} = \mathcal{A}_{\sigma,\tau} + \mathcal{A}_{\tau,T}$$
$$c_{[\sigma,T]}(\mathbb{Q}) = c_{[\sigma,\tau]}(\mathbb{Q}) + \mathbb{E}_{\mathbb{Q}}[c_{[\tau,T]}(\mathbb{Q}) \mid \mathcal{F}_\sigma]$$

cocycle property of Bion-Nadal and Penner.

We refer to the dynamic programming principle: $u_0(\xi) = u_0(u_\sigma(\xi))$

Regularity of trajectories

$u$ is *relevant* if $\xi \leq 0$ and $\mathbb{P}[\xi < 0] > 0$ gives:

$u_0(\xi) < 0$.

(see also Cheridito's susceptibility)

If $u$ is relevant there are càdlàg versions of

$c_{t,T}(\mathbb{Q})$ ($c_{t,T}(\mathbb{Q})$ is a $\mathbb{Q}-$potential of class D)

and of

$u_t(\xi)$.

Example (Brownian filtration)

$\mathbb{Q} \sim \mathbb{P}$ then we get $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{E}(q \cdot W)_t$

$\mathcal{E}(q \cdot W)_t = \exp\left(\int_0^t q_u \, dW_u - \frac{1}{2}\int_0^t q_u^2 \, du\right)$

$f : \mathbb{R} \to [0, +\infty]$, convex, lsc, $f(0) = 0$

Fenchel-Legendre transform is $g$

$c_{[t,T]}(\mathbb{Q}) = \mathbb{E}_{\mathbb{Q}}[\int_t^T f(q_u) \, du \mid \mathcal{F}_t]$

$u_t(\xi) = \text{ess.inf}_{\mathbb{Q}\sim\mathbb{P}} \, \mathbb{E}_{\mathbb{Q}}\left[\xi + \int_t^T f(q_u) \, du \mid \mathcal{F}_t\right]$

For $f(q) = \frac{1}{2}q^2$ we have $g(x) = \frac{1}{2}x^2$

$c_0(\mathbb{Q}) = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]$, the so-called entropy.

$$u_t(\xi) = -\log \mathbb{E}[\exp(-\xi) \mid \mathcal{F}_t]$$

$\exp(-u_t(\xi))$ is a martingale.

It is the only law determined time consistent utility function (Kupper-Schachermayer, valid in a much more general context).

Remarks

In case $g$ is not sub-quadratic, the BSDE does not always have a bounded solution.

In case there is a bounded solution, there are infinitely many and they all satisfy $Y_. \leq u_.(\xi)$. $u_.(\xi)$ is not necessarily a solution.

In case $\xi = \phi(W_T)$ $u_.(\xi)$ is related to the quasi-linear equation (well studied in analysis, VHJ) (Ben-Artzi, Laurençot, Souplet, ...)

$u(T,x) = \phi(x)$ and

$$\frac{1}{2}\partial_{x,x}u + \partial_t u = g(-\partial_x u)$$

$$\frac{1}{2}\Delta u + \partial_t u = g(-\nabla u)$$

The general structure is known:

**Theorem 2** *(FD,Peng,Rosazza-Gianin)*
*Suppose* $c_0(\mathbb{P}) = 0$*. There is a function*

$$f\colon \mathbb{R}^d \times [0,T] \times \Omega \to \overline{\mathbb{R}_+}; f \in \mathcal{R}^d \otimes \mathcal{P}$$

*for all* $(t,\omega), f$ *is convex,lsc in* $q \in \mathbb{R}^d$

$$f(0,.,.) = 0$$

*for all* $q \in \mathbb{R}^d, f$ *is predictable in* $(t,\omega)$

$$c_0(\mathbb{Q}) = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T f(q_t(.),t,.)dt\right]$$

There are different (and long) proofs. One proof uses truncation and the result of Coquet, Hu, Memin, Peng, for "dominated" utility functions. The other proof uses a measure theoretic characterisation of the penalty function and gives as a corollary the result of CHMP

Measure theoretic characterisation

**Theorem 3** *Let* $\mathbb{Q} \sim \mathbb{P}$ *with* $c_0(\mathbb{Q}) < \infty$*. Let* $\alpha(\mathbb{Q})$ *be the* $\mathbb{Q}-$*potential of the* $\mathbb{Q}-$*supermartingale* $c(\mathbb{Q})$*. For each* $\xi \in L^\infty$ *let* $du_t(\xi) = dA^\xi_t - Z^\xi_t\, dW_t$ *be the Doob-Meyer decomposition (under* $\mathbb{P}$*) of the utility process* $u(\xi)$*. The measure* $d\alpha_t(\mathbb{Q})$ *is the smallest measure that is bigger than the family (*$\xi \in L^\infty$*)*

$$q_t Z^\xi_t\, dt - dA^\xi_t.$$

*Consequently the potential* $\alpha$ *has a predictable density with respect to Lebesgue measure.*

$$d\alpha_t(\mathbb{Q}) = f_t(\mathbb{Q})\, dt$$

Using more convexity and localisation, one then shows that $f$ has the good form.

$g_t$ is the Fenchel Legendre transform of $f_t$.

In case there is $\mathbb{Q} \sim \mathbb{P}$ (not just $\mathbb{Q} \ll \mathbb{P}$) with

$$u_0(\xi) = \min_{\mathbb{Q} \sim \mathbb{P}} \mathbb{E}\left[\xi + \int_0^T f_u(q_u)\, du\right]$$

we can say more:

$$u_t(\xi) = u_0(\xi) + \int_0^t g_u(Z_u)\, du - \int_0^t Z_u\, dW_u$$

Hence $u_t(\xi)$ is a bounded solution of the BSDE:

$dY_t = g_t(Z_t)\, dt - Z_t\, dW_t, \quad Y_T = \xi$

In some cases $u_{\cdot}(\xi)$ is the unique bounded solution of the BSDE.

In general

$$du_t(\xi) = g_t(Z_t)\,dt + dC_t - Z_t\,dW_t$$

where $C$ is non-decreasing càdlàg .

Related to the Kramkov optional decomposition theorem in finance.
(Take $u_t(\xi)$ = ask price of $\xi$ at time $t$)

There are examples where $dC \ll dt$

$u_t(\xi) = \min(t, \tau)$

with $\tau$ a "well chosen" stopping time.

In case

$$0 \le g_t(z,\omega) \le K(\omega)(1+|z|^2),$$

convex in $z$, $g(0,t,\omega)=0$ we can prove

**Theorem 4** *For each $\xi \in L^\infty$, $u_.(\xi)$ is a bounded solution of*

$$dY_t = g_t(Z_t)\,dt - Z_t\,dW_t; \quad Y_T = \xi$$

However, uniqueness is not guaranteed. Even when the model is Markovian.

Proof uses Bishop-Phelps theorem, preceding result on minimizer + (probably) well known result on submartingales.

**Theorem 5** *Suppose $X^n$ uniformly bounded (in $L^\infty$) sequence of (continuous) submartingales. Suppose $X^n = A^n + M^n$ is the Doob-Meyer decomposition. Suppose*

$$\| \sup_t |X_t^n - X_t| \|_\infty \to 0.$$

*Then $X = A + M$ where $M^n \to M$ in $BMO$.*

**Remark 1** If $\xi_n$ is uniformly bounded and tends to 0 a.s. this does not imply that $\xi_n \to 0$ in BMO. So we (almost) need $\|.\|_\infty$ convergence.

**Condensed Counter-example:** The mapping $H^1 \to L^1$ is not weakly compact. Hence $L^\infty \to BMO$ is not weakly compact.

$dL_t = -L_t^2 \, dW_t, \; L_0 = 1; \; L = \frac{1}{R}; \; BES^3$

$\xi = \frac{1}{1+L_T} = \frac{R_T}{1+R_T}$

$g_t(z) = \frac{L_t^2}{1+L_t}|z| = \frac{1}{R_t(R_t+1)}|z|$

There are at least two solutions:

1. $\frac{1}{1+L_t}$ where $\frac{1}{1+L_0} = 1/2$ and

2. $u_t(\xi)$

Both are of the form $\psi(t, L_t) = \phi(t, R_t)$

They are different since

$$\lim_{T \to \infty} u_0(\xi) = 1$$

(non-trivial)

What is the PDE?

Itô's formula gives that $\phi(t,x); x > 0$ satisfies (at least in viscosity sense)

$$\left|\frac{1}{x(1+x)}\partial_x\phi\right| = \partial_t\phi + \frac{1}{2}\partial_{x,x}\phi + \frac{1}{x}\partial_x\phi$$

and for

$$\phi(T,x) = \frac{x}{1+x}$$

$$\frac{1}{1+x}\partial_x\phi + \partial_t\phi + \frac{1}{2}\partial_{x,x}\phi = 0$$

For a Markov process

$$dX_t = \sigma(X_t)\, dW_t + b(X_t) dt$$
$$X_0 = x_0$$

and $\xi \in L^\infty$ we treat the equation

$$Y_T = \xi; \quad Y \text{ bounded}$$

$$dY_t = g(X_t, Z_t)\, dt - Z_t\, dW_t$$

and if $\xi = \phi(T, X_T)$:

$$\partial_t \phi + \frac{1}{2}\sigma^2(x)\partial_{x,x}\phi + b(x)\partial_x \phi = g(x, -\sigma(x)\partial_x \phi)$$

with $\phi$ bounded, terminal condition $\phi(T, x)$

In more dimensions, there are subtleties if $\sigma$ can be degenerate.

- El Karoui-Quenez: non-linear pricing

- El Karoui-Quenez-Peng paper in Math Fin

- Peng's book on BSDE, $g-$expectation

- Kobylanski quadratic case Ann Proba

- Barrieu-El Karoui, Ravanelli

- Horst, Cheridito on general equilibrium

- Cheridito and Stadje on (non-)approximation